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CANTOR SET IN ONE DIMENSION

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ITERATED FUNCTIONS AND THE CANTOR SET IN ONE DIMENSION

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Abstract. In this paper we consider the long-term behavior of points in \mathbb{R} under iterations of continuous functions. We show that, given any Cantor set Λ^* embedded in \mathbb{R} , there exists a continuous function $F^* : \mathbb{R} \rightarrow \mathbb{R}$ such that the points that are bounded under iterations of F^* are just those points in Λ^* . In the course of this, we find a striking similarity between the way in which we construct the Cantor middle-thirds set, and the way in which we find the points bounded under iterations of certain continuous functions.

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1 Introduction

In this paper we consider the long-term behavior of points in \mathbb{R} under iterations of continuous functions. For some motivation as to why iterated functions are interesting to study, we can look to the *Mandelbrot Set* (Figure 1). Consider the function $F_c : \mathbb{C} \rightarrow \mathbb{C}$ given by $F_c(z) = z^2 + c$. If we let $z = 0$, let $F_c^1 = F_c$, and let $F_c^{n+1} = F_c \circ F_c^n$, then the behavior of $F_c^n(0)$ as $n \rightarrow \infty$ will depend on the value of c . The Mandelbrot set is the set of values for $c \in \mathbb{C}$ such that $F_c^n(0)$ remains bounded as $n \rightarrow \infty$. In Figure 1, these are the points of the complex plane that are colored black. The Mandelbrot set is the canonical example of a *fractal*. As you zoom in on one area of the boundary new, finer detail will always emerge. The points outside the set are colored based on how quickly F_c^n diverges as n grows. The beauty and the simplicity of the Mandelbrot set helped popularize the study of fractals and iterated functions in the 1980s. For a more in depth discussion of iterated functions see [2]. Sections 2 and 3 of this paper draw from the discussion there.

In this paper, we will relate iterated functions to a different kind of set. It is common in an undergraduate analysis course to construct the *Cantor middle-thirds set*, by removing a countably infinite set of successively smaller open intervals from the set $[0, 1]$. The Cantor middle-thirds set is an example of a certain kind of topological space called a *Cantor set*. In this paper, we show that, given any Cantor set Λ^* embedded in \mathbb{R} , there exists a continuous function $F^* : \mathbb{R} \rightarrow \mathbb{R}$ such that the points that are bounded under iterations of F^* are just those points in Λ^* . In the course of this, we find a striking similarity between the way in which we construct the Cantor middle-thirds set, and the way in which we find the points bounded under iterations of certain continuous functions.

After defining the relevant terms in Section 3, we further motivate our main result. We prove it in Section 4. More formally, our main result is:

Theorem 1 *Given a Cantor set $\Lambda^* \subset \mathbb{R}$, there exists a continuous function $F^* : \mathbb{R} \rightarrow \mathbb{R}$ such that both of the following hold:*

- (i) *if $x \in \Lambda^*$ then $(F^*)^n(x) \in \Lambda^*$ for all n ,*
- (ii) *if $x \notin \Lambda^*$ then $(F^*)^n(x)$ diverges to infinity.*

2 Preliminaries

Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function, let $F^1 = F$ and $F^{n+1} = F \circ F^n$. Note that, depending on what F is, some points $x \in \mathbb{R}$ are such that $|F^n(x)| \rightarrow \infty$ as $n \rightarrow \infty$, while some points are such that $F^n(x) \in [a, b]$ for all n , where $a, b \in \mathbb{R}$. In the first case, we say that $F^n(x)$ *diverges to infinity*. We choose a specific function and study this type of behavior in that function.

Let $F_c : \mathbb{R} \rightarrow \mathbb{R}$ be given by $F_c(x) = x^2 + c$. We can study the long-term behavior of F_c by plotting F_c alongside the function $f(x) = x$. Given x , we start at $F_c(x)$, draw a line horizontally to the graph of f , then vertically to the graph of F_c , then horizontally to the graph of f , and so on. This heuristic method of *graphical analysis* gives a quick check for

