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Sectioning Angles Using Hyperbolic Curves

Julie Fink[a] Nicholas Molbert[b]

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Rose-Hulman Institute of Technology
Department of Mathematics
Terre Haute, IN 47803
Email: mathjournal@rose-hulman.edu
http://www.rose-hulman.edu/mathjournal

[a]Mathematics Department, Tulane University
[b]Mathematics Department, University of Louisiana at Lafayette

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Abstract. In this paper, we construct a single hyperbola Γ that, along with a straight edge and compass, allow for the trisection of any angle. Descartes constructed a parabola with this property in his original treatment of analytic geometry. Unlike Descartes's proof, the proof that all angles can be trisected with the hyperbola Γ is a geometric rather than an algebraic argument.

1 Introduction

The problem of trisecting an angle goes back to the time of the Ancient Greeks. Given an angle θ, the Ancient Greeks wanted to be able to construct another angle that is 1/3 of the original angle. Constructing such an angle is closely tied with the notion of showing existence of the angle and is reflected in Euclid's approach in the *Elements*. Trisecting the angle is one of the three classic problems of Greek antiquity; two others are creating a square with the same area as a circle and constructing a cube with twice the volume of a given cube.

The following information pertaining to the history of geometric constructions are gathered from Kazarinoff's *Ruler and the Round: Classic Problems in Geometric Constructions* [4] and A. Jackter's work done at Rutgers University titled *The Problem of Angle Trisection in Antiquity* [2]. Throughout history many mathematicians have attempted to find a general method of trisection. Many tried but only a few made notable contributions. Hippocrates (460-380 BC) was the first person to attempt the trisection of an angle but was unsuccessful. He contributed to the field of geometry by labeling points and lines with letters. Hippias (460-399 BC) is credited with the first successful trisection with a curve called the quadratix. The quadratix may have been the first curve introduced into geometry other than lines and circles. Menaechmus (380-320 BC) introduced conic sections for the purpose of studying the classic questions. He successfully doubled the cube and may have trisected the angle. Menaechmus's methods used curves shown to exist by classical methods but Hippias's did not. Since the cone is a solid, constructions using conic sections are called *solid constructions*. Solid constructions for the trisection of an angle by Apollonius (250-175 BC) and Pappus (early fourth century) are known. Many other trisection constructions were given by the ancient Greeks that, like Hippias's construction, were beyond the methods and curves given in Euclid's *Elements*. These include constructions by Archimedes (287-217 BC) and Nicomedes (280-210 BC) which required the use of a twice marked straightedge.

The classic problems continued to be studied after the ancient period. Descartes (1596-1650) used his new invention of analytic geometry to show that using one single parabola along with a straight-edge and compass one may trisect any angle. The lack of a straight-edge and compass construction for the trisection of the general angle was eventually explained by Pierre Wantzel (1814-1848). He proved it impossible in his 1837 article titled (translated from the French), *Research on the Means of Knowing If a Problem of Geometry Can Be Solved with Compass and Straight Edge*.

In this paper, the authors show that using one single hyberbola, one may trisect any angle. The construction used in this paper is a variation on the construction of Pappus. This paper is an outgrowth of the summer SMILE program at LSU that was part of the NSF VIGRE program. The authors wish to express their gratitude to the NSF; the organizer, Mark Davidson; and their graduate mentor, Jose Ceniceros.

2 Basic Constructions

Basic straight-edge and compass constructions are contained in this section. A standard reference for these constructions is Kazarinoff's *Ruler and the Round* [4]. The necessary constructions are listed in order to provide a self-contained treatment.

Before listing constructions needed in trisecting an angle, let us introduce some notation used in the following proofs:

- $\overleftrightarrow{XY}$ denotes the line through points X and Y.
- $\overline{XY}$ denotes the segment from point X to point Y.
- $C(X, Y)$ denotes a circle with center X and radius $\overline{XY}$.
- XY denotes the magnitude of segment $\overline{XY}$
- $\angle XYZ$ denotes the measure or name of an angle that is less than 180°.

