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**TANGENT CIRCLES IN THE  
HYPERBOLIC DISK**

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# TANGENT CIRCLES IN THE HYPERBOLIC DISK

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**Abstract.** Constructions of tangent circles in the hyperbolic disk, interpreted in Euclidean geometry, give us examples of four mutually tangent circles. These are shown to satisfy Descartes's Theorem for tangent circles. We also show that the Archimedes twin circles in the hyperbolic arbelos are usually not hyperbolic congruent, even though they are Euclidean congruent. We include a few construction instructions because all items under consideration require surprisingly few steps.

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## 1 Introduction

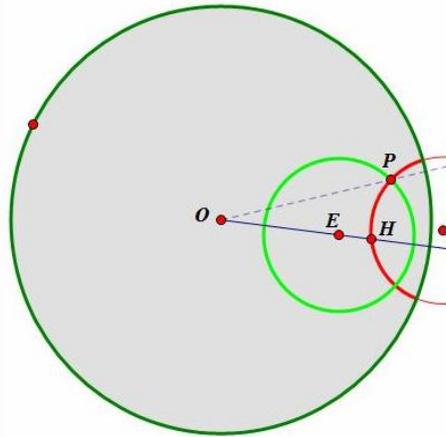
In this quick trip to the Poincaré disk model of hyperbolic geometry, we use compass and straightedge to create unusual combinations of tangent circles, circles that touch at a single point, where if you take the tangent line of that point for each circle, they will be the exact same line, and use nifty shortcuts to keep the creation processes brief. We prove the short-cuts and their applications, including a proof that the Archimedes Twin Circles are not congruent in hyperbolic geometry. Our work uses constructions to create our examples and support our proofs. We invite the reader to make her or his own versions using compass and straightedge or dynamic geometry software.

## 2 Background

Among the axioms underlying Euclidean geometry, there are five that get special attention. Other geometries use some of these same axioms, but they may not use all of them. Axioms are statements that are universally accepted as true without requiring a proof. Hyperbolic geometry follows the first four of these axioms, but not the fifth axiom, the Parallel Postulate.

One negation of the Parallel Postulate says, “Through a point not on a line, there are many lines parallel to the given line.” The only way for Euclidean-mode creatures like us to see such lines is to let them appear bent. The geometry for this bent space is called hyperbolic geometry. Henri Poincaré defined a disk model of hyperbolic geometry in which Euclidean axioms, like two points determine a line, are true, along with the negation of the Parallel Postulate mentioned above.

The Poincaré disk is made of the Euclidean points inside a unit disk, not including the boundary. These points have no size of their own, they just mark locations within the disk. The hyperbolic lines are diameters of the disk and arcs of circles orthogonal to the boundary of the disk, that is, the angles at the points where the disk and hyperbolic line meet are  $\pi/2$  radians. The center of the disk is called  $O$ . In hyperbolic geometry, the angle sum of a triangle can be anything between 0 and  $\pi$  radians. Hyperbolic distance is not the same as Euclidean distance because the length of a segment depends on its location in the disk relative to the boundary. A small segment near  $O$  is much shorter than a segment of the same Euclidean size near the boundary. A hyperbolic circle looks Euclidean, but the position of its hyperbolic center depends on the circle’s location. That means any hyperbolic circle whose center is not  $O$  has two centers: one Euclidean and one hyperbolic,  $E$  and  $H$  in Figure 1 respectively.



**Figure 1:** Hyperbolic circle with Euclidean and hyperbolic centers.

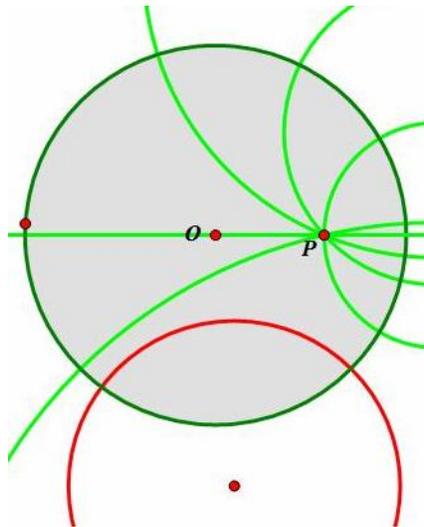
The first axiom states that two points determine a line. Suppose you have two points  $A$  and  $B$  inside the disk. When the two points are collinear with the center  $O$ , the hyperbolic line is simply the diameter of the disk. When the two points are not collinear with  $O$ , it can be confirmed that we really have a hyperbolic line by making sure that the line goes through a point and that point's inverse, which is outside of the hyperbolic space. If  $A$  is the point we are starting with, then let the inversion of  $A$  across the boundary be called  $A^{-1}$ . All hyperbolic lines that pass through  $A$  are also going to pass through  $A^{-1}$ . Point  $B$  can be anywhere else inside the disk. We then have three points and can make a Euclidean circle that will be orthogonal to the disk and will therefore be a hyperbolic line. Drawing the circle simply requires finding the intersection of the perpendicular bisectors of two segments made from our three points.