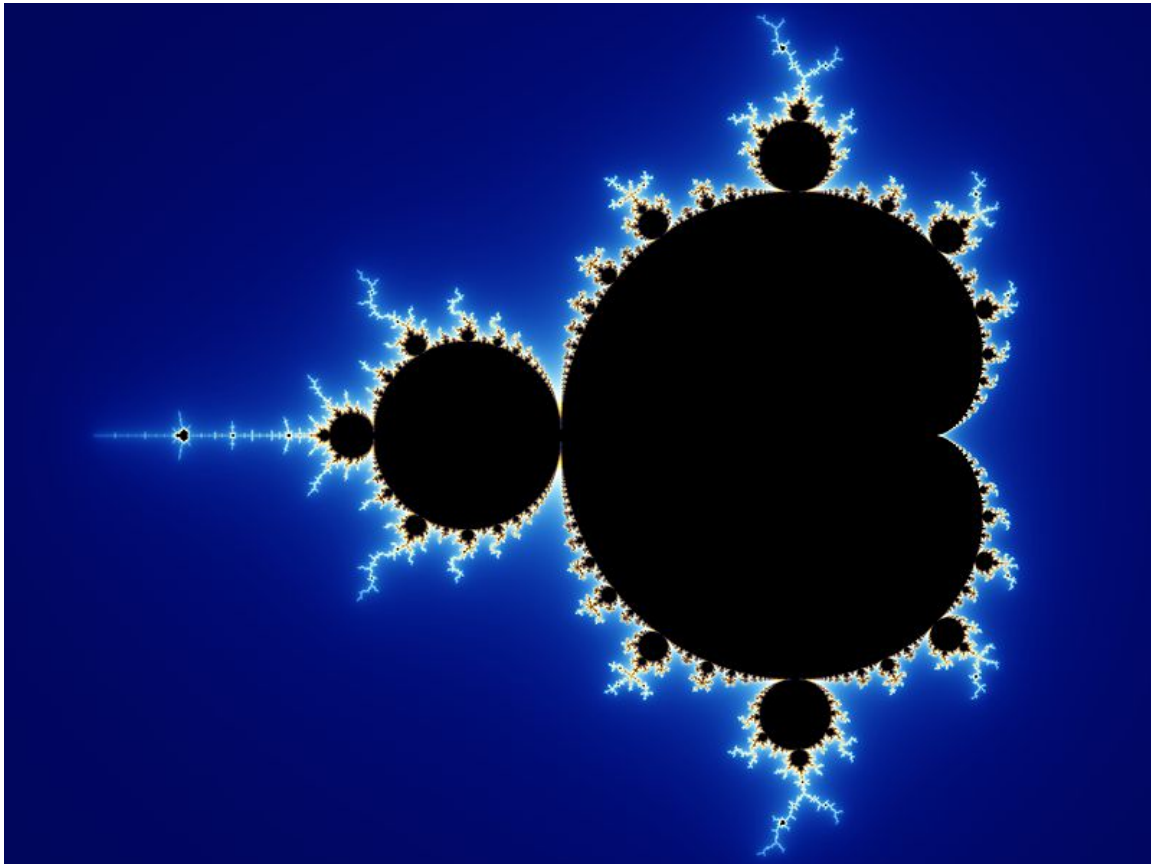


Figure 1: The Mandelbrot set. Points within the set are black, points outside the set are colored based on how quickly F_c diverges. Image created by Wolfgang Beyer and used under the Creative Commons Attribution-Share Alike 3.0 Unported license.

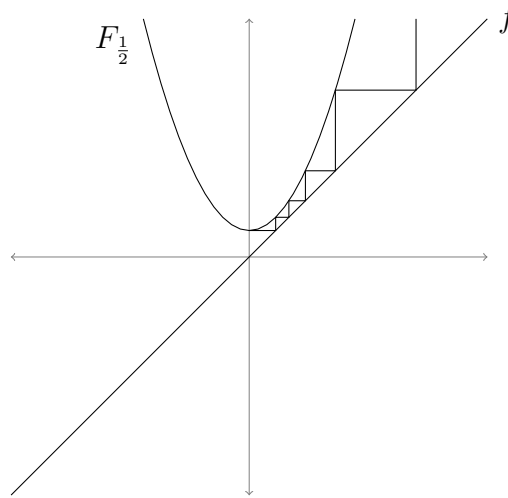


Figure 2: Graphical analysis showing that $F_{\frac{1}{2}}(0) \rightarrow \infty$ as $n \rightarrow \infty$.

the long-term behavior of points under iteration of F_c . For instance, we see from graphical analysis that when $c > \frac{1}{4}$, $F_c^n(x)$ diverges to infinity for all x (Figure 2). When $c = \frac{1}{4}$, $x = \frac{1}{2}$ is a fixed point of F_c , since $F_c(\frac{1}{2}) = \frac{1}{2}$. $F_c^n(x)$ diverges to infinity for all $x \notin [-\frac{1}{2}, \frac{1}{2}]$.

When $c < \frac{1}{4}$, note that there are two distinct solutions to $F_c(x) = x$. Let p denote the larger of the two. Graphical analysis tells us that, letting $I = [-p, p]$, if $x \notin I$, then $F_c^n(x)$ diverges to infinity (Figure 3).