We assume the standard results of Euclidean geometry, but we identify a few items for future reference. First we remind the reader of the generalization of Thales's Theorem that is often called the *Inscribed Angle Theorem*: An arc of a circle that is subtended by an inscribed angle is twice the measure of the angle. Next we remind the reader of the following construction axioms:

1. Given two points A and B, we can construct the line and the segment that includes points A and B (i.e. $\overleftrightarrow{AB}$ and $\overline{AB}$ respectively).
2. Given two points A and B, we can construct the circle with center A and radius $\overline{AB}$, i.e., $C(A, B)$.

Points that are intersections of constructed curves are *constructed points* and may be taken as given for the purpose of further constructions. The following propositions are the basic constructions needed in our final trisection construction.

Lemma 2.1. ***(Rusty Compass Theorem)*** *Given points A, B, and C, we wish to construct a circle centered at point A with radius equal to BC.*

Proof. The argument may be followed in Figure 1. First, draw $C(A, B)$ and $C(B, A)$ and obtain point D which forms the equilateral triangle $\triangle ABD$. Then, construct $C(B, C)$. Extend $\overline{DB}$ past point B and call $\overline{DB} \cap C(B, C)$, point E. Construct $C(D, E)$. Then extend $\overline{DA}$ past point A and label $\overline{DA} \cap C(D, E)$, point F. Construct $C(A, F)$. Because E lies on $C(B, C)$, $BE = BC$. Then, because $\triangle ABD$ is equilateral, $DA = DB$. Also, because E and F lie on a circle with center D, $DE = DF$. Therefore, $AF = BE = BC$. □

Proposition 2.2. ***(Copying an Angle)*** *Given $\angle ABC$ and a line l containing a point D, we can find E on l and a point F such that $\angle ABC = \angle EDF$. Furthermore, F can be taken on either side of l and E can be taken on either side of D.*

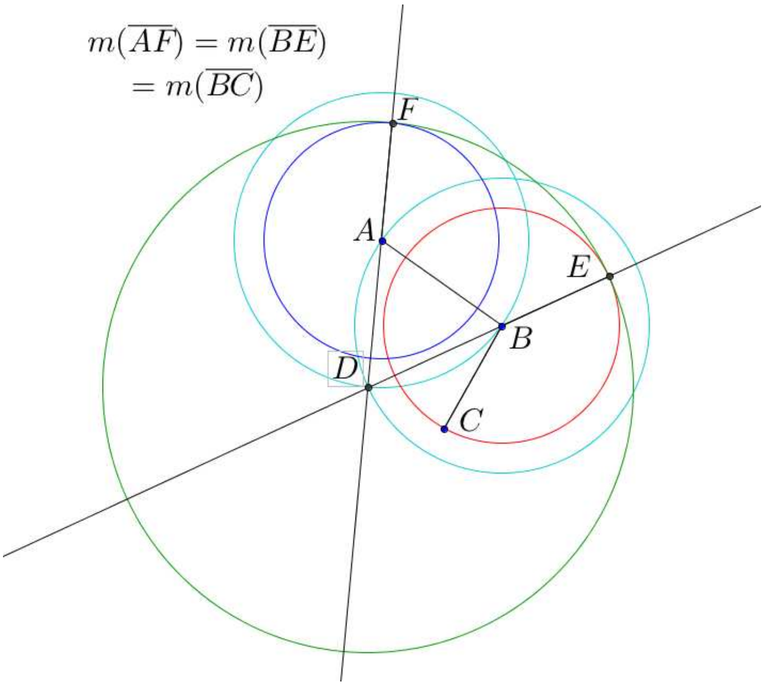


Figure 1: Rusty Compass Theorem

Proof. Using Proposition 2.1 we construct a point E on the line l with $BA = DE$. Construct a circle with center D with radius length BC. Construct another circle with center E with radius length AC. We get the point F. Therefore, $\triangle ABC \cong \triangle EDF$ by the side-side-side congruent triangle theorem. So, $\angle ABC = \angle EDF$. □

Proposition 2.3. ***(Bisecting an Angle)*** *Given $\angle ABC$, there is a point D such that $\angle ABD \cong \angle DBC$.*

Proof. Extend $\overline{AB}$ to get the line l. Construct the circle $C(B, C)$ and intersect with l. Call the interesection on the A side of B, E. Construct $C(E, B)$ and $C(C, B)$ to get a new point D. Then $\angle ABD \cong \angle DBC$. Therefore, $\overline{BD}$ bisects $\angle ABC$. □