Euclid's second axiom is that lines extend indefinitely. Although this appears not to be the case in hyperbolic geometry, hyperbolic lines only have finite Euclidean length. The hyperbolic distance formula where  $A$  and  $B$  are points inside the space and  $C$  and  $D$  are where the hyperbolic line through  $A$  and  $B$  intersects the boundary is the absolute value of  $\ln\left(\frac{A-C}{A-D} + \frac{B-C}{B-D}\right)$ , with complex coordinates.

That circles exist is the third axiom. Hyperbolic circles are Euclidean circles, but their hyperbolic centers are not the same as the Euclidean centers unless the hyperbolic circle is centered at  $O$ . The Euclidean center of a hyperbolic circle has no hyperbolic significance, but it is useful in constructing hyperbolic circles or finding hyperbolic centers.

The fourth axiom says that all right angles are equal. Angles in hyperbolic geometry are measured by their tangent lines, and of course right angles are always  $\pi/2$  radians.

Euclid's fifth axiom, the Parallel Postulate, states that through a point not on a line, there is exactly one line through that point which is parallel to the original line. Due to the curvature of hyperbolic space though, there are many hyperbolic lines that can pass through a point not on a given line which are parallel to that line. Figure 2 depicts a few hyperbolic lines through a point  $P$  not on the given red hyperbolic line, which never touch the red hyperbolic line and are therefore parallel to that hyperbolic line.



**Figure 2:** Hyperbolic parallel lines.

Constructions in the hyperbolic disk employ *inversion* of points across the boundary circle. For a point  $P$  in the disk, there exists a point  $P^{-1}$  on ray  $\overrightarrow{OP}$  such that  $r^2 = 1 = \overline{OP} * \overline{OP^{-1}}$ . We will not need the construction of inversion [Goodman-Strauss], but we do need its properties: any circle through a point and its inverse is orthogonal to the boundary circle. This is a handy way to make sure a circle we construct actually contains a hyperbolic line. We may use points like  $P^{-1}$  on ray  $\overrightarrow{OP}$  outside the disk because we are using Euclidean construction tools to make our hyperbolic objects.

### 3 Our Hyperbolic Toolbox

There are some excellent tricks for constructing hyperbolic objects and geometers have probably known many moves without publishing them [Goodman-Strauss]. Certainly constructions for the hyperbolic midpoint, angle bisector, and other basics have been known quite a while [Ross], [McDaniel]. Here are construction steps for a few structures that are quick and easy to do, and we prove why these shortcut steps work.

Let's start by making a simple hyperbolic line so we have something to work with. All we need to build a hyperbolic line is two tangent lines. Specifically, for two points  $A$  and  $B$  on the boundary, their tangents meet at the center,  $C$ , of a circle orthogonal to circle  $O$ , through  $A$  and  $B$ .

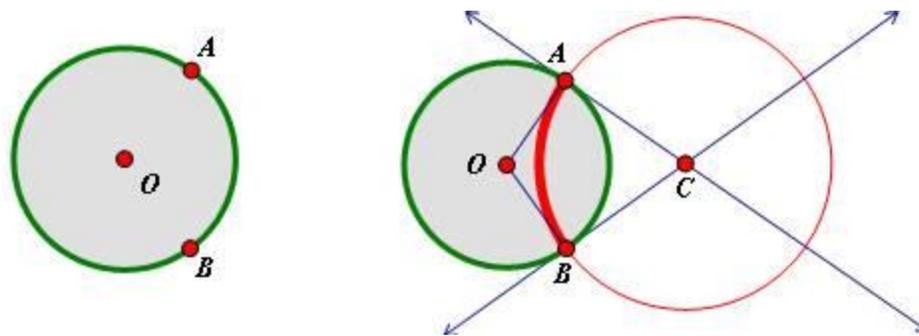


Figure 3: Simple hyperbolic line construction.

**Proposition 1:** Given hyperbolic segment  $\overline{AB}$ , construct the hyperbolic midpoint  $M$  by first constructing  $E$ , the intersection of the tangents at  $A$  and  $B$ . The intersection of  $\overrightarrow{OE}$  and  $\overline{AB}$  is  $M$ .