Note that, when $c < -2$, the set $A_0 = \{x \in I : F_c(x) < -p\}$ is nonempty (Figure 4), and that if $x \in A_0$ then $F_c^n(x)$ diverges to infinity. Graphically, we find A_0 by drawing a square centered at the origin with sides parallel to the x and y -axis, and which passes through the points $(p, 0)$ and $(-p, 0)$. A_0 is just those points $x \in I$ such that $F_c(x)$ lies *outside* this square. Note that A_0 is an open interval, and that $F_c([-p, 0] \setminus A_0) = F_c([0, p] \setminus A_0) = I$. Now, for $n > 0$ we define $A_n = \{x \in I : F_c^n(x) \in A_0\}$.

Proposition 2 A_n consists of 2^n open intervals (Figure 5).

Proof: Assume for induction that, for some n , A_n consists of 2^n open intervals and $I \setminus \bigcup_{m=0}^n A_m$ consists of 2^{n+1} closed intervals B_j , $1 \leq j \leq 2^{n+1}$ such that for each j , $F_c^n(B_j) = F_c([-p, 0] \setminus A_0)$ or $F_c^n(B_j) = F_c([0, p] \setminus A_0)$. Then we have that B_j consists of two closed intervals B_j^-, B_j^+ and an open interval B_j^0 such that $F_c^{n+1}(B_j^-) = [-p, 0] \setminus A_0$, $F_c^{n+1}(B_j^+) = [0, p] \setminus A_0$, and $F_c^{n+1}(B_j^0) = A_0$. Then A_{n+1} consists of the open intervals B_j^0 , $1 \leq j \leq 2^{n+1}$ and $I \setminus \bigcup_{m=0}^{n+1} A_m$ consists of 2^{n+2} closed intervals B_j^-, B_j^+ , $1 \leq j \leq 2^{n+1}$, such that each of these intervals is mapped by F_c^{n+1} to $[-p, 0] \setminus A_0$ or $[0, p] \setminus A_0$. \square

Now, if $x \in I$ is such that there is no n where $F_c^n(x) \in A_0$, then there is no point at which x is mapped outside of I . The set $\Lambda_c = \bigcap_{n=0}^{\infty} (I \setminus A_n) = \{x \in I : F_c^n(x) \in I \text{ for all } n \geq 0\}$ is

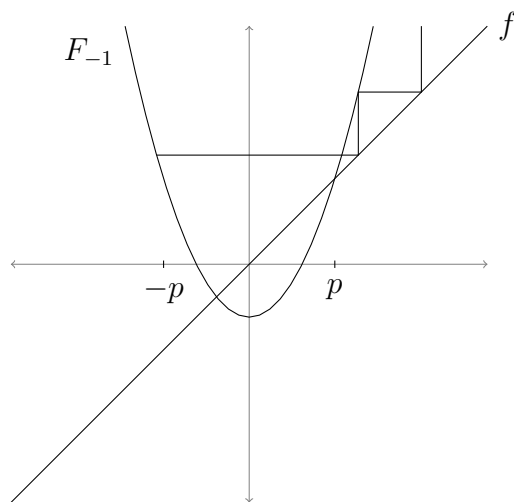


Figure 3: Graphical analysis showing that for $x \notin [-p, p]$, $F_{-1}(x)$ diverges to infinity.

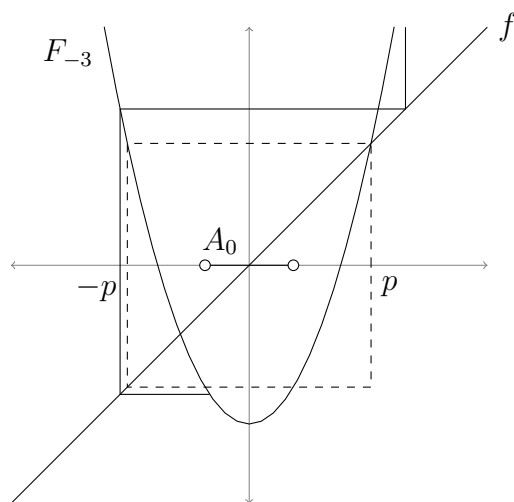


Figure 4: Graphical analysis showing that there exists an open interval $A_0 \subset [-p, p]$ such that for $x \in A_0$, $F_{-3}(x)$ diverges to infinity.