Proposition 2.4. ***(Dropping a Perpendicular)*** *Given a line l and a point P not on l, we can construct a line l' which is perpendicular to l and passes through P.*

Proof. There exists a point A on l. If $\overleftrightarrow{PA} \perp l$, we are done. If not, construct $C(P, A)$ to get B. Next, construct $C(A, B)$ and $C(B, A)$, which intersect in two points C and D. Then $\overleftrightarrow{CD}$ is perpendicular to l'. □

Proposition 2.5. ***(Raising a Perpendicular)*** *Given a point P on a line l, then you can construct l' through the point P and perpendicular to the line l.*

Proof. There exists a point A on line l distinct from point P. Construct $C(P, A)$ to obtain $l \cap C(P, A) = B$. Construct $C(A, B)$ and $C(B, A)$ to get C. Draw $\overleftrightarrow{PC}$. This line is l'. □

Proposition 2.6. ***(Drawing a Parallel)*** *Given a line l and P not on l, we can construct l' through P and parallel to l.*

Proof. Construct l'' such that it is perpendicular to l and passes through P by Proposition 2.4. Now construct l' through P and perpendicular to l'' by Proposition 2.5. l' is parallel to l through P (Notice that the notion of parallel here is as stated in Playfair's axiom, an axiom that can be derived from Euclid's first five postulates). □

Given these basic constructions, we now prove that we can construct rational lengths.

Theorem 2.7. *If on the Cartesian plane the points* $(0, 0)$ *and* $(1, 0)$ *are given, then all points with rational coordinates are constructible.*

Proof. Let $A = (0, 0)$ and $B = (1, 0)$. Construct $\overleftrightarrow{AB}$. This is the x-axis. The y-axis is constructed by raising a perpendicular to the x-axis at point A. If any segments of lengths a and b are constructible, then the points $(\pm a, \pm b)$ are constructible (combine the Rusty Compass Theorem with Proposition 2.5, raising a perpendicular). Since we have an segment $\overline{AB}$ of length 1, we can copy it end to end along a line to construct a segment whose length is any desired natural number. Hence we can construct all points with integer coordinates. It remains to construct segments of any rational length. Given $n, m \in \mathbb{N}$, consider the triangle with vertices $A = (0, 0)$, $C = (m, 0)$, and $D = (m, n)$. Raise a perpendicular to the x-axis at $B = (1, 0)$ and let the intersection of the perpendicular and $\overline{AD}$ be denoted Q. By examining the similar triangles ΔACD and ΔABQ, we see that the length of $\overline{BQ}$ is $\frac{n}{m}$. □

3 Hyperbolas As Conic Sections And The Hyperbola Γ

Non-degenerate conic sections are curves that are obtained by intersecting a double cone with a plane that does not pass through the vertex. These sections may be circles, ellipses, hyperbolas, or parabolas. In particular, hyperbolas are created when the intersecting plane intersects both halves of the double cone.

Unlike the constructions of circles and lines, the ancient Greeks did not give precise rules to allow the construction of a hyperbola. They did, however, give examples of constructions. Pappus related constructions using a property of hyperbolas that is often used as the defining property. A hyperbola is the locus of all points for which the ratio of distances from a fixed point, the *focus*, to a fixed line, the *directrix*, is a constant e, the *eccentricity*, with $e > 1$. The discovery of this property is usually attributed to Apollonius. We take the following as the postulate for constructing hyperbolas:

1. Given a point A, a line l, and a number $e > 1$, then we can construct a the hyperbola with focus A, directrix l, and eccentricity e. We will only use $e = 2$.

Algebraically, hyperbolas are realized as the real solution sets to quadratic equations

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

with

$$\begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{vmatrix} \neq 0 \text{ and } \begin{vmatrix} 2A & B \\ B & 2C \end{vmatrix} < 0$$

[6].

Hyperbolas whose axes are parallel to the coordinate axes are commonly expressed in standard form.