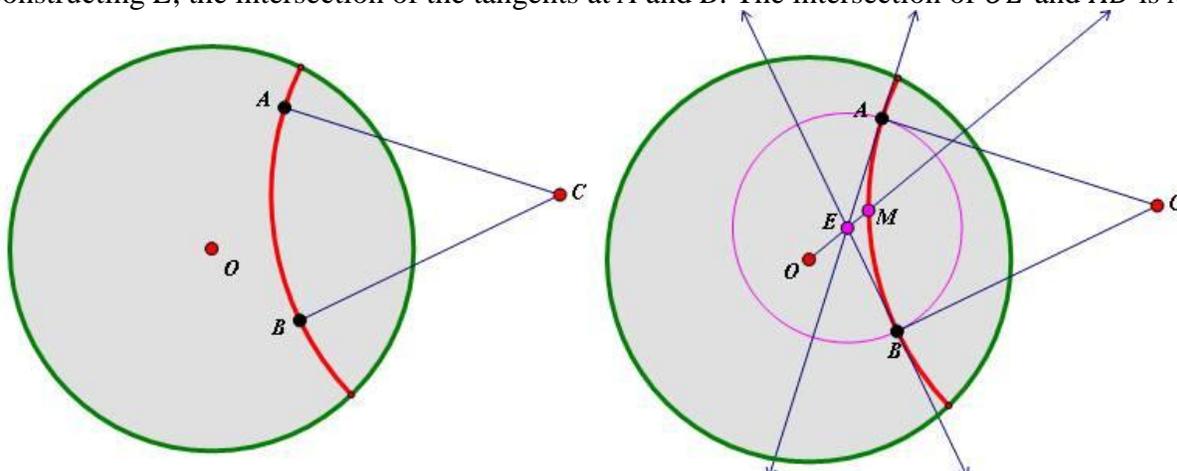
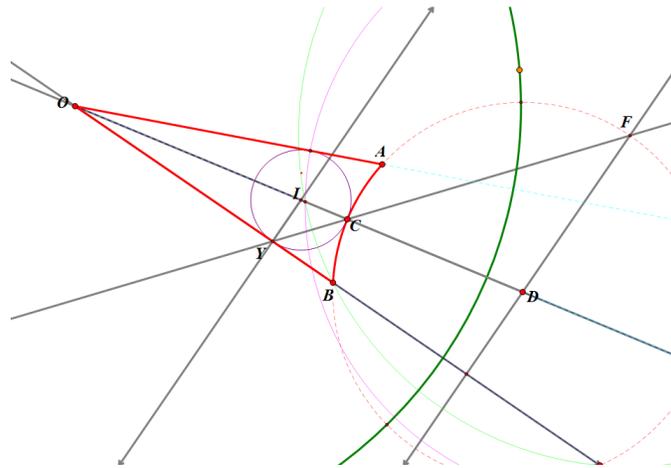


Figure 4: The hyperbolic midpoint construction.

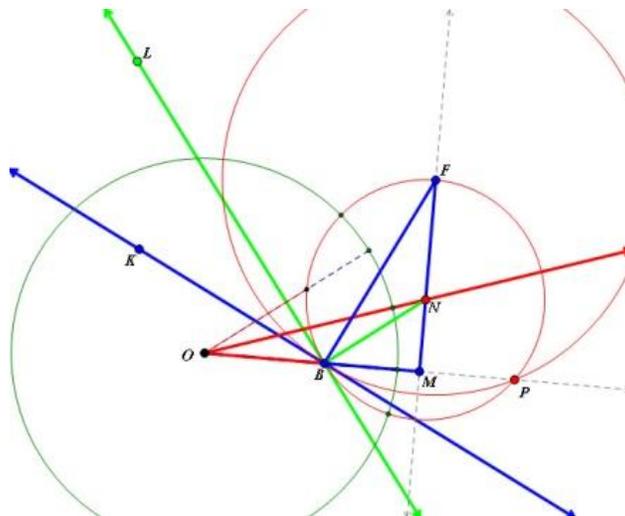
**Proof:** The ray  $\overrightarrow{OE}$  is orthogonal to circle  $E$  since  $E$  is the Euclidean center. Then the ray  $\overrightarrow{OE}$  contains the hyperbolic center of circle  $E$ . The segment  $\overline{AB}$  is also a diameter of circle  $E$  since it is part of the given hyperbolic line and we built circle  $E$  orthogonal to this line. Therefore, their intersection,  $M$ , is the hyperbolic center of circle  $E$ . This makes  $\overline{MA}$  and  $\overline{MB}$  hyperbolic radii of the same circle, having the same hyperbolic length. ■

**Proposition 2:** Given a hyperbolic isosceles triangle with vertex  $O$ , the incircle may be constructed as follows. The Euclidean line perpendicular to  $\overline{OB}$  through  $D$ , the Euclidean center of the base of the isosceles triangle, intersects the Euclidean circle containing the base at the point  $F$  on the other side of  $D$  from ray  $\overrightarrow{OB}$ . The point  $C$  is the Euclidean and hyperbolic midpoint of the base, easily found by drawing the Euclidean segment  $\overline{OD}$ , intersecting  $\overline{AB}$  at  $C$ . The Euclidean ray  $\overrightarrow{FC}$  intersects  $\overline{OB}$  at  $Y$ , a point of tangency for the desired incircle. The Euclidean line perpendicular to  $\overline{OB}$  at  $Y$  intersects  $\overline{OC}$  at the incenter  $I$ .



