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On the Extension of Complex Numbers

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Abstract. This paper proposes an extension of the complex numbers, adding further imaginary units and preserving the idea of the product as a geometric construction. These 'supercomplex numbers', denoted $\mathbb{S}$, are studied, and it is found that the algebra contains both new and old phenomena. It is established that equal-dimensional subspaces of $\mathbb{S}$ containing $\mathbb{R}$ are isomorphic under algebraic operations, whereby a symmetry within the space of imaginary units is illuminated. Certain equations are studied, and also a connection to special relativity is set up and explored. Finally, abstraction leads to the notion of a 'generalised supercomplex algebra'; both the supercomplex numbers and the quaternions are found to be such algebras.

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1 Introduction

When the complex numbers were first introduced, they were considered rather mysterious entities; this, however, partly changed through the construction of the complex plane, which added one more axis to the real – the imaginary. It contributed to a more visualisable approach to the complex numbers and the operations among them, since it set up a connection between the pure algebra and the geometry of $\mathbb{R}^2$. In some sense, the complex numbers were the 'end of the story', since, as the Fundamental Theorem of Algebra states, the field of complex numbers is algebraically closed, i.e. any non-constant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$. Nevertheless, the connection between $\mathbb{C}$ and $\mathbb{R}^2$ inspires to look for algebraic systems which can in similar manners be identified with $\mathbb{R}^n$, and as has come to my attention when beginning the work presented here, there have historically indeed been others struck by such inspiration. One of the most well-known is perhaps Irish Sir William Hamilton who in 1843, after having tried for years with the case $n = 3$, found a system for $n = 4$, the *quaternions*.[1] However, even though the quaternions partly rely on geometric principles (the quaternion product may be expressed in terms of vector operations with natural geometric interpretations), these are not strong enough to carry over to an arbitrary $n \neq 4$.
Later the same year, though, a country fellow of Hamilton, John Thomas Graves, put forward yet another extension, for the case $n = 8$, now called the *octonions*. As we shall see later the octonions are not 'geometric'[2] in nature as are the quaternions, but the two systems do have something in common; a theorem by Adolf Hurwitz, dating back to 1898, states that the real numbers, complex numbers, quaternions and octonions are (up to isomorphism) the only four *normed division algebras* – this essentially means that the product commutes with the operation of forming the *norm*, a quantity associated with each number, and that any non-zero number has a multiplicative inverse (see [1]).

This paper presents an extension of the complex numbers, which is necessarily not a normed division algebra, but which in continuation of ordinary complex numbers, relies on geometrical ideas; the principles set up allow us to consider algebraic systems which can be identified with arbitrary $\mathbb{R}^n$ – this flexibility, however, comes with a cost. The 'supercomplex numbers' thus developed have algebraic properties not as desirable as the ones for complex numbers (the product fails, in general, to be commutative, associative and right-distributive); but they are interesting because they preserve and expose some profound structures of ordinary complex numbers, and moreover enriches them with new concepts.
We shall first develop the fundamental algebraic operations (multiplication, that is, in particular), and then explore some of the thereby inherited phenomena; supercomplex exponentials are treated and an investigation of certain equations is presented. Towards the end, some relations to physics are illustrated, and in particular we establish a connection to the Lorentz

[1] The historical content presented here is inspired by [1], whose introduction is very recommendable, also for a more detailed overview.

[2] In fact, we shall see in the last section exactly in what sense this statement is true.

transformations of special relativity. Lastly, we abstract from our work some general ideas, thus constructing the notion of a 'generalised supercomplex algebra', which – as we shall see – is broad enough to encapsulate the quaternions as well as the supercomplex numbers, and thereby finally describe in which sense and to what extent these structures are connected.

Recommendable preliminaries are mainly linear algebra (in particular inner product spaces and orthogonal transformations), but also the notions of homo-, iso-, (endo-) and automorphisms, and – of course – complex numbers.

2 Foundations

First, let us make the following definitions:

Definition 2.1. *The set of supercomplex numbers is denoted by $\mathbb{S}$ and is a countably infinite-dimensional vector space over $\mathbb{R}$ with a standard basis denoted $(\kappa_0, \kappa_1, \kappa_2, \ldots)$; the basis elements $\kappa_n, n \in \mathbb{N}$ are called the imaginary units and furthermore $\kappa_0 := 1 \in \mathbb{R}$.*

An element ξ of $\mathbb{S}$ is called a supercomplex number and can be written as $\xi = \sum_{n=0}^{\infty} a_n \kappa_n$, where finitely many of the coordinates $a_n \in \mathbb{R}$ are non-zero; these are called the components of ξ, and a_0 is called the real part of ξ, denoted α_ξ, whereas $\xi - a_0$ is called the imaginary part of ξ, denoted σ_ξ.