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = \pm 1 \tag{3.1}$$

The characteristics of a hyperbola given by Equation 3.1 with constant $+1$ are as follows. The *vertices* are $(h+a, 0)$ and $(h-a, 0)$ and can be described as the points where the distance between the two branches is the least. The *transverse axis* is the line segment which connects the *vertices*. The *center*, given by coordinates (h, k), is the point on the transverse axis that bisects the segment between the vertices. The *conjugate axis* is perpendicular to the *transverse axis* through the center. The center of a hyperbola is also located at the intersection of the *asymptotes*, which are given by equations $y - k = \pm\frac{b}{a}(x - h)$. The eccentricity is $e = \sqrt{1 + \frac{b^2}{a^2}}$. There are two choices for the focus and directrix: $(h + ae, k)$ and $x = h + \frac{a}{e}$ or the pair $(h - ae, k)$ and $x = h - \frac{a}{e}$. Now that we have clearly defined certain characteristics of hyperbolas, we can derive the hyperbola Γ needed for the purpose of this project [3].

Theorem 3.1. *Let $A = (0, 0)$ and $B = (1, 0)$. The set $\{C = (x, y) : x, y > 0$ and $\angle CBA = 2\angle CAB\}$ is the first quadrant branch of the hyperbola defined by*

$$\frac{(x - \frac{1}{3})^2}{\left(\frac{1}{3}\right)^2} - \frac{y^2}{\left(\frac{1}{\sqrt{3}}\right)^2} = 1. \tag{3.2}$$

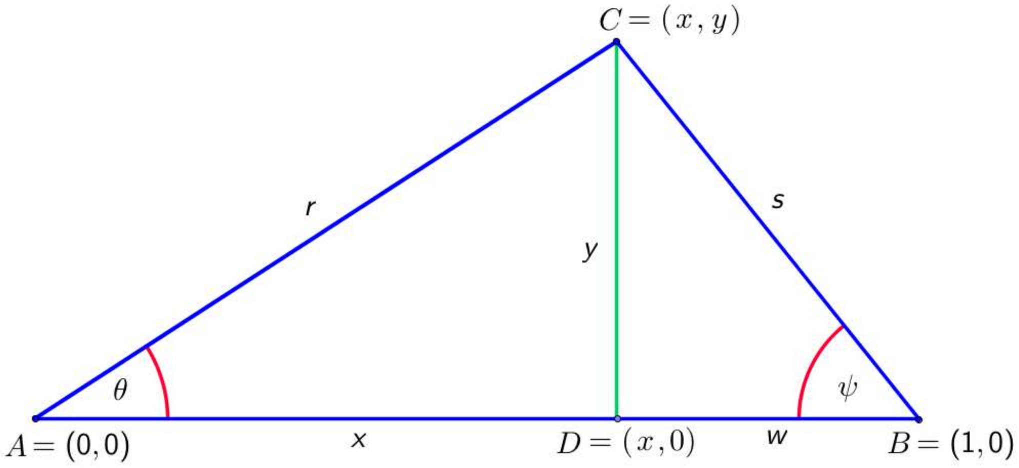
Notation 3.2. Let Γ denote the hyperbola described above by Equation 3.2.

Before proceeding to the proof, it is worth remarking that the whole hyperbola, Equation 3.2, can be described in a similar manner. Suppose l_A and l_B are variable lines rotating through the points A and B. They both start as the x-axis. The line l_A rotates counter clockwise while l_B rotates clockwise at twice the rate as l_A. The intersection of the distinct lines, $l_A \cap l_B$, sweeps out the hyperbola Γ punctured at the vertices.

Proof. We derive the equation for Γ. Let $C \in \{C = (x, y) : x, y > 0$ and $\angle CBA = 2\angle CAB\}$ be $C = (x, y)$. Let $D = (x, 0)$, r be the length of segment $\overline{AC}$, and w be the length of segment $\overline{DB}$. We now have right triangles $\triangle CAD$ and $\triangle CDB$. Let $\theta = \angle CAB$ and s be the length of the segment $\overline{CB}$.