**Figure 5:** A point of tangency.

**Proof:** Given an isosceles hyperbolic triangle with a vertex at  $O$ , it can be proved that line  $\overleftrightarrow{OY}$  is perpendicular to hyperbolic segment  $\overline{OB}$ , and thus that  $Y$  is the point of tangency on  $\overline{OB}$  of the incircle. The ray  $\overrightarrow{OC}$  is the angle bisector of  $\angle AOB$ , so the Euclidean incenter lies somewhere along that ray between  $O$  and  $C$ . Point  $F$  is the center of the hyperbolic angle bisector of  $\angle OBA$ . To see that this is so, we modify Figure 5 and draw the perpendiculars which go with the tangents in question.



**Figure 6:** Point  $F$  is center of hyperbolic angle bisector.

The following angle sums are all 90 degrees:  
 $\angle LBF + \angle FBN = \angle KBL + \angle LBF$  and  $\angle OBK + \angle FBN + \angle NBM = \angle NFB + \angle FBN + \angle NBM$ . The Euclidean isosceles triangle  $FBN$  gives us  $\angle NFB = \angle FBN$ . These equations imply  $\angle OBK = \angle KBL$ . The remainder of the proof uses Figure 5.

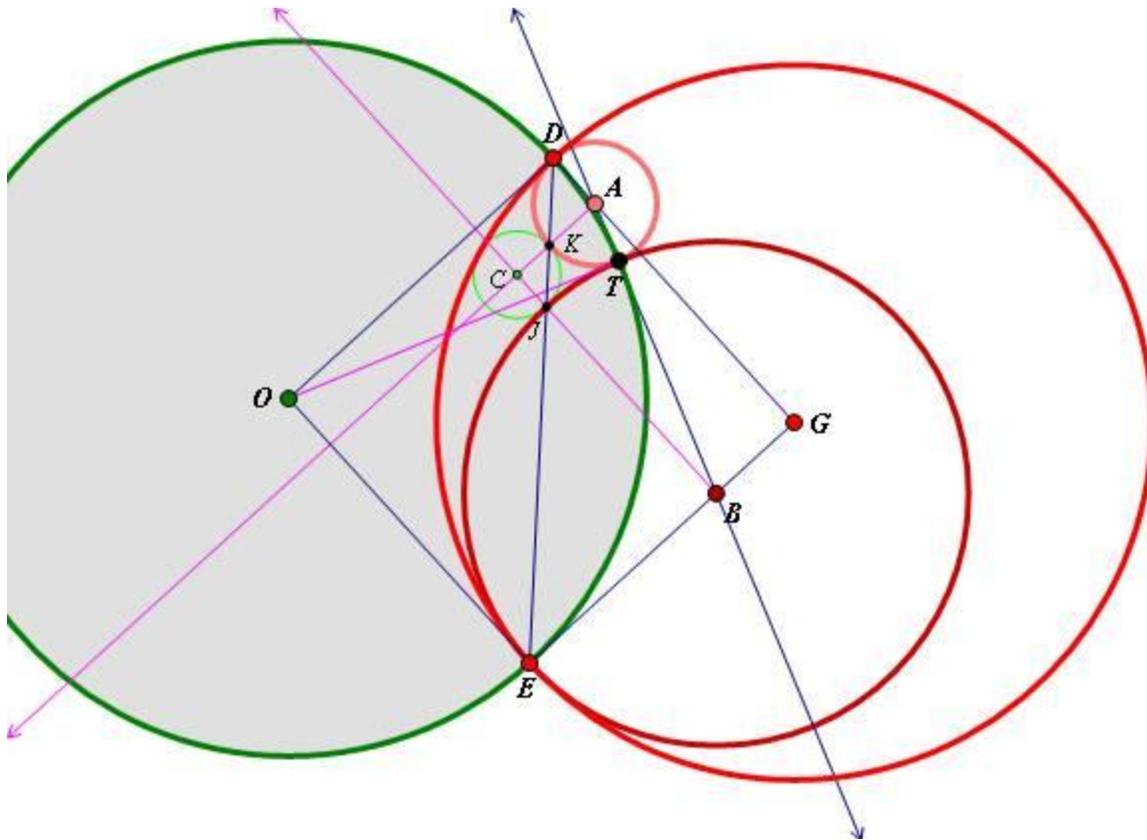
Thus,  $C$  is the intersection of hyperbolic line  $\overleftrightarrow{AB}$  and  $\angle AOB$ 's angle bisector. Point  $Y$  is the intersection of Euclidean line  $\overleftrightarrow{FC}$  and hyperbolic segment  $\overline{OB}$ , so points  $F$ ,  $C$ , and  $Y$  are collinear. Segments  $\overline{DC}$  and  $\overline{DF}$  are radii of circle  $D$  and are therefore the same Euclidean length. Segment  $\overline{FC}$  is then the third side of isosceles triangle  $CDF$ , meaning  $\angle FCD$  and  $\angle CFD$  are equal. Lines  $\overline{OD}$  and  $\overline{FY}$  intersect at point  $C$ , making  $\angle FCD$  and  $\angle YCI$  vertical angles. Since  $Y$  and  $C$  are both points on the circle with center  $I$ ,  $\overline{IY}$  and  $\overline{IC}$  are Euclidean radii and the same Euclidean length. Segment  $\overline{CY}$  is then the third side of a second isosceles triangle, and since  $\angle YCI$  was shown to be the same as  $\angle FCD$  and  $\angle CFD$ , then  $\angle CYI$  must also be the same size. The angles formed by lines  $\overline{YI}$ ,  $\overline{YF}$ , and  $\overline{DF}$  show that the lines are an example of alternate interior angles, and that  $\overline{YI}$  is therefore parallel to line  $\overline{DF}$ . Line  $\overline{YI}$  must be perpendicular to  $\overline{OB}$  then, because line  $\overline{DF}$  is the perpendicular bisector of  $\overline{BB^{-1}}$  which is on line  $\overline{OB}$ . ■