We let $\mathbb{S}_m, m \in \mathbb{N}_0$ denote the $(m+1)$-dimensional subspace of $\mathbb{S}$ with basis $(\kappa_0, \kappa_1, \ldots, \kappa_m)$; we may identify κ_1 with i and $\mathbb{S}_1$ with $\mathbb{C}$. Moreover, $\mathbb{S}_0 = \mathbb{R}$ (since $\kappa_0 = 1$), and we shall almost exclusively write '$\mathbb{R}$' for $\mathbb{S}_0$; for $\lambda \in \mathbb{R}, \xi \in \mathbb{S}$, we identify the product $\lambda\xi$, thinking of λ as an element of $\mathbb{S}$, with the product $\lambda\xi$, thinking of λ as a scalar.

Accordingly, if $\xi = \sum_{n=0}^{\infty} a_n \kappa_n, \phi = \sum_{n=0}^{\infty} b_n \kappa_n \in \mathbb{S}$ their sum can be written as $\xi + \phi = \sum_{n=0}^{\infty}(a_n + b_n)\kappa_n$, and for $\lambda \in \mathbb{R}(= \mathbb{S}_0)$, we can similarly write $\lambda\xi = \sum_{n=0}^{\infty}(\lambda a_n)\kappa_n$.

Definition 2.2. *For $\xi \in \mathbb{S}_n$, the $(n+1) \times 1$ column vector which has as its entries the components of ξ from top to bottom, is called the vector representation of ξ and is denoted by $\check{\xi}$; occasionally, for the sake of clarity, the $\check{}$-symbol shall be written after a parenthesis, for instance as in $(\xi + \phi)\check{}$.*

Given a finite collection of numbers $\xi_1, \xi_2, \ldots, \xi_k \in \mathbb{S}$ there always exists, since we regard only numbers with finitely many non-zero components, an $n \in \mathbb{N}$ such that $\xi_1, \ldots, \xi_k \in \mathbb{S}_n$. When we speak of the vector representation of a general $\xi \in \mathbb{S}$, or write $\check{\xi}$, it is implicit, and rarely mentioned, that such an n can be and has already been chosen.

We now turn to the problem of defining multiplication and division; parts of the following discussion contain somewhat vague statements, since they are meant primarily as an intuitive motivation for the definitions that are to be given. To halve the challenge, so to speak, we concentrate on the development of a product, and then propose that (left-)division by ξ is the operation that brings the product $\xi\phi$ back to ϕ (if such an operation, independent of ϕ, turns out to exist).[3]

First of all, we wish for any equation in ordinary complex numbers to maintain its validity under the replacement of i by any other imaginary unit, $\kappa_n, n \geq 1$; in other words, any subspace $\mathbb{R} + \mathbb{R}\kappa_n, n \geq 1$ of $\mathbb{S}$ should be algebraically an exact copy of the complex numbers.[4] In order to describe multiplication in a refined way, let us agree to call the factors of a product by different names; thus in the product denoted $\xi\phi$ (which is yet to be defined), we shall call ξ the multiplicator and ϕ the multiplicand. Also, we make the following generalisation:

Definition 2.3. *Let* $\sum_{n=0}^{\infty} a_n\kappa_n = \xi \in \mathbb{S}$. *We define the modulus, or absolute value of* ξ, *denoted by* $|\xi|$, *as*

$$|\xi| := \sqrt{\sum_{n=0}^{\infty} a_n^2}. \tag{1}$$

(Note that only finitely many terms are non-zero.)

Now, the product of ordinary complex numbers is a binary operation $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ which has a neat geometrical interpretation, as is well-known; if $z, w \in \mathbb{C}$ are visualised as points in the complex plane, the product zw is constructed from w by a counterclockwise rotation through the angle $arg(z)$ about the origin and a re-scaling of the modulus by $|z|$. Of course, the rôles of z and w could be exchanged in the previous statement, reflecting commutativity under multiplication of ordinary complex numbers.

In defining multiplication for supercomplex numbers we shall pursue the idea of the product as a geometric construction, involving rotation and re-scaling; for that purpose, we might picture the basis elements $\kappa_0, \kappa_1, \kappa_2, \ldots$ as constituting an orthonormal system, i.e. visualise $\mathbb{S}_k$ as k-dimensional Euclidean space endowed with mutually orthogonal axes (one 'real' and several 'imaginary'). As mentioned before, we wish for multiplication within the plane $\mathbb{R} + \mathbb{R}\kappa_n$ to be similar to multiplication within $\mathbb{C}$; actually, given any non-real multiplicator, ξ, it determines a plane of rotation, $\mathbb{R} + \mathbb{R}\sigma_\xi$, and it is tempting to define the product as the number obtained by re-scaling and rotating the projection of the multiplicand onto this plane by an amount that would take 1 to ξ, while only altering directions orthogonal to the plane by a re-scaling. This would make multiplication within *any* plane containing the real axis similar to that within the complex plane. Later, after having properly defined multiplication, we shall formalise and specify this idea. For now, we make the following definitions based on the usual inner product on $\mathbb{R}^{n+1}$, inherited by $\mathbb{S}_n$:

Definition 2.4. *Two numbers,* $\xi = \sum_{n=0}^{\infty} a_n\kappa_n \in \mathbb{S}$ *and* $\phi = \sum_{n=0}^{\infty} b_n\kappa_n \in \mathbb{S}$ *are said to be orthogonal, or perpendicular, to each other, if*

[3] What we shall actually do in practice, is to omit the concept of division in favour of the introduction of an inverse, ξ^{-1}, and demand the equivalent property $\xi^{-1}(\xi\phi) = \phi$.