There are three cases to consider. The first is $x = 1$ in which case $D = B$ and $C = (1, 1)$. The second is the case $x < 1$. This case is needed for the trisection construction and we prove it below. The third case is $x > 1$, is not need for the trisection construction and the proof is very similar to $x < 1$. In comparison to the case of $x < 1$, one would only need to note that the sine of $180° - 2\theta$ is $\sin(2\theta)$ to get started, $x - w = 1$, and be careful of signs. We will not go into the details and concentrate on $x < 1$. We continue with the case $x < 1$. See Figure 3 where $\psi = 2\theta$. Note that implies $2\theta < 90°$. In analyzing the triangles, we have the following:

$$1 = x + w \tag{3.3}$$
$$y = r \sin\theta \tag{3.4}$$
$$x = r \cos\theta \tag{3.5}$$
$$s = \frac{r \sin\theta}{\sin 2\theta}. \tag{3.6}$$



By the Pythagorean Theorem, we find the following relationship for w, $w^2 = s^2 - y^2$. By substituting y and s values from equations (3.4) and (3.6),

$$w^2 = \left(\frac{r\sin\theta}{\sin 2\theta}\right)^2 - (r\sin\theta)^2 \tag{3.7}$$

$$= \frac{r^2\sin^2\theta}{\sin^2 2\theta} - r^2\sin^2\theta \tag{3.8}$$

$$\begin{aligned}
w &= r\sin\theta\sqrt{\left(\frac{1}{\sin^2 2\theta} - 1\right)} \\
&= r\sin\theta\sqrt{\csc^2 2\theta - 1} \\
&= r\sin\theta\sqrt{\cot^2 2\theta} \\
&= r\sin\theta\cot 2\theta \\
&= r\sin\theta\left(\frac{\cos 2\theta}{\sin 2\theta}\right) \\
&= r\sin\theta\left(\frac{\cos 2\theta}{2\sin\theta\cos\theta}\right) \\
&= \frac{r\cos 2\theta}{2\cos\theta} \\
&= \frac{r(\cos^2\theta - \sin^2\theta)}{2\cos\theta} \\
&= \frac{r\cos^2\theta}{2\cos\theta} - \frac{r\sin^2\theta}{2\cos\theta}
\end{aligned}$$

$$w = \frac{r\cos\theta}{2} - \frac{r\sin^2\theta}{2\cos\theta} \tag{3.9}$$

Using Equations (3.3), (3.5), and (3.9) , we obtain

$$1 = r\cos\theta + \frac{r\cos\theta}{2} - \frac{r\sin^2\theta}{2\cos\theta}$$

Multiplying both sides of the equation by $2r\cos\theta$,

$$2r^2\cos^2\theta + r^2\cos^2\theta - r^2\sin^2 = 2r\cos\theta.$$

Using substitution with equations (3.2) and (3.3),

$$2x^2 + x^2 - y^2 = 2x.$$

Completing the square,

$$\begin{aligned}
3x^2 - 2x - y^2 &= 0 \\
x^2 - \frac{2}{3}x - \frac{1}{3}y^2 &= 0 \\
(x - \frac{1}{3})^2 - \frac{y^2}{3} &= \frac{1}{9}
\end{aligned}$$

Putting this equation in standard form, we arrive at the curve,

$$\frac{(x-\frac{1}{3})^2}{\left(\frac{1}{3}\right)^2} - \frac{y^2}{\left(\frac{1}{\sqrt{3}}\right)^2} = 1.$$

We have derived Γ.

Suppose that C is any point on the first quadrant branch of Γ, $A = (0,0)$, and $B = (1,0)$. Let $\theta = \angle CAB$ and $\psi = \angle CBA$ We wish to show that $\psi = 2\theta$. Again, we show case 2, i.e., $x < 1$. For this case, both angles are acute and $\psi = 2\theta$ if and only if $\sin(\psi) = 2\sin(\theta)\cos(\theta)$. Using that the equation for Γ, i.e., $y^2 = 3x^2 - 2x$, and following Figure 3, we have

$$\begin{aligned}
2\sin(\theta)\cos(\theta) &= 2\frac{y}{\sqrt{x^2+y^2}}\frac{x}{\sqrt{x^2+y^2}} \\
&= 2\frac{xy}{x^2+y^2} \\
&= 2\frac{xy}{4x^2-2x} \\
&= \frac{y}{2x-1} \\
&= \frac{y}{\sqrt{4x^2-4x+1}} \\
&= \frac{y}{\sqrt{4x^2-4x+1}} \\
&= \frac{y}{\sqrt{(x-1)^2+3x^2-2x}} \\
&= \frac{y}{\sqrt{(x-1)^2+y^2}} \\
&= \sin(\psi)
\end{aligned}$$