## 4 Descartes's Theorem in the Hyperbolic Disk

In this section we find mutually tangent circles where three of the circles are also hyperbolic lines. Our first excursion into the tangent circles which we find in hyperbolic geometry will be an example of Descartes's Theorem for mutually tangent circles [Wells]. The theorem says if four circles are mutually tangent then the curvatures  $k_1$ ,  $k_2$ ,  $k_3$ , and  $k_4$  satisfy the equation

$$(k_1 + k_2 + k_3 + k_4)^2 = 2(k_1^2 + k_2^2 + k_3^2 + k_4^2). \quad (1)$$

Any reader handy with constructions is urged to construct four mutually tangent circles: some planning is required. All the construction marks for our example are in the figure below: a remarkably economic use of a few circles and lines in order to get four circles, each one of which is tangent to the other three. These are circles centered at  $A$ ,  $B$ ,  $G$  and  $C$ . (There are other ways to arrange four mutually tangent circles besides one circle containing the other three.)



**Figure 7:** Four tangent circles  $A, C, G, B$ .

The construction starts with circle  $O$  and a pair of perpendicular radii. Finishing the square gives us  $G$ , the center of a circle orthogonal to circle  $O$  and with the same size Euclidean radius as circle  $O$ . The position of point  $A$  can be anywhere along that side of the square and the construction holds. The choice of  $A$  decides the tangent at  $T$  used to find point  $B$ , giving two hyperbolic lines, circles  $A$  and  $B$ .

**Proposition 3:** For this arrangement of hyperbolic lines in a quadrant of the hyperbolic disk, the radical axis of circle  $O$  and the hyperbolic line the same Euclidean size as circle  $O$  intersects these two interior hyperbolic lines at the points of tangency of a fourth circle, which is also tangent to the first hyperbolic line.

**Proof.** Let's assign the radii  $a$  and  $b$  to circles  $A$  and  $B$ , respectively. We will set the radius of the hyperbolic disk, which has the same radius of the first hyperbolic line, to 1. The Pythagorean Theorem applied to the right triangle  $ABC$  gives us

$$(a + b)^2 = (1 - a)^2 + (1 - b)^2. \tag{2}$$

Equation (2) simplifies to

$$ab = 1 - a - b. \tag{3}$$

Our construction uses the meeting of the Euclidean lines  $\overline{AK}$  and  $\overline{BJ}$  to give the center  $C$  of a circle tangent to circles  $A$  and  $B$ . The dimensions of rectangle  $AGBC$  show the radius of the

fourth circle is also  $1 - a - b$ ! We built circle  $C$  tangent to circles  $A$  and  $B$ ; we will now verify it is tangent to the outer circle.