[4] **Note on notation**: By $\mathbb{R}\kappa_n$ we mean $\{\alpha\kappa_n | \alpha \in \mathbb{R}\}$; also, in general, given sets $\mathscr{A}$ and $\mathscr{B}$, whose elements can be added, we define $\mathscr{A} + \mathscr{B} := \{a + b \,|\, a \in \mathscr{A}, b \in \mathscr{B}\}$.

$$\check{\xi} \cdot \check{\phi} := \sum_{n=0}^{\infty} a_n b_n = 0. \tag{2}$$

(Note that only finitely many terms are non-zero.) If on the other hand

$$\check{\xi} \cdot \check{\phi} = \pm|\xi||\phi|, \tag{3}$$

or, equivalently, if $\xi = \lambda\phi$ *or* $\phi = \lambda'\xi$ *for some* $\lambda, \lambda' \in \mathbb{R}$, *they are said to be parallel. Accordingly,* 0 *is both parallel and orthogonal to any other number.*

We already defined products with real multiplicators, after imposing a vector space structure on $\mathbb{S}$; hence we may now assume that the multiplicator ξ is non-real. From the geometric description above, the map $(\xi, \phi) \mapsto \xi\phi$ to be defined as multiplication should be *linear* in the second argument.[5] From the general theory of linear algebra, any vector, in particular any multiplicand $\phi \in \mathbb{S}$, may be written uniquely as the sum of a number $\phi_\parallel \in \mathbb{R} + \mathbb{R}\sigma_\xi$ and a number $\phi_\perp := \phi - \phi_\parallel \in (\mathbb{R} + \mathbb{R}\sigma_\xi)^\perp$; in the language just introduced, $\sigma_{\phi_\parallel}$ is parallel to σ_ξ, whereas $\phi_\perp$ has no real part and is orthogonal to σ_ξ. We may use this property to obtain an expression for the product based on the presented principles, since linearity implies that we need only specify $\xi\phi_\parallel$ and $\xi\phi_\perp$.
$\phi_\parallel$ is easily found as the orthogonal projection onto $\mathbb{R} + \mathbb{R}\sigma_\xi$; the latter is spanned by 1 and σ_ξ (mutually orthogonal), and hence

$$\check{\phi_\parallel} = \frac{\check{\phi} \cdot \check{1}}{\check{1} \cdot \check{1}}\check{1} + \frac{\check{\phi} \cdot \check{\sigma_\xi}}{\check{\sigma_\xi} \cdot \check{\sigma_\xi}}\check{\sigma_\xi} = \alpha_\phi \check{1} + \frac{\check{\sigma_\xi} \cdot \check{\sigma_\phi}}{|\xi|^2 - {\alpha_\xi}^2}\check{\sigma_\xi}, \tag{4}$$

since obviously $\check{\sigma_\xi} \cdot \check{\sigma_\xi} = |\sigma_\xi|^2 = |\xi|^2 - {\alpha_\xi}^2$. Now, we want $\xi\phi_\parallel$ to be simply a rotation and scaling of $\phi_\parallel$ in the common plane of $\phi_\parallel, \xi$ and 1 – the algebraic expression for this product is easily found, since it should be as if among ordinary complex numbers;[6] thus

$$\xi\phi_\parallel = \alpha_\xi \alpha_{\phi_\parallel} - \check{\sigma_\xi} \cdot \check{\sigma_{\phi_\parallel}} + \alpha_\xi \sigma_{\phi_\parallel} + \alpha_{\phi_\parallel}\sigma_\xi.$$

The product $\xi\phi_\perp$ is even more trivial:

$$\xi\phi_\perp = |\xi|\phi_\perp = |\xi|\phi - |\xi|\phi_\parallel.$$

Altogether we therefore have

$$\xi\phi = \xi\phi_\perp + \xi\phi_\parallel = |\xi|\phi - |\xi|\phi_\parallel + \alpha_\xi\alpha_{\phi_\parallel} - \check{\sigma_\xi}\check{\sigma_{\phi_\parallel}} + \alpha_\xi\sigma_{\phi_\parallel} + \alpha_{\phi_\parallel}\sigma_\xi. \tag{5}$$

Finally, substitution of Eq. (4) into this yields after some manipulation a formula for the product, $\xi\phi$, for non-real ξ:

[5]If this is not obvious to the reader, it may be taken simply as an additional wish for the operation of multiplication; it will ensure left-distributivity.

[6]Compare with $(a_1 + b_1 i)(a_2 + b_2 i) = a_1 a_2 - b_1 b_2 + (a_1 b_2 + a_2 b_1)i$.