□

Theorem 3.3. *The hyperbola Γ has eccentricity $e = 2$ and the vertices are $(\frac{2}{3}, 0)$ and $(0,0)$. One focus-directrix pair is given by $(1,0)$ and $x = \frac{1}{2}$ and the second by $(-\frac{1}{3}, 0)$ and $x = \frac{1}{6}$. The asymptotes are given by the equations, $y = \pm\sqrt{3}(x - \frac{1}{3})$.*

Proof. From the equation for Γ given in Theorem 3.1, we see that $h = \frac{1}{3}$, $k = 0$, $a = \frac{1}{3}$, and $b = \frac{1}{\sqrt{3}}$. The results follow by applying the formulas given after Equation 3.1. □

Theorem 3.4. *Given two points, then we may construct a coordinate system and the hyperbola Γ.*

We will construct Γ and a coordinate system around it. Any further coordinate analysis will be done with respect to these Cartesian coordinates.

Proof. Label the two known points A and B and give them the coordinates $A = (0,0)$ and $B = (1,0)$. Note that we have now set a scale where the distance between A and B is 1. Construct $\overleftrightarrow{AB}$. This is the x-axis. Raise a perpendicular to the x-axis at A by Proposition 2.5. This is the y-axis. Next construct the point $(\frac{1}{3}, 0)$ using Theorem 2.7, and raise a perpendicular to the x-axis by Proposition 2.5. This line is the directrix. Since the directrix and focus have been constructed and the eccentricity is 2, the hyperbola Γ may be constructed. □

4 Trisecting An Angle Using The Hyperbola Γ

Theorem 4.1. *Any angle may be trisected with a straight-edge, compass, and the hyperbola* Γ.

Our argument is based on an argument presented by Pappus given in Sir Thomas Heath's *A History of Greek Mathematics* [1]. You may find it helpful to reference the figure below titled *Trisection Construction* while reading the following proof.

Proof. Starting with $A = (0,0)$, $B = (1,0)$, and an angle θ with measure less than 180°, we trisect θ. If one wishes to trisect an angle with measure greater than 180°, then one can remove a multiples of 90° first and separately trisect a 90° angle.

1. By Theorem 3.4, construct Γ with right branch's focus at point B.
2. Using Theorem 2.7, construct point D at $(\frac{1}{2}, 0)$.
3. Using Proposition 2.4, construct l perpendicular to $\overline{AB}$ at point D.
4. Bisect angle θ using Proposition 2.3 to obtain an angle with measure $\frac{\theta}{2}$.
5. Let E have a negative y coordinate on l, then using Proposition 2.2 copy $\frac{\theta}{2}$ measured clockwise from l having vertex at point E. Construct point F such that $\angle DEF = \frac{\theta}{2}$.
6. Construct $l' \parallel \overleftrightarrow{EF}$ through point B using Proposition 2.6.
7. Obtain point O such that $\angle DOB = \frac{\theta}{2}$ (i.e. point $O = l' \cap l$).
8. Construct $\overline{BO}$.
9. Using Proposition 2.2, reflect $\angle DOB$ about line l such that it creates $\angle AOD$.
10. Construct $\overline{AO}$ yielding isosceles triangle $\triangle AOB$.
11. Construct $C(O, A)$.
12. Obtain the point P from the intersection of the arc $\overset{\frown}{AB}$ and Γ.