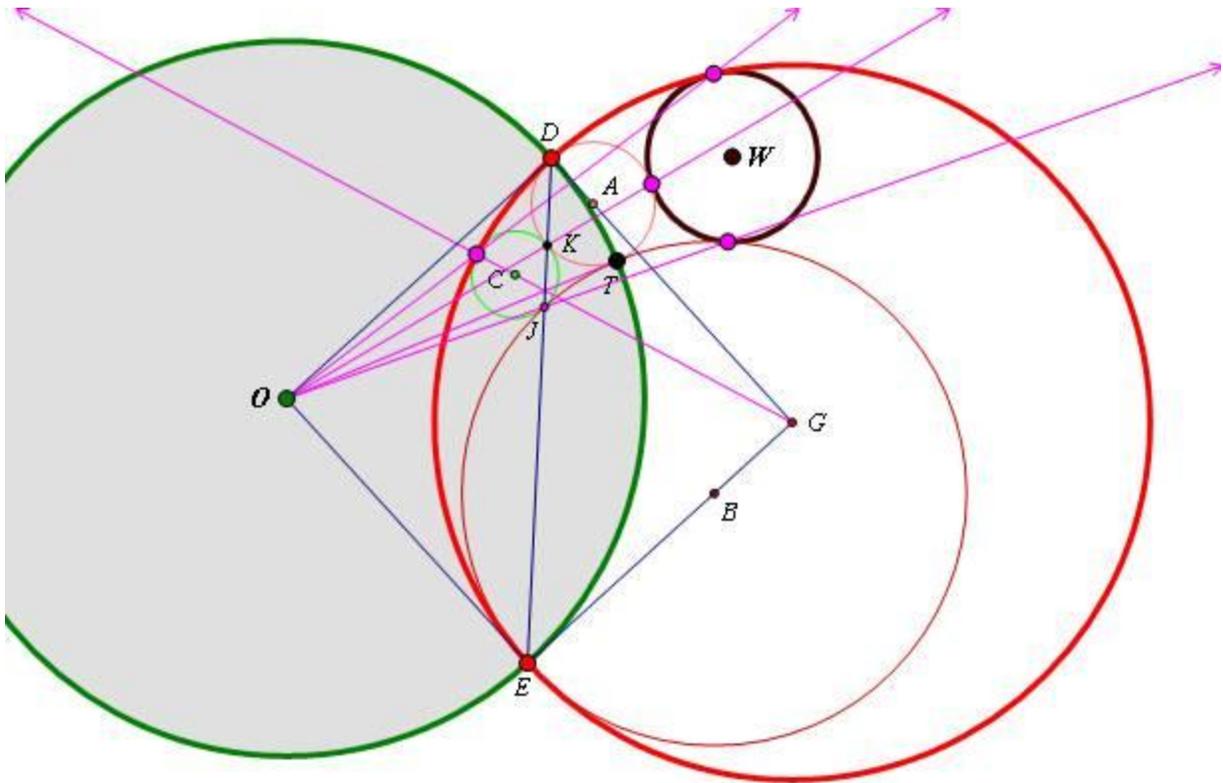
We can show that circle  $C$  contains the intersection  $(x, y)$  of the line  $\overleftrightarrow{GC}$  and circle  $G$ . The equations  $1 - y = \left(\frac{1-b}{1-a}\right)(1 - x)$  and  $(1 - x)^2 + (1 - y)^2 = 1$  combined with  $ab = 1 - a - b$  give us  $x = \frac{2a+b-1}{a+b}$  and  $y = \frac{2b+a-1}{a+b}$ . All we have to do now is establish this ordered pair is  $ab$  from point  $C$ .

The square of the distance is

$$\begin{aligned} & \left(a - \frac{2a+b-1}{a+b}\right)^2 + \left(b - \frac{2b+a-1}{a+b}\right)^2 \\ &= \left(\frac{a^2 - a + 2ab}{a+b}\right)^2 + \left(\frac{b^2 - b + 2ab}{a+b}\right)^2 \\ &= \left(\frac{a(a+b-1) + ab}{a+b}\right)^2 + \left(\frac{b(a+b-1) + ab}{a+b}\right)^2 \\ &= \left(\frac{ab(1-a)}{a+b}\right)^2 + \left(\frac{ab(1-b)}{a+b}\right)^2 \\ &= \left(\frac{ab}{a+b}\right)^2 ((1-a)^2 + (1-b)^2) \\ &= \left(\frac{ab}{a+b}\right)^2 (a+b)^2 = a^2b^2. \quad \blacksquare \end{aligned}$$

There is another circle we can construct tangent to three hyperbolic lines in this figure, with very little work. Since any diameter of the disk is a hyperbolic line, we could send two more hyperbolic lines through  $O$  which intersect the known points of tangency  $J$  and  $K$ . This allows us to apply our toolbox shortcut for the incircle of an isosceles hyperbolic triangle with a vertex at  $O$ . Our new circle has center  $W$  in Figure 8. Since inversion preserves incidence, we know circle  $W$  is tangent to the circles  $G$ ,  $B$  and  $A$ .