Definition 2.5. *Given $\xi, \phi \in \mathbb{S}, \xi \notin \mathbb{R}$ we define their product $\xi\phi$ by*

$$\xi\phi := \alpha_\xi\alpha_\phi - \check{\sigma_\xi} \cdot \check{\sigma_\phi} + |\xi|\sigma_\phi + \sigma_\xi(\alpha_\phi - \frac{\check{\sigma_\xi} \cdot \check{\sigma_\phi}}{|\xi| + \alpha_\xi}). \tag{6}$$

From this (and by construction), it is clear that the product is left-distributive:

Proposition 2.6. *The product between supercomplex numbers is left-distributive, i.e. $\xi(\phi \pm \chi) = \xi\phi \pm \xi\chi$ for any $\xi, \phi, \chi \in \mathbb{S}$.*

The expression in Eq. (6) is independent of particular coordinates in the imaginary space – it only depends explicitly on the real component, stressing its distinction from imaginary components. Of course, however, considering a multiplicator and multiplicand in $\mathbb{S}_n$, referring to the natural basis $(\kappa_0, \kappa_1, \ldots, \kappa_n)$ we might, because of linearity in the multiplicand, represent the multiplicator by a matrix:

Definition 2.7. *The $(n+1) \times (n+1)$ matrix representing a multiplicator, $\xi \in \mathbb{S}_n$, is denoted $\tilde{\xi}$, and defined by $(\xi\phi)\check{} = \tilde{\xi}\check{\phi}$ for any $\phi \in \mathbb{S}_n$; $\tilde{\xi}$ is called the matrix representation of ξ. Occasionally, for the sake of clarity, the $\tilde{}$-symbol shall be written after a parenthesis, for instance as in $(\xi + \phi)\tilde{}$.*

As for the vector representations, it will from now on be implicit, that an appropriate subspace $\mathbb{S}_n$ has already been chosen, when speaking of the matrix representation, $\tilde{\xi}$, of a given $\xi \in \mathbb{S}$.

We may quite easily read off the components of $\tilde{\xi}$ from Eq. (6); the matrix representation for $\xi \in \mathbb{S}_n, \xi \notin \mathbb{R}$ may be written in block form as

$$\tilde{\xi} = \begin{pmatrix} \alpha_\xi & -\beta_\xi^T \\ \beta_\xi & |\xi|\tilde{1} - \frac{1}{|\xi|+\alpha_\xi}\beta_\xi\beta_\xi^T \end{pmatrix}, \tag{7}$$

where β_ξ – here and in the following – denotes the $n \times 1$ matrix (column vector) obtained from $\check{\sigma_\xi}$ by removing the top entry (which is zero), where superscript T denotes transposition, $\tilde{1}$ is the $n \times n$ identity matrix, and $\beta_\xi\beta_\xi^T$ is the matrix product of the $n \times 1$ matrix β_ξ with the $1 \times n$ matrix β_ξ^T. In $\mathbb{S}_3$, for instance, letting $\xi := \alpha + \beta_1\kappa_1 + \beta_2\kappa_2 + \beta_3\kappa_3$ be non-real, we have

$$\tilde{\xi} = \begin{pmatrix} \alpha & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & |\xi| - \frac{\beta_1{}^2}{|\xi|+\alpha} & -\frac{\beta_1\beta_2}{|\xi|+\alpha} & -\frac{\beta_1\beta_3}{|\xi|+\alpha} \\ \beta_2 & -\frac{\beta_1\beta_2}{|\xi|+\alpha} & |\xi| - \frac{\beta_2{}^2}{|\xi|+\alpha} & -\frac{\beta_2\beta_3}{|\xi|+\alpha} \\ \beta_3 & -\frac{\beta_1\beta_3}{|\xi|+\alpha} & -\frac{\beta_2\beta_3}{|\xi|+\alpha} & |\xi| - \frac{\beta_3{}^2}{|\xi|+\alpha} \end{pmatrix}, \tag{8}$$

which obviously agrees with multiplication in $\mathbb{C}$ when $\beta_2 = \beta_3 = 0$. Even though we expected none of the above expressions to hold for real ξ, in fact both Eq. (6) and (7) do, if ξ is positive; for non-positive numbers[7], however, we must always use the product defined for real numbers, and any proof involving products should take special care of this case. If we wish, we may write the matrix representation of any real ξ as $\xi\tilde{1}$.

