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Abstract. We give a simple, direct proof of the easy fact about the Weierstrass Representation, namely, that it always gives a minimal surface. Most presentations include the much harder converse that every simply connected minimal surface is given by the Weierstrass Representation.

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1 Introduction

A soap film on a wire frame seeks to minimize area for the given boundary (Figure 1). Such films are delicately balanced, with their principal curvatures (maximum bendings) equal in magnitude and opposite in sign, that is, with average or mean curvature zero. Such surfaces are called minimal surfaces. Minimal surfaces long have fascinated and inspired mathematicians [1], physicists, biologists, and material scientists, with recent application to the structure of block polymers [2].

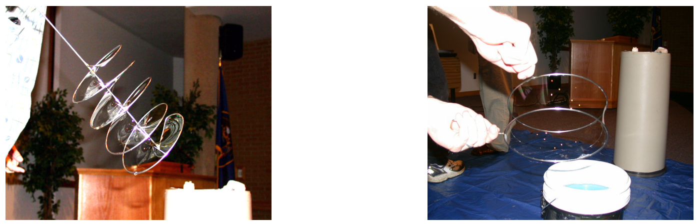

Figure 1: A helicoid (left) and a catenoid (right) formed by a soap film for the wire boundary are examples of minimal surfaces. academic.csuohio.edu/oprea_j/utah/Prospects.html Accessed 8/15/12. Used by permission, all rights reserved.

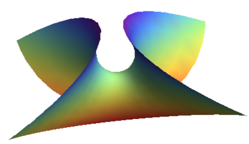

Figure 2: Enneper's Surface - another minimal surface, created using Mathematica.

Karl Weierstrass (1866, [3, p. 146]) found a way to represent minimal surfaces in terms of complex functions of a complex variable, such as $f(z) = z^2$ or $f(z) = e^z$. The Weierstrass representation theorem states that for any complex differentiable functions $f(z)$ and $g(z)$ on the unit disk or complex plane, the surface $\mathbf{x}(z)$ is minimal, where $\mathbf{x}$ is the real part of an integral of

$$\phi = \begin{bmatrix} f(1-g^2) \\ if(1+g^2) \\ 2fg \end{bmatrix}.$$

In this paper we give a simple direct proof of the easy direction: that every surface given by Weierstrass's formula is a minimal surface.

Section 2 provides some background and definitions and gives the example of Enneper's Surface. Section 3 presents the key lemma. Section 4 proves that the Weierstrass representation always gives a minimal surface.

2 Background and the Example of Enneper's Surface

We start with some definitions. There are many ways to define a surface in $\mathbb{R}^3$, for example as the graph of a real-valued function of two variables or as a level set of a real-valued function of three variables. Perhaps the most convenient definition is as a map from the plane into $\mathbb{R}^3$.

Definition 1. A smooth surface is a smooth map $\mathbf{x}(z) : X \to \mathbb{R}^3$, where X could be the entire complex plane or the unit disk in the complex plane, with complex coordinate $z = u + iv$. We assume that the map is nonsingular in the sense that the area element dA, given by calculus in terms of the partial derivatives $\mathbf{x}_u$ and $\mathbf{x}_v$ by

$$dA^2 = \left(\mathbf{x}_u^2\, \mathbf{x}_v^2 - (\mathbf{x}_u \cdot \mathbf{x}_v)^2\right)\; du^2 dv^2, \tag{1}$$

is nonzero.

For example, Enneper's Surface (Figure 2) is given by

$$\mathbf{x}(z) = \begin{bmatrix} u - \dfrac{u^3}{3} + uv^2 \\ -v - u^2 v + \dfrac{v^3}{3} \\ u^2 - v^2 \end{bmatrix}. \tag{2}$$

Remark. A smooth surface generally curves different amounts in different directions. The mean curvature is the mean or average of the largest and smallest such curvatures. For a minimal surface, these curvatures must be equal in magnitude and opposite in sign. A *minimal surface* is a smooth surface that has mean curvature zero at every point [5]. A surface which minimizes area, such as a soap film in equilibrium, must have this delicate balance of mean curvature zero. If a soap film were curving upward in all directions, it would move upward and get smaller. If it were curving downward in all directions, it would move downward and get smaller. Thus for a soap film, trying to minimize area, the *mean* curvature must be zero at every point. That is why mean-curvature-zero surfaces are called minimal. We note that this idea does not apply to soap bubbles because a soap bubble curves around a volume of air trapped inside. All that we'll use in this paper is the formula [5] for the mean curvature H of a surface $\mathbf{x}(z)$:

$$H = P_\perp \frac{\mathbf{x}_v^2 \mathbf{x}_{uu} - 2(\mathbf{x}_u \cdot \mathbf{x}_v)\mathbf{x}_{vv} + \mathbf{x}_u^2 \mathbf{x}_{vv}}{\mathbf{x}_u^2 \mathbf{x}_v^2 - (\mathbf{x}_u \cdot \mathbf{x}_v)^2}, \tag{3}$$

where $P_\perp$ denotes projection onto the line normal to the surface. Note that the denominator is nonzero by the assumption that dA given in (1) is nonzero.