Knowing they are mutually tangent means we can use Descartes's Theorem to calculate the radius of circle  $W$ . Recall the theorem says that the curvatures  $k_1$ ,  $k_2$ ,  $k_3$ , and  $k_4$  of four mutually tangent circles satisfy the equation (1). There is one more little hassle: when one of the mutual tangent circles contains the other three, its curvature takes a negative sign in the equation. Some authors call this signed curvature the *bend*.

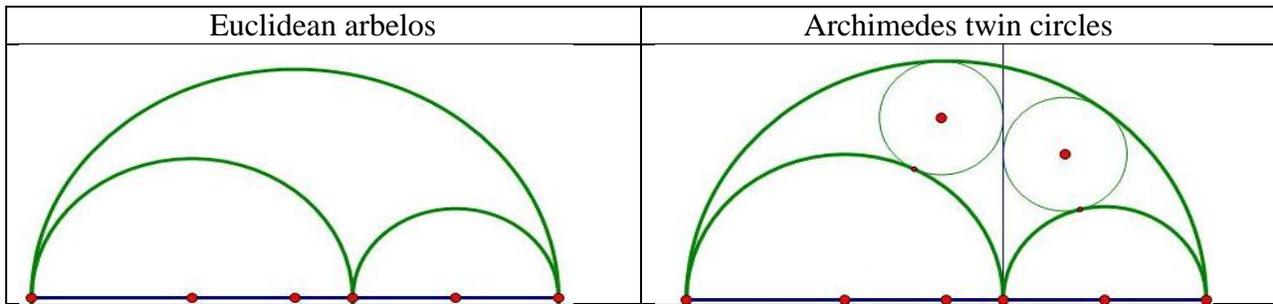


**Figure 8:** Another example of Descartes's theorem.

The radius of circle  $W$  turns out to be  $\frac{a(1-a)}{1-3a+4a^2}$ ; the interested reader may find this value by letting the unknown radius be  $x$  in Descartes's theorem and rewriting the theorem as a quadratic, (the other root of the quadratic is the radius of the first circle,  $ab$ , because we are using the same three circles to find the fourth.) So, starting with circles  $A$ ,  $B$ , and  $G$ , Descartes's theorem allows two possible sizes and we found a circle of each size.

We can apply Proposition 2 to this same drawing to place another tangent circle in this same figure. Our construction steps are all in Euclidean terminology. Reflecting  $\overline{OT}$  across  $\overline{OG}$  creates the isosceles triangle  $OPN$ . The newer marks in purple show the steps to the incircle of triangle  $OPN$ :  $\overline{GF}$  perpendicular to  $\overline{ON}$  to find  $F$ ,  $\overline{FR}$  intersects  $\overline{ON}$  at the point of tangency  $M$ , and the perpendicular to  $\overline{OM}$  at  $M$  finds the Euclidean center  $L$  of our desired circle. Figure 9 contains this new tangent circle.