Lemma 2.8. *For any $\xi \in \mathbb{S}_n, \xi \neq 0$, the $(n+1)\times(n+1)$ matrix $\frac{1}{|\xi|}\tilde{\xi}$ is orthogonal.*

Proof. For $n = 0$, i.e. $\xi \in \mathbb{R}$, this is trivial. For $\xi \notin \mathbb{R}$, we may use the block matrix representation to compute $\tilde{\xi}\tilde{\xi}^T$; we have

$$\tilde{\xi}\tilde{\xi}^T = \begin{pmatrix} \alpha_\xi & -\beta_\xi^T \\ \beta_\xi & |\xi|\tilde{1} - \frac{1}{|\xi|+\alpha_\xi}\beta_\xi\beta_\xi^T \end{pmatrix} \begin{pmatrix} \alpha_\xi & \beta_\xi^T \\ -\beta_\xi & |\xi|\tilde{1} - \frac{1}{|\xi|+\alpha_\xi}\beta_\xi\beta_\xi^T \end{pmatrix}$$

$$= \begin{pmatrix} {\alpha_\xi}^2 + |\sigma_\xi|^2 & (\alpha_\xi - |\xi| + \frac{|\sigma_\xi|^2}{|\xi|+\alpha_\xi})\beta_\xi^T \\ (\alpha_\xi - |\xi| + \frac{|\sigma_\xi|^2}{|\xi|+\alpha_\xi})\beta_\xi & |\xi|^2\tilde{1} + (1 + \frac{|\sigma_\xi|^2}{(|\xi|+\alpha_\xi)^2} - \frac{2|\xi|}{|\xi|+\alpha_\xi})\beta_\xi\beta_\xi^T \end{pmatrix},$$

since $\beta_\xi^T\beta_\xi = |\sigma_\xi|^2$. However, $\alpha_\xi - |\xi| + \frac{|\sigma_\xi|^2}{|\xi|+\alpha_\xi} = 0$, since $|\sigma_\xi|^2 = |\xi|^2 - {\alpha_\xi}^2$, and similarly

$$1 + \frac{|\sigma_\xi|^2}{(|\xi|+\alpha_\xi)^2} - \frac{2|\xi|}{|\xi|+\alpha_\xi} = \frac{(|\xi|+\alpha_\xi)^2 + |\xi|^2 - {\alpha_\xi}^2 - 2|\xi|(|\xi|+\alpha_\xi)}{(|\xi|+\alpha_\xi)^2} = 0.$$

Therefore $\tilde{\xi}\tilde{\xi}^T = |\xi|^2\tilde{1}$, or $\frac{1}{|\xi|^2}\tilde{\xi}\tilde{\xi}^T = \tilde{1}$, demonstrating orthogonality. □

The operation of multiplication commutes with that of forming moduli, a property known from ordinary complex numbers:

Proposition 2.9. *For any $\xi, \phi \in \mathbb{S}$,*

$$|\xi\phi| = |\xi||\phi|.$$

Proof. When ξ is non-positive it is trivial, and for any other ξ it may be verified from Eq. (6). Alternatively, since orthogonal matrices preserve inner products, we have $|\xi\phi|^2 = (\tilde{\xi}\check{\phi}) \cdot (\tilde{\xi}\check{\phi}) = |\xi|^2(\frac{\tilde{\xi}}{|\xi|}\check{\phi}) \cdot (\frac{\tilde{\xi}}{|\xi|}\check{\phi}) = |\xi|^2\check{\phi}\cdot\check{\phi} = |\xi|^2|\phi|^2$, or $|\xi\phi| = |\xi||\phi|$. □

We furthermore note the following features of the matrix representation of a supercomplex number:

Theorem 2.10. *Let $\xi \in \mathbb{S}_{n-1}, \xi \notin \mathbb{R}^-$. For the determinant and trace of $\tilde{\xi}$, it holds that*

$$\det\tilde{\xi} = |\xi|^n, \tag{9}$$

$$\operatorname{tr}\tilde{\xi} = 2\alpha_\xi + (n-2)|\xi|. \tag{10}$$

[7] From now on, the term 'non-positive', will implicitly mean 'real and non-positive'.

Proof. Since $\frac{1}{|\xi|}\tilde{\xi}$ is orthogonal, all of its eigenvalues has an absolute value of 1 – therefore the eigenvalues of $\tilde{\xi}$ all have absolute value $|\xi|$. Thus $|\det\tilde{\xi}| = |\xi|^n$; the determinant is obviously real, and therefore $\det\tilde{\xi} = \pm|\xi|^n$. Since the determinant must be a continuous function of the components of ξ and since it is obviously positive when $\xi \in \mathbb{R}^+$, we conclude that $\det\tilde{\xi} = |\xi|^n$. The trace may easily be computed from Eq. (7) itself. □