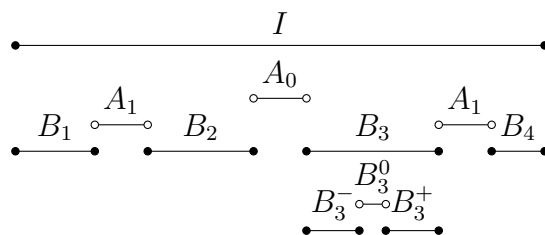


Figure 5: An illustration of the argument made in Proposition 2, for $n = 1$.

then the set of $x \in \mathbb{R}$ such that $F_c^n(x)$ does *not* diverge to infinity. The construction of Λ_c is reminiscent of the construction of the Cantor middle-thirds set. We find that this is indeed the case; Λ_c is a Cantor set.

3 A Cantor set as the set of points bounded under iterations of a function

In this section, we show that for a certain function F , the set of points which are bounded under iterations of F form a Cantor set. The process we use to find these points bears a strong resemblance to the classic construction of the Cantor middle-thirds set. A topological space Λ is a *Cantor set* iff it is totally disconnected, perfect, and compact. A topological space X is *totally disconnected* iff the connected components of X are just the one-point sets of X , and *perfect* iff X is closed and every point in X is a limit point.

Proposition 3 *All Cantor sets are homeomorphic. If Λ is a Cantor set, then $|\Lambda| = \mathfrak{c}$, the cardinality of the continuum.*

Proof: See [4] for a proof of this proposition. □

Note that there exists a c_* such that for all $c < c_*$ there is some λ such that, $|F'_c(x)| > \lambda > 1$ for any $x \in I \setminus A_0$. We prove that for such a c , Λ_c is a Cantor set. This result holds for all $c < -2$, though this will not be proved here. For the remainder of this section, we fix $c < c_* < -2$ and drop the subscripts from F and Λ .

Theorem 4 *If $c < c_*$, then Λ is a Cantor set.*

Proof: Since for all n , A_n consists of 2^n open intervals, Λ is the intersection of closed sets. Hence, Λ is closed. Since $\Lambda \subset I = [-p, p]$, Λ is bounded and hence compact.

We show Λ is totally disconnected. To show that a subspace of \mathbb{R} is totally disconnected, it suffices to show that it contains no open intervals. Assume that $[x, y] \subset \Lambda$. Then $F^n([x, y]) \subset I \setminus A_0$ for all n . Also, by assumption, for all $z \in I \setminus A_0$, $|F'(z)| > \lambda > 1$. So $|(F^n)'(z)| = |(F^{n-1})'(z) \cdot F'(F^{n-1}(z))| > \lambda |(F^{n-1})'(z)|$. Repeating this argument gives that $|(F^n)'(z)| >$

λ^n . By the mean value theorem, there exists some $z \in [x, y]$ such that $|F^n(y) - F^n(x)| = |(y - x)(F^n)'(z)| > (y - x)\lambda^n$. F^n then expands $[x, y]$ by a factor of at least λ^n , so if $x \neq y$ then there exists an n such that $F^n([x, y]) \not\subset I$. Thus, $x = y$ and the only closed intervals of Λ are single point sets. Thus, Λ contains no open intervals and is totally disconnected.

To show that Λ is perfect, we show that if $x \in \Lambda$, then x is a limit point. Consider an open interval T containing x . By the argument in the previous paragraph, there exists an n such that $-p \in F^n(T)$ or $p \in F^n(T)$. Since $-p \in \Lambda$ and $p \in \Lambda$, we have that $(F^n)^{-1}(-p) \subset \Lambda$ and $(F^n)^{-1}(p) \subset \Lambda$. Also, $((F^n)^{-1}(-p) \cup (F^n)^{-1}(p)) \cap T \neq \emptyset$. Thus, any open set containing x contains points in Λ other than x and hence x is a limit point of Λ . Thus, Λ is perfect. \square

We now reformulate the definition of Λ slightly, in a way that will be useful in the next section of this paper. Let $C_0 = I$ and $C_{n+1} = C_n \setminus A_n$. C_n then consists of 2^n closed intervals, and $\Lambda = \bigcap_{i=0}^{\infty} C_n$. We then obtain the following corollary:

Corollary 5 *Given $\epsilon > 0$, there exists N such that for all $n \geq N$ the segments of C_n are all less than ϵ in length.*

Proof: Assume this is false. Then there must exist a system of intervals $\{T_n\}_{n=0}^{\infty}$, where T_n is one of the 2^n intervals in C_n , and where $T_m \subset T_n$ whenever $m \geq n$. But then $\bigcap_{i=0}^{\infty} T_i \subset \Lambda$ is of length greater than ϵ , contradicting the total disconnectedness of Λ . \square

4 Proof of the main theorem

In this section we prove the following result, generalizing some of the ideas from the last section to arbitrary Cantor sets in \mathbb{R} .

Theorem 6 *Given a Cantor set $\Lambda^* \subset \mathbb{R}$, there exists a continuous function $F^* : \mathbb{R} \rightarrow \mathbb{R}$ such that both of the following hold:*

- (i) *if $x \in \Lambda^*$ then $(F^*)^n(x) \in \Lambda^*$ for all n ,*
- (ii) *if $x \notin \Lambda^*$ then $(F^*)^n(x)$ diverges to infinity.*

We require several results for this to obtain.

Proposition 7 *There exists a Cantor set $\Lambda \in \mathbb{R}$ and a continuous function F such that (i) and (ii) in Theorem 6 hold.*

Proof: Λ and F were constructed in the previous section. \square

Proposition 8 *Let $\Lambda^* \subset [a, b] \in \mathbb{R}$ be a Cantor set such that $a, b \in \Lambda^*$. Then there is a nested sequence of closed sets C_n^* such that C_n^* consists of 2^n segments each of length less than or equal to $(\frac{2}{3})^n(b - a)$. Furthermore, $\partial(C_n^*) \subset \Lambda^*$.*

Proof: We first show that, given $[c, d] \subset [a, b]$, $c \neq d$, there exists an open interval $(e, f) \subset [c, d]$ such that $(e, f) \cap \Lambda^* = \emptyset$. Assume not. That is, assume that there exists some $[c, d] \subset [a, b]$, $c \neq d$ such that for all open intervals $(e, f) \subset [c, d]$, $(e, f) \cap \Lambda^* \neq \emptyset$. Then there exists a point $g \in (c, d)$ such that $g \in \Lambda^*$. Because Λ^* is totally disconnected, there is a point $h \in (c, d)$ such that $h \notin \Lambda^*$. Then, by assumption, for all $\epsilon > 0$, $(h - \epsilon, h + \epsilon)$ contains a point from Λ^* . But then h is a limit point of Λ^* , and so $h \in \Lambda^*$ because Λ^* is closed. Hence, the claim holds and given $[c, d] \subset [a, b]$, $c \neq d$, there exists an open interval $(e, f) \subset [c, d]$ such that $(e, f) \cap \Lambda^* = \emptyset$.

Now, given $[c, d] \subset [a, b]$, choose an interval $[g, h] \subset [c, d]$ such that $g - c \geq \frac{1}{3}(d - c)$ and $d - h \geq \frac{1}{3}(d - c)$. From the previous paragraph, we may choose an open interval $(e, f) \subset [g, h]$ such that $(e, f) \cap \Lambda^* = \emptyset$, and $[c, e], [f, d]$ are such that $e - c < \frac{2}{3}(d - c)$ and $d - f < \frac{2}{3}(d - c)$. Also, given an interval (e, f) that satisfies these conditions, since Λ^* is compact, $f' = \inf\{x \in \Lambda^* : f < x\}$ and $e' = \sup\{x \in \Lambda^* : x < e\}$ exist. And (e', f') also is such that $(e', f') \cap \Lambda^* = \emptyset$, $e' - c < \frac{2}{3}(d - c)$, and $d - f' < \frac{2}{3}(d - c)$. Thus, we may assume that $e, f \in \Lambda^*$.

We now define the sets C_n^* . Let $C_0^* = [a, b]$, which contains $2^0 = 1$ interval of length $(\frac{2}{3})^0(b - a)$. Given C_n^* consisting of 2^n closed intervals of length each less than or equal to $(\frac{2}{3})^n(b - a)$ and greater than 0, and such that $\partial(C_n^*) \subset \Lambda^*$, by the previous paragraph we may split each of these intervals into two intervals of length less than or equal to $(\frac{2}{3})^{n+1}(b - a)$. Furthermore, when we pick an open interval (e, f) to remove we may do so such that $e, f \in \Lambda^*$. This guarantees that $\partial(C_{n+1}^*) \subset \Lambda^*$. Also, since Λ^* is perfect, no intervals of C_n^* are of length 0; otherwise there would be some $x \in \Lambda^*$ that is not a limit point of Λ^* .

Let $C^* = \bigcap_{i=0}^{\infty} C_i^*$. Showing that $C^* = \Lambda^*$ completes the proof. We first fix some notation. C_n^* is composed of 2^n closed intervals, denoted $[a_{n,j}^*, b_{n,j}^*]$ where $1 \leq j \leq 2^n$. Given C_{n-1}^* , to get C_n^* we remove 2^{n-1} open intervals, denoted $(c_{n,j}^*, d_{n,j}^*)$, where $1 \leq j \leq 2^{n-1}$. Since there is a corresponding construction for Λ , we similarly denote the 2^n closed intervals of C_n by $[a_{n,j}, b_{n,j}]$, and the 2^{n-1} open intervals removed from C_{n-1} by $(c_{n,j}, d_{n,j})$.

Let $x \in \Lambda^*$. Then if $x \notin C^*$, it was removed in some step of the construction of C^* . But this cannot be so, since we only removed points not in Λ^* . Let $x \in C^*$. Then for all n there exists an interval $[a_{n,j}^*, b_{n,j}^*] \subset C_n^*$ such that $x \in [a_{n,j}^*, b_{n,j}^*]$. Since $a_{n,j}^* \in \Lambda^*$ and $|x - a_{n,j}^*| \leq (\frac{2}{3})^n(a - b)$ for all n , x is a limit point of $a_{n,j}$ and hence $x \in \Lambda^*$. Thus, $C^* = \Lambda^*$. \square

Corollary 9 *Given $\epsilon > 0$, there exists N such that for all $n \geq N$ the segments of C_n^* are all less than ϵ in length.*

Proof: Follows immediately from the construction of C_n^* . \square

We now construct a homeomorphism $\phi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\phi(\Lambda) = \Lambda^*$ (Figure 6). Define $\phi_0 : \mathbb{R} \setminus C_0 \rightarrow \mathbb{R} \setminus C_0^*$ by

$$\phi_0(x) = \begin{cases} x + (a_{0,1}^* - a_{0,1}) & \text{if } x \leq a_{0,1} \\ x + (b_{0,1}^* - b_{0,1}) & \text{if } x \geq b_{0,1} \end{cases} \quad (1)$$

For $n \geq 1$ let $D_n = \text{cl}(\{(c_{n,j}, d_{n,j}) : 1 \leq j \leq 2^n\}) = \text{cl}(C_{n-1} \setminus C_n)$, and $D_n^* = \text{cl}(\{(c_{n,j}^*, d_{n,j}^*) : 1 \leq j \leq 2^n\}) = \text{cl}(C_{n-1}^* \setminus C_n^*)$. Define $\phi_n : D_n \rightarrow D_n^*$ by

$$\phi_n(x) = \left(\frac{d_{n,j}^* - c_{n,j}^*}{d_{n,j} - c_{n,j}} \right) (x - c_{n,j}) + c_{n,j}^* \quad \text{for } x \in [c_{i,j}, d_{i,j}] \text{ where } 1 \leq j \leq 2^{n-1}. \quad (2)$$

Recalling that functions are sets of ordered pairs, let $\hat{\phi} = \bigcup_{n=0}^{\infty} \phi_n$. Note that $\hat{\phi} : \mathbb{R} \setminus \hat{\Lambda} \rightarrow \mathbb{R} \setminus \hat{\Lambda}^*$ is a function, where $\hat{\Lambda} = \{x \in \Lambda : \forall n, j (x \neq a_{n,j} \wedge x \neq b_{n,j})\}$ is the set of all points in Λ that are not the endpoints of an interval in some C_n , and $\hat{\Lambda}^* = \{x \in \Lambda^* : \forall n, j (x \neq a_{n,j}^* \wedge x \neq b_{n,j}^*)\}$ is the analogous subset of Λ^* .

Define $\phi : \mathbb{R} \rightarrow \mathbb{R}$ by:

$$\phi(x) = \begin{cases} \hat{\phi}(x) & \text{if } x \in \mathbb{R} \setminus \hat{\Lambda} \\ \lim_{k_n \rightarrow x} \hat{\phi}(k_n) & \text{if } x \in \hat{\Lambda} \end{cases} \quad (3)$$

where $k_n = a_{n,j}$ or $b_{n,j}$ is a fixed sequence of the boundary points of C_n converging to x , which exists by Corollary 5. The choice of k_n doesn't matter; we see in Proposition 11 that ϕ is continuous everywhere. The proof of this statement holds independent of our choice of k_n , and hence $\lim_{k_n \rightarrow x} \phi(k_n) = \lim_{k_n \rightarrow x} \hat{\phi}(k_n)$ will be the same for any sequence of points $k_n \rightarrow x$.

We now show that ϕ is a homeomorphism. It suffices to show that ϕ is a continuous bijection.

Lemma 10 *If $x \in [a_{n,j}, b_{n,j}] \subset C_n$, then $\phi(x) \in [a_{n,j}^*, b_{n,j}^*] \subset C_n^*$.*

Proof: If there is some n such that $x \in D_n$, the result follows from the construction of $\hat{\phi}$. Otherwise, $x \in \hat{\Lambda}$. Consider the sequence $k_m \rightarrow x$ that defines $\phi(x)$. Then k_m is eventually contained in the interval $[a_{n,j}, b_{n,j}]$, and since $k_m \in D_m$, $\hat{\phi}(k_m)$ is eventually contained in $[a_{n,j}^*, b_{n,j}^*]$. But $\hat{\phi}(k_m) \rightarrow \phi(x)$ by definition, and $[a_{n,j}^*, b_{n,j}^*]$ is closed, so $\phi(x) \in [a_{n,j}^*, b_{n,j}^*]$. \square

Proposition 11 *ϕ is continuous everywhere.*

Proof: If $x \notin \Lambda$, then ϕ is continuous on an open interval around x , by construction of $\hat{\phi}$. Let $x \in \Lambda$. Take a sequence $p_i \rightarrow x$. For each p_i , if $p_i \neq x$ let $M_i = [a_{k(i),j}, b_{k(i),j}]$ be the interval of $C_{k(i)}$ containing both x and p_i , where $k(i)$ gives the index of the last C_n in which both x and p_i are contained in the same interval; $k(i)$ is well-defined by Corollary 5. If $p_i = x$, let $M_i = \{x\}$. By Corollary 5, $k(i) \rightarrow \infty$ as $p_i \rightarrow x$, whenever $k(i)$ is defined. By Lemma 10, as well as the construction of $\hat{\phi}$ when $p_i \neq x$, $\phi(p_i) \in \phi(M_i)$. By Corollary 9, $\phi(p_i) \rightarrow \phi(x)$ and thus ϕ is continuous at x . Hence, ϕ is continuous everywhere. \square

Proposition 12 *ϕ is a bijection.*

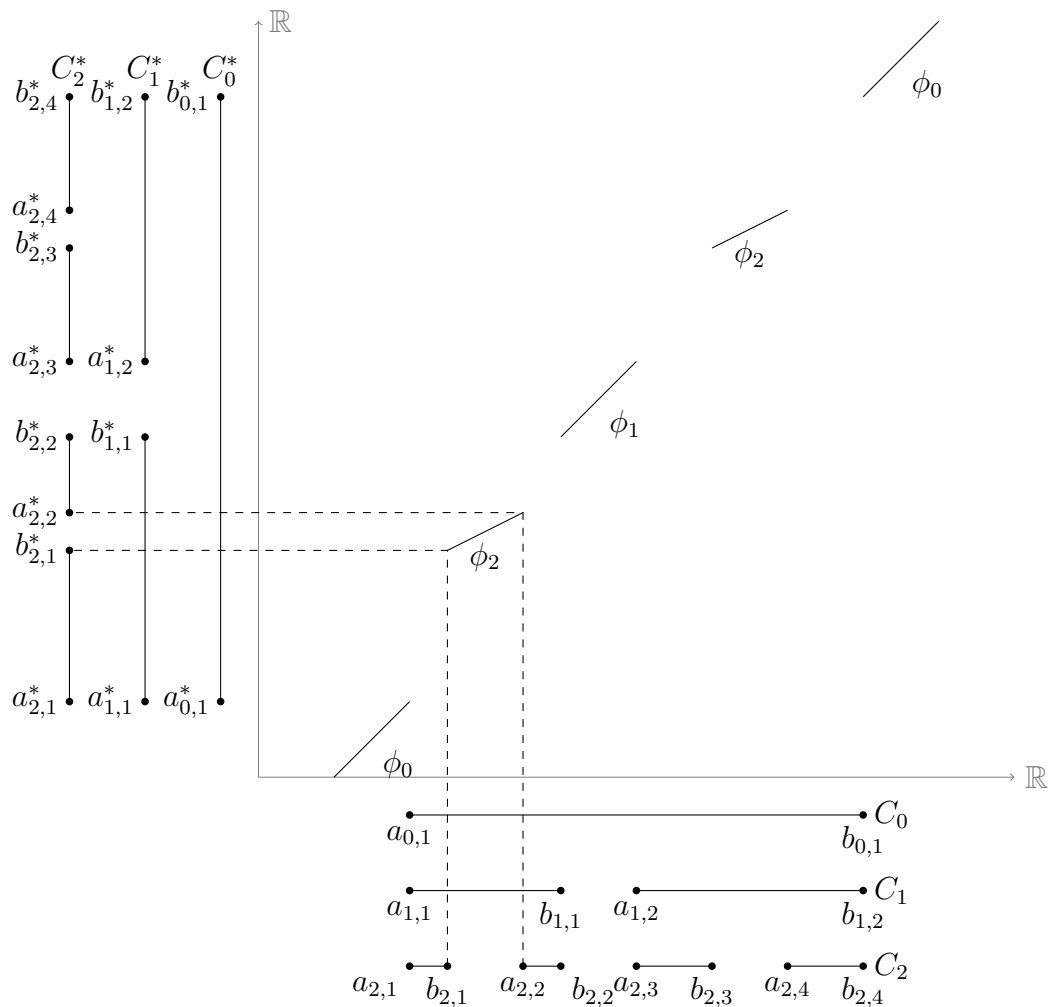


Figure 6: Definition of ϕ_0 , ϕ_1 , and ϕ_2 based on the first three stages of the constructions of Λ and Λ^* .

$$\begin{array}{ccc}
 \mathbb{R} \setminus \Lambda \cup \Lambda & \xrightarrow{\phi} & \mathbb{R} \setminus \Lambda^* \cup \Lambda^* \\
 \downarrow F & & \downarrow F^* \\
 \mathbb{R} \setminus \Lambda \cup \Lambda & \xrightarrow{\phi} & \mathbb{R} \setminus \Lambda^* \cup \Lambda^*
 \end{array}$$

Figure 7: Definition of F^*

Proof: Note that ϕ is strictly increasing on $\mathbb{R} \setminus \Lambda$, i.e. if $x, y \notin \Lambda$ and $x < y$, then $\phi(x) < \phi(y)$. Now, given arbitrary $x, y \in \mathbb{R}$, since Λ is totally disconnected there exists sequences $x_n \rightarrow x$ and $y_n \rightarrow y$ where $x_n, y_n \notin \Lambda$ and $x_n < x, y_n > y$. What is more, there exists $z_1, z_2 \notin \Lambda$ such that $x < z_1 < z_2 < y$. We then have that $\phi(x_n) < \phi(z_1) < \phi(z_2) < \phi(y_n)$ for all n . Hence, by continuity $\phi(x) \leq \phi(z_1) < \phi(z_2) \leq \phi(y_n)$. So ϕ is strictly increasing everywhere. Since ϕ is continuous, $\phi(x) \rightarrow -\infty$ as $x \rightarrow -\infty$, and $\phi(x) \rightarrow \infty$ as $x \rightarrow \infty$, it follows that ϕ is a bijection. \square

Corollary 13 $\phi(\Lambda) = \Lambda^*$

Proof: This follows from the previous proposition, as well as the fact that $\phi(\mathbb{R} \setminus \Lambda) = \mathbb{R} \setminus \Lambda^*$. \square

Corollary 14 $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is a homeomorphism.

Proof: Since $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous bijection, by Brouwer’s invariance of domain theorem (see [1] and page 172 of [3]) it is a homeomorphism. \square

Now, define F^* so that the diagram in Figure 7 commutes. It is easy to check that F^* is a continuous function that satisfies criteria (i) and (ii) in Theorem 6. Indeed, since $F^* = \phi \circ F \circ \phi^{-1} : \mathbb{R} \rightarrow \mathbb{R}$, ϕ is a homeomorphism, and F is continuous, F^* is continuous. If $x \in \Lambda^*$, then $\phi^{-1}(x) \in \Lambda$ by the construction of ϕ , and so $F^n(\phi^{-1}(x)) \in \Lambda$ for all n . Thus, $\phi(F^n(\phi^{-1}(x))) = F^*(x) \in \Lambda^*$ for all n , again by the construction of ϕ . Similarly, if $x \notin \Lambda^*$, then $\phi^{-1}(x) \notin \Lambda$. For any $s \in \mathbb{R}$, there exists some N such that for all $n > N$, $F^n(\phi^{-1}(x)) > \phi^{-1}(s)$. Since ϕ is strictly increasing, it is order preserving and so $\phi(F^n(\phi^{-1}(x))) = (F^*)^n(x) > \phi(\phi^{-1}(s)) = s$. Thus, $(F^*)^n(x)$ diverges to infinity.

We conclude by stating an additional conjecture.

Conjecture 15 Given a Cantor set $\Lambda^* \subset \mathbb{R}^m$, there exists a continuous function F^* such that both of the following hold:

- (i) if $x \in \Lambda^*$ then $(F^*)^n(x) \in \Lambda^*$ for all n ,
- (ii) if $x \notin \Lambda^*$ then $\|(F^*)^n(x)\|$ diverges to infinity.

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