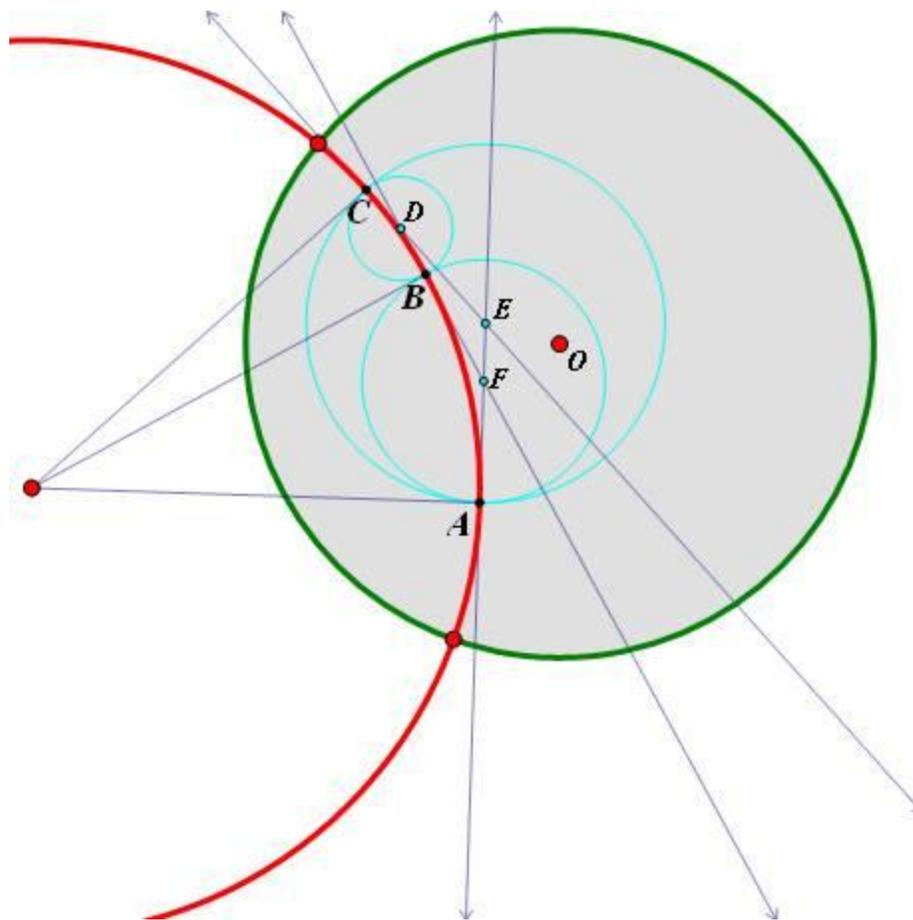


**Figure 10:** The arbelos and the twin circles.

The arbelos popped up on mathematicians' radar recently when Harold Boas published an article on the arbelos in the *American Mathematical Monthly*, March 2006. With references in Greek, Japanese, German, French, and English and a title with a math pun, *Reflections on the Arbelos* is a truly outstanding read. Boas won the 2009 Lester R. Ford writing award from the MAA for the article [Boas].

The construction of the twin circles pictured in the figure above is not our own construction, so we did not include the construction marks. Each twin is tangent to two of the three circles in the arbelos. They are called twins because they are congruent [Boas].

The hyperbolic version of the arbelos actually results in two different looking arbeloi from the same construction. A hyperbolic line contains the hyperbolic centers and the points of tangency for the hyperbolic circles that make the boundaries of the arbeloi. One arbelos falls on each side of the hyperbolic line. In this case, each of the twin circles is tangent to the hyperbolic line perpendicular to the hyperbolic line containing the diameters. Figure 11 below uses just a few construction lines to create the hyperbolic arbelos on a hyperbolic line which is not a diameter. Points  $A$ ,  $B$ , and  $C$  are chosen on the given hyperbolic line and tangent lines are constructed at each point. The intersections of the tangents,  $D$ ,  $E$ , and  $F$ , are the Euclidean centers of the circles in the hyperbolic arbelos and the hyperbolic centers are where rays  $\overrightarrow{OD}$ ,  $\overrightarrow{OE}$ , and  $\overrightarrow{OF}$  would intersect the hyperbolic line. Someone new to the hyperbolic disk might think the arbelos with the Euclidean centers inside is different from the other arbelos. But remember: the Euclidean centers are merely construction aids. They have no special status as hyperbolic points. The arbelos closest to  $O$  is congruent to the other arbelos, just as in the Euclidean case.



**Figure 11:** The hyperbolic arbelos.

We will continue thinking about the Euclidean and hyperbolic arbelos figures to prove our last claim.

**Proposition 4:** The Archimedes twin circles in a hyperbolic arbelos are congruent in the hyperbolic metric if and only if the inner circles of the arbelos are congruent in that metric.

**Proof:** Any hyperbolic arbelos may be translated so that the point of intersection between the two inner circles becomes  $O$ . Now the Euclidean picture of the arbelos matches the hyperbolic. Moreover, the Euclidean construction of the Archimedes twins works the same, producing the picture of the twins in Figure 10. The Euclidean rays from  $O$  through each twin's center must contain their respective hyperbolic centers. This gives us a way to compare the hyperbolic length of their diameters. These diameters are Euclidean congruent. The only way they can be hyperbolic congruent is if these diameters are the same Euclidean distance from  $O$ . That only happens when the inner circles of the arbelos are congruent. ■

## References

- Boas, Harold P. "Reflections on the Arbelos." *The American Mathematical Monthly*. 113 (2006): 236-249.
- Goodman-Strauss, Chaim. "Compass and Straightedge in the Poincaré Disk." *The American Mathematical Monthly*. 108 (2001): 38-49.
- McDaniel, Michael. "The Hyperbolic Tourist." *Math Horizons*. 14.1 (2006): 18-21.
- Ross, Skyler. *Non-Euclidean Geometry*, M.A. Thesis, University of Maine, 2000.
- Wells, David. The Penguin Dictionary of Curious and Interesting Geometry, Penguin, 1992.