Remark (Completion of $\mathbb{S}$): As the reader may have noticed, the space $\mathbb{S}$ of supercomplex numbers is not complete with respect to the metric induced by the 'modulus-norm' given in Def. 2.3: For any countable collection of real numbers $\{a_n\}_{n=0,1,\dots}$ with $\sum_{n=0}^{\infty} {a_n}^2 < \infty$, we may consider the sequence $\xi_0, \xi_1, \dots$ with $\xi_k := \sum_{n=0}^{k} a_n\kappa_n \in \mathbb{S}$; it is easily verified to be Cauchy, but it does not have a limit in $\mathbb{S}$ unless only finitely many a_i are non-zero. However, if we consider the extended space

$$\overline{\mathbb{S}} := \{ \sum_{n=0}^{\infty} a_n\kappa_n \,|\, a_i \in \mathbb{R}, \sum_{n=0}^{\infty} {a_n}^2 < \infty \},$$

it necessarily contains the limit of the above sequence. Actually, defining $|\xi| := \sqrt{\sum_{n=0}^{\infty} {a_n}^2}$ for $\xi = \sum_{n=0}^{\infty} a_n\kappa_n \in \overline{\mathbb{S}}$, thus extending the modulus-norm from Def. 2.3, $\overline{\mathbb{S}}$ becomes essentially the space $\ell^2_{\mathbb{R}}$, which is known to be complete;[8] $\overline{\mathbb{S}}$ is therefore the completion of $\mathbb{S}$. As $\ell^2_{\mathbb{R}}$, it is closed under addition and multiplication by real numbers (which may thereby be extended from $\mathbb{S}$ to $\overline{\mathbb{S}}$), and it is also easily verified that the product as defined by Eq. (6) extends to all of $\overline{\mathbb{S}}$, letting $\check{\sigma_\xi} \cdot \check{\sigma_\phi} := \sum_{n=1}^{\infty} a_n b_n$, for $\xi = \sum_{n=0}^{\infty} a_n\kappa_n, \phi = \sum_{n=0}^{\infty} b_n\kappa_n \in \overline{\mathbb{S}}$ (this quantity is finite by the Cauchy-Schwarz Inequality).
Thus, it is indeed possible to complete $\mathbb{S}$ and define on the completion addition and multiplication. Purely abstractly, we may regard any multiplicator $\xi \in \overline{\mathbb{S}}$ (or, equivalently, an element of $\ell^2_{\mathbb{R}}$) as a linear map from the space to itself (namely $\phi \mapsto \xi\phi$), i.e. an endomorphism of $\ell^2_{\mathbb{R}}$ under addition and multiplication by reals. The identification of the elements of $\ell^2_{\mathbb{R}}$ with such endomorphisms thus becomes a map, Π, from $\ell^2_{\mathbb{R}}$ to its dual space, ${\ell^2_{\mathbb{R}}}^*$ (in fact, since orthogonal matrices are invertible, $\Pi(\xi)$ is an *automorphism* for non-zero ξ). This view, of course, also applies to the non-complete space $\mathbb{S}$ and, as we shall later remark, much more generally.

In this paper, we shall not systematically treat the completion of $\mathbb{S}$, but only regard numbers with finitely many non-zero components; we may however, now that we have mentioned the construction, bear in mind that it is in many cases possible to extend the notions presented within the realm of $\mathbb{S}$ to all of $\overline{\mathbb{S}}$.

2.1 Eigennumbers, Conjugation and Inverses

We may note some basic interesting properties of the product; we see that for instance $\kappa_1\kappa_2 = (-\kappa_1)\kappa_2 = \kappa_2$ and $\kappa_2\kappa_1 = \kappa_1 \neq \kappa_1\kappa_2$, since multiplication by κ_1 rotates in a

[8] See [2] for a treatment of the more general L^p-spaces demonstrating completeness and closure under addition and multiplication by reals.

plane perpendicular to κ_2 (and vice versa). From this, we learn that multiplication is in general neither commutative nor associative; also we note the curious fact, that there exist multiplicators different from 1, whose product with a given multiplicand is the multiplicand itself.

Definition 2.11. *A number $\phi \in \mathbb{S}, \phi \neq 0$ is called an eigennumber of $\xi \in \mathbb{S}$, if for some $\tau \in \mathbb{R}$,*

$$\xi\phi = \tau\phi. \tag{11}$$

τ is said to be the corresponding eigenvalue. By $\mathscr{E}(\xi)$ we denote the entire set of eigennumbers of $\xi \in \mathbb{S}$, i.e. $\mathscr{E}(\xi) := \{\phi \in \mathbb{S} \,|\, \phi$ is an eigennumber of ξ $\}$.

Proposition 2.12. *If $\xi \in \mathbb{R}$, any number $\phi \in \mathbb{S} \setminus \{0\}$ is an eigennumber of ξ with eigenvalue ξ.*

Theorem 2.13. *A number $\phi \in \mathbb{S}, \phi \neq 0$ is an eigennumber of $\xi \in \mathbb{S} \setminus \mathbb{R}$ if and only if ϕ is orthogonal to both ξ and the real axis (i.e. is purely imaginary). Its eigenvalue will be $|\xi|$.*

Proof. Assume ϕ is an eigennumber of $\xi \notin \mathbb{R}$; we must have $|\xi||\phi| = |\tau||\phi|$, or $\tau = \pm|\xi|$. Eq. (6) now reads

$$\alpha_\xi\alpha_\phi - \check{\sigma_\xi} \cdot \check{\sigma_\phi} + |\xi|\sigma_\phi + \sigma_\xi(\alpha_\phi - \frac{\check{\sigma_\xi} \cdot \check{\sigma_\phi}}{|\xi| + \alpha_\xi}) = \pm|\xi|\alpha_\phi \pm |\xi|\sigma_\phi.$$

For $\tau = -|\xi|$, the real part of the equation implies $\check{\sigma_\xi} \cdot \check{\sigma_\phi} = \alpha_\phi(|\xi| + \alpha_\xi)$, which, when substituted into the imaginary part, gives $|\xi|\sigma_\phi = -|\xi|\sigma_\phi$, or $\sigma_\phi = 0$; but then, from the real part, $\alpha_\phi = 0$, and thus $\phi = 0 \notin \mathscr{E}(\xi)$ – hence there can be no eigennumbers with eigenvalue $-|\xi|$.

For $\tau = +|\xi|$ the imaginary part of the equation implies, since $\sigma_\xi \neq 0$, that $\check{\sigma_\xi} \cdot \check{\sigma_\phi} = \alpha_\phi(|\xi| + \alpha_\xi)$, which, on substitution into the real part, demands $-|\xi|\alpha_\phi = |\xi|\alpha_\phi$, or $\alpha_\phi = 0$. Hence ϕ is purely imaginary. From the former we thus have $\check{\sigma_\xi} \cdot \check{\sigma_\phi} = 0$, implying orthogonality between σ_ξ and σ_ϕ, which, for $\alpha_\phi = 0$, is equivalent to orthogonality between ξ and ϕ themselves. Conversely, it is obvious that the conditions $\check{\sigma_\xi} \cdot \check{\sigma_\phi} = 0$ and $\alpha_\phi = 0$ ensure $\phi \in \mathscr{E}(\xi)$.

□

Let us introduce the complex conjugate:

Definition 2.14. *A supercomplex number $\xi = \alpha_\xi + \sigma_\xi$ has associated with it a number $\xi^* := \alpha_\xi - \sigma_\xi$, called the complex conjugate of ξ.*

Proposition 2.15. *Let $\xi \in \mathbb{S}$. Then*

$$\xi^*\xi = \xi\xi^* = |\xi|^2. \tag{12}$$

Proof. This can be verified by simple substitution into Eq. (6); alternatively, observe that $\tilde{\xi^*} = \tilde{\xi}^T$, and therefore $(\xi^*\xi)\breve{} = \tilde{\xi}^T(\tilde{\xi}\breve{1}) = (\tilde{\xi}^T\tilde{\xi})\breve{1} = |\xi|^2\breve{1}$, by Lem. 2.8, and similarly for $\xi\xi^*$. □

We shall now try to find an expression for the inverse, ξ^{-1}, of a supercomplex number $\xi \neq 0$, with the property that $\xi^{-1}(\xi\phi) = \phi$ for all $\phi \in \mathbb{S}$. If this is really possible, the equation must in particular hold for $\phi = \xi^*$; the requirement then becomes $\xi^{-1}|\xi|^2 = \xi^*$, or $\xi^{-1} = \xi^*/|\xi|^2$.

Proposition 2.16. *Given any $\xi \in \mathbb{S}, \xi \neq 0$, the number $\xi^{-1} := \xi^*/|\xi|^2$, called the inverse of ξ, has the property that $\xi^{-1}(\xi\phi) = \phi$ for all $\phi \in \mathbb{S}$; furthermore $(\xi^{-1})^{-1} = \xi$.*

Proof. For $\xi \in \mathbb{R}, \xi \neq 0$ the statement is obviously true. Given $\xi, \phi \in \mathbb{S}, \xi \notin \mathbb{R}$, observe that $(\xi^{-1})\tilde{} = (\xi^*/|\xi|^2)\tilde{} = 1/|\xi|^2\tilde{\xi}^T$ (the second equality is the statement that $(\lambda\xi^*)\tilde{} = \lambda\tilde{\xi^*}$ for any $\lambda \in \mathbb{R}^+$, as may easily be verified); by Lem. 2.8, $(\xi^{-1})\tilde{}\tilde{\xi} = 1/|\xi|^2\tilde{\xi}^T\tilde{\xi} = \tilde{1}$, implying that $(\xi^{-1})\tilde{} = \tilde{\xi}^{-1}$; this confirms the desired property, since $(\xi^{-1})\tilde{}(\tilde{\xi}\breve{\phi}) = ((\xi^{-1})\tilde{}\tilde{\xi})\breve{\phi} = \breve{\phi}$.
It is obvious from the definition that $(\xi^{-1})^{-1} = \xi$. □

The following statements about complex conjugation are apparent from the given definitions and Eq. (6):

Proposition 2.17. *For any $\xi, \phi \in \mathbb{S}$, it is true that $(\xi \pm \phi)^* = \xi^* \pm \phi^*$, $(\xi\phi)^* = \xi^*\phi^*$, $(\xi^{-1})^* = (\xi^*)^{-1}$ and $(\xi^*)^* = \xi$.*

We conclude by giving a formula that extracts from ξ its n^{th} component (real or imaginary) – inspired from the corresponding formulae for ordinary complex numbers, $Re(z) = \frac{z+z^*}{2}$ and $Im(z) = -\frac{iz+(iz)^*}{2}$, it is easily verified that

$$a_n = ({\kappa_n}^2)\frac{\kappa_n\xi + (\kappa_n\xi)^*}{2}, \tag{13}$$

for $\xi = \sum_{n=0}^{\infty} a_n\kappa_n$. This also allows for an expression for the ordinary inner product between vectors in terms of the product between supercomplex numbers; observing that the real part of $\xi\phi^*$ (or $\phi\xi^*$) is exactly $\breve{\xi} \cdot \breve{\phi}$, we see that

$$\breve{\xi} \cdot \breve{\phi} = \frac{\xi\phi^* + \xi^*\phi}{2} = \frac{\phi\xi^* + \phi^*\xi}{2}. \tag{14}$$

3 Algebraic Structure of the Supercomplex Numbers

We now turn to some further properties of the product between supercomplex numbers; we shall investigate commutativity, associativity and distributivity.

Definition 3.1. *Let $\xi, \phi, \chi \in \mathbb{S}$. The commutator between the numbers ξ and ϕ is defined as*

$$[\xi, \phi]_C := \xi\phi - \phi\xi. \tag{15}$$

Two numbers are said to commute if their commutator is 0*. Similarly, we define the associator and distributor of* ξ, ϕ *and* χ *as*

$$[\xi, \phi; \chi]_A := (\xi\phi)\chi - \xi(\phi\chi), \tag{16}$$

$$[\xi, \phi; \chi]_{D\pm} := (\xi \pm \phi)\chi - (\xi\chi \pm \phi\chi), \tag{17}$$

respectively.

3.1 The set of Commutants and the Weak Theorem of Isomorphism

Starting with commutativity, we obviously have, from Eq. (6), that $[\xi, \phi]_C = 0$ for real ξ or ϕ. Simple calculation shows that for non-real ξ, ϕ,

$$[\xi, \phi]_C = \sigma_\phi(|\xi| - \alpha_\xi + \frac{\check{\sigma_\xi} \cdot \check{\sigma_\phi}}{|\phi| + \alpha_\phi}) - \sigma_\xi(|\phi| - \alpha_\phi + \frac{\check{\sigma_\xi} \cdot \check{\sigma_\phi}}{|\xi| + \alpha_\xi}); \tag{18}$$

thus the real parts of $\xi\phi$ and $\phi\xi$ are equal. We state the following result:

Theorem 3.2. $\xi, \phi \in \mathbb{S}$ *commute if and only if* σ_ξ *and* σ_ϕ *are parallel.*

Proof. First assume commutativity; if either ξ or ϕ are real, the imaginary parts are obviously parallel (at least one is zero) – if on the other hand $\xi, \phi \notin \mathbb{R}$ we infer from Eq. (18), since $\sigma_\xi, \sigma_\phi \neq 0$, that either $\check{\sigma_\xi} \parallel \check{\sigma_\phi}$, which demonstrates the criterion, or else

$$|\xi| - \alpha_\xi + \frac{\check{\sigma_\xi} \cdot \check{\sigma_\phi}}{|\phi| + \alpha_\phi} = |\phi| - \alpha_\phi + \frac{\check{\sigma_\xi} \cdot \check{\sigma_\phi}}{|\xi| + \alpha_\xi} = 0.$$

Multiplying by $(|\xi| + \alpha_\xi)(|\phi| + \alpha_\phi)$, invoking $|\xi|^2 - {\alpha_\xi}^2 = |\sigma_\xi|^2$ and a similar relationship for ϕ yields

$$(|\phi| + \alpha_\phi)|\sigma_\xi|^2 + (|\xi| + \alpha_\xi)\check{\sigma_\xi} \cdot \check{\sigma_\phi} = (|\xi| + \alpha_\xi)|\sigma_\phi|^2 + (|\phi| + \alpha_\phi)\check{\sigma_\xi} \cdot \check{\sigma_\phi} = 0.$$

Note that $\check{\sigma_\xi} \cdot \check{\sigma_\phi} \neq 0$ (since otherwise the above implies $\sigma_\xi = \sigma_\phi = 0$), and therefore we now have

$$\frac{|\sigma_\xi|^2}{\check{\sigma_\xi} \cdot \check{\sigma_\phi}} = -\frac{|\xi| + \alpha_\xi}{|\phi| + \alpha_\phi} = \frac{\check{\sigma_\xi} \cdot \check{\sigma_\phi}}{|\sigma_\phi|^2},$$

implying $\check{\sigma_\xi} \cdot \check{\sigma_\phi} = \pm|\sigma_\xi||\sigma_\phi|$, and thus according to Def. 2.4 that σ_ξ is parallel to σ_ϕ. Now, assume parallelism; if either ξ or ϕ is real, they commute – if $\xi, \phi \notin \mathbb{R}