Definition 2. *The square of a complex vector* $\mathbf{v} = (v_1, v_2, v_3)$ *is defined as*

$$\mathbf{v}^2 = \mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + v_3^2.$$

Example. Before formally stating and proving that the Weierstrass representation yields minimal surfaces, we'll give an example. Start with two complex differentiable functions, such as f(z) = 1 and g(z) = z. The Weierstrass representation then gives a minimal surface by the formula

$$\mathbf{x}(z) = \text{Re} \begin{bmatrix} \int f(z)(1-(g(z))^2)\ dz \\ \int if(z)(1+(g(z))^2)\ dz \\ \int 2f(z)g(z)\ dz \end{bmatrix} = \text{Re} \begin{bmatrix} z - \dfrac{z^3}{3} \\ iz + \dfrac{iz^3}{3} \\ z^2 \end{bmatrix}.$$

This is Enneper's Surface as given in (2). Let us compute the mean curvature by formula (3). First note that

$$\mathbf{x}_u = \frac{\partial}{\partial u}\left\{ \text{Re} \begin{bmatrix} z - \dfrac{z^3}{3} \\ iz + \dfrac{iz^3}{3} \\ z^2 \end{bmatrix} \right\} = \text{Re} \left\{ \frac{d}{dz} \begin{bmatrix} z - \dfrac{z^3}{3} \\ iz + \dfrac{iz^3}{3} \\ z^2 \end{bmatrix} \frac{\partial z}{\partial u} \right\} = \text{Re} \begin{bmatrix} 1-z^2 \\ i(1+z^2) \\ 2z \end{bmatrix},$$

because $\dfrac{\partial z}{\partial u} = 1$ as $z = u + iv$. Thus

$$\mathbf{x}_u = \begin{bmatrix} 1-u^2+v^2 \\ -2uv \\ 2u \end{bmatrix}.$$

Similarly

$$\mathbf{x}_v = \text{Re} \left\{ \frac{d}{dz} \begin{bmatrix} z - \dfrac{z^3}{3} \\ iz + \dfrac{iz^3}{3} \\ z^2 \end{bmatrix} \frac{\partial z}{\partial v} \right\} = \text{Re} \begin{bmatrix} i(1-z^2) \\ -(1+z^2) \\ 2iz \end{bmatrix} = \begin{bmatrix} 2uv \\ -1-u^2+v^2 \\ -2v \end{bmatrix},$$

because $\dfrac{\partial z}{\partial v} = i$ as $z = u + iv$. Therefore

$$\mathbf{x}_u \cdot \mathbf{x}_v = 2uv - 2u^3v + 2uv^3 + 2uv + 2u^3v - 2uv^3 - 4uv = 0. \tag{4}$$

Furthermore

$$\begin{aligned} \mathbf{x}_u^2 &= \mathbf{x}_u \cdot \mathbf{x}_u \\ &= 1 + u^4 + v^4 + 2v^2 - 2u^2 - 2u^2v^2 + 4u^2v^2 + 4u^2 \\ &= (1+u+v)^2. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbf{x}_v^2 &= \mathbf{x}_v \cdot \mathbf{x}_v \\ &= 1 + u^4 + v^4 - 2v^2 + 2u^2 - 2u^2v^2 + 4u^2v^2 + 4v^2 \\ &= (1 + u + v)^2. \end{aligned}$$

Thus

$$\mathbf{x}_u^2 = \mathbf{x}_v^2. \tag{5}$$

Furthermore, since

$$\mathbf{x}_{uu} = \frac{\partial}{\partial u} \text{Re} \begin{bmatrix} 1 - z^2 \\ i(1 + z^2) \\ 2z \end{bmatrix} = \text{Re} \left\{ \frac{d}{dz} \begin{bmatrix} 1 - z^2 \\ i(1 + z^2) \\ 2z \end{bmatrix} \frac{\partial z}{\partial u} \right\} = \text{Re} \begin{bmatrix} -2z \\ 2iz \\ 2 \end{bmatrix}$$

and

$$\begin{aligned} \mathbf{x}_{vv} &= \frac{\partial}{\partial v} \text{Re} \begin{bmatrix} i(1 - z^2) \\ -(1 + z^2) \\ 2iz \end{bmatrix} = \text{Re} \left\{ \frac{d}{dz} \begin{bmatrix} i(1 - z^2) \\ -1(1 + z^2) \\ 2iz \end{bmatrix} \frac{\partial z}{\partial v} \right\} \\ &= \text{Re} \left\{ \begin{bmatrix} -2iz \\ -2z \\ 2i \end{bmatrix} i \right\} = \text{Re} \begin{bmatrix} 2z \\ -2iz \\ -2 \end{bmatrix}, \end{aligned}$$

therefore

$$\mathbf{x}_{uu} = -\mathbf{x}_{vv}. \tag{6}$$

Plugging (4), (5), (6) into (3) gives $H = 0$, confirming that Enneper's Surface is a minimal surface.

3 The Key Lemma

The following lemma gives the key property of the Weierstrass Representation.

Lemma 3. *For complex numbers f and g, and* $\phi = \begin{bmatrix} f(1 - g^2) \\ if(1 + g^2) \\ 2fg \end{bmatrix}$, *the following statements are true.*

(1) $(\text{Re } \phi)^2 - (\text{Im } \phi)^2 = 0.$

(2) $(\text{Re } \phi) \cdot (\text{Im } \phi) = 0.$

Proof. First we note that

$$\begin{aligned} \phi^2 &= f^2 - 2f^2g^2 + f^2g^4 - f^2 - 2f^2g^2 - f^2g^4 + 4f^2g^2 \\ &= 0. \end{aligned}$$

Therefore, we get

$$0 = \text{Re }\ \phi^2 = (\text{Re }\ \phi)^2 - (\text{Im }\ \phi)^2,$$

because Re $(a+ib)^2 = a^2 - b^2$. Similarly,

$$0 = \text{Im }\ \phi^2 = 2(\text{Re }\ \phi) \cdot (\text{Im }\ \phi),$$

because Im $(a+ib)^2 = 2ab$. □

We note that Lemma 3 also applies for any complex-valued function on any domain.

4 Weierstrass Representation Gives a Minimal Surface

The following theorem is our main result.

Theorem 4. *For any complex differentiable functions $f(z)$ and $g(z)$ on the unit disk or complex plane, the surface $\mathbf{x}(z)$ is minimal, where $\mathbf{x}$ is the real part of an integral of*

$$\phi = \begin{bmatrix} f(1-g^2) \\ if(1+g^2) \\ 2fg \end{bmatrix}.$$

Actually you can allow $g(z)$ to have poles as long as fg^2 is differentiable.

Proof of theorem. Letting Φ denote an integral of ϕ, we compute that

$$\mathbf{x}_u = \text{Re }[\Phi_u] = \text{Re }\ \left[\frac{d\Phi}{dz}\frac{\partial z}{\partial u}\right] = \text{Re }\ \phi,$$

as $z = u + iv$ so $\dfrac{\partial z}{\partial u} = 1$. Similarly,

$$\mathbf{x}_v = \text{Re }[\Phi_v] = \text{Re }\ \left[\frac{d\Phi}{dz}\frac{\partial z}{\partial v}\right] = \text{Re }(i\phi) = -\text{ Im }\ \phi,$$

because Re $[i(a+ib)] = -b$. By Lemma 3(2), we get

$$\mathbf{x}_u \cdot \mathbf{x}_v = 0. \tag{7}$$

Since

$$\mathbf{x}_u^2 \quad = \quad (\text{Re }\ \phi)^2$$

and

$$\mathbf{x}_v^2 \quad = \quad (\text{Im } \phi)^2,$$

by Lemma 3(1), we get

$$\mathbf{x}_u^2 = \mathbf{x}_v^2. \tag{8}$$

Furthermore, since

$$\mathbf{x}_{uu} = \frac{d\mathbf{x}_u}{du} = \frac{d\mathbf{x}_u}{dz}\frac{\partial z}{\partial u} = \text{Re } (\phi')$$

and

$$\mathbf{x}_{vv} = \frac{d\mathbf{x}_v}{dv} = \frac{d\mathbf{x}_v}{dz}\frac{\partial z}{\partial v} = -\text{ Re } (\phi'),$$

therefore,

$$\mathbf{x}_{uu} + \mathbf{x}_{vv} = 0. \tag{9}$$

Plugging Equations (7), (8), (9) into formula (3) yields $H = 0$, the definition of a minimal surface. □

Remark. The converse of Theorem 3 also holds: every simply connected minimal surface is given by the Weierstrass representation [4, Theorem 2.66].

References

[1] Stefan Hildebrandt and Antony Tromba. *The Parsimonious Universe*, Springer, 1996.

[2] Edwin L. Thomas, David M. Anderson, Chris S. Henkee, and David Hoffman. "Periodic area-minimizing surfaces in block copolymers," Nature 334 (1988), 598-602.

[3] Johannes C. C. Nitsche. *Lectures on Minimal Surfaces (Volume I)*. Cambridge University Press, Cambridge, 1989.

[4] Michael Dorff. "Soap films, differential geometry, and minimal surfaces," *Explorations in Complex Analysis*, 85-159, Math. Assoc. Amer., Washington, DC, 2012.

[5] Frank Morgan. *Riemannian Geometry: A Beginner's Guide.* A K Peters, Wellesley, second edition, 1998.