13. Construct $\overline{AP}$ and $\overline{PB}$ giving us a triangle such that $\angle PBA = 2\angle PAB$.

14. Construct $\overline{OP}$.

Our claim is that $\overline{OP}$ trisects $\angle AOB$. Before we prove this claim, we will rename angles so that $\angle PAB = \phi$, $\angle AOP = \alpha$ and $\angle POB = \psi$. Notice that $\alpha + \psi = \theta$. By Theorem 3.1, Γ, it holds that $\angle PBA = 2\phi$. Note that ϕ and ψ subtend the same arc, $\overset{\frown}{PB}$. Since ϕ is an inscribed angle and ψ is a central angle, we have $2\phi = \psi$ by the Inscribed Angle Theorem. Similarly, the angles subtending the arc $\overset{\frown}{AP}$ show that $\phi = \alpha$. Therefore $\psi = 2\alpha$. Notice that $\angle AOB = \alpha + \psi = \alpha + 2\alpha = 3\alpha$. Hence, $\overline{OP}$ trisects $\angle AOB$ and separately trisect each $90°$ angle. □

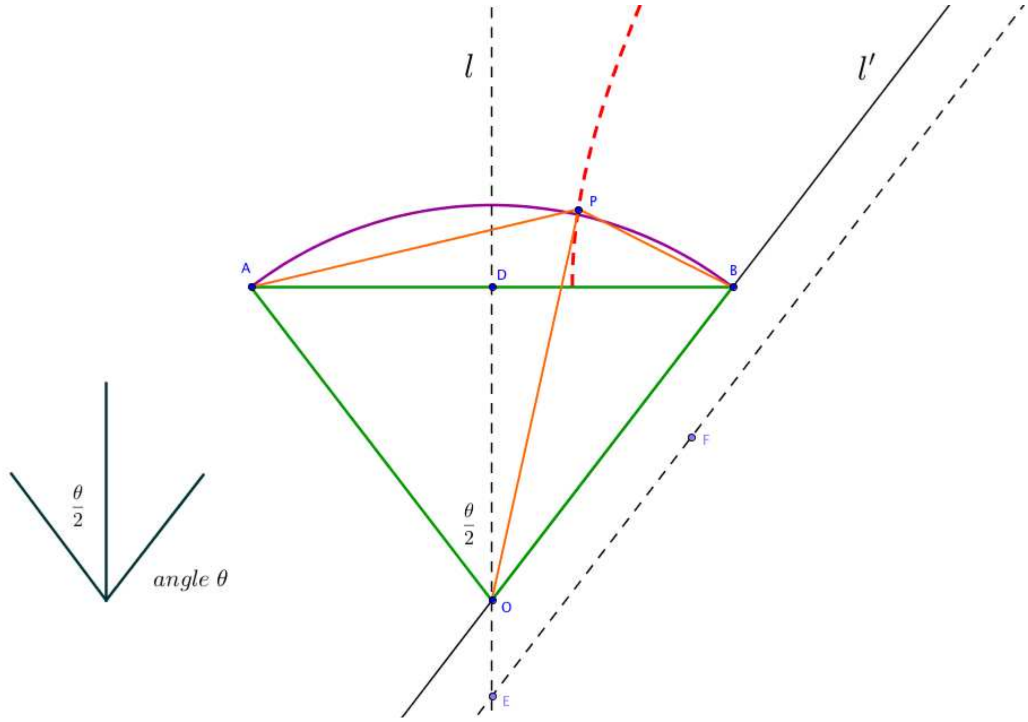


Figure 2: Trisection Construction

5 Further investigation

One direction for future investigation is to study curves that allow for sectioning an angle into a larger number of equal angles than three, i.e., sectioning an angle into $n+1$ equal angles for $n > 2$. If $A = (0,0)$ and $B = (1,0)$, then the set $\Gamma_n = \{C = (x,y) : x, y > 0 \text{ and } \angle CBA = n\angle CAB\}$ is a component of

$$(1-x)\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}\cos\left(\frac{(n-k)\pi}{2}\right) - y\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}\sin\left(\frac{(n-k)\pi}{2}\right) = 0. \quad (5.1)$$

The sine and cosine terms in Equation 5.1 are $0, 1, \text{or} -1$ varying with the remainder of $n-k$ divided by 4. The polynomial equation Equation 5.1 is derived in same manner as Equation 3.2 in Theorem 3.1 except that the double angle formulas are replaced by n-th angle formulas [5]. Minor alterations to the proof of Theorem 4.1 show that Γ_n along with the use of a straight edge and compass allow the $n+1$ sectioning of a angle. One can examine the algebraic properties of the curves Γ_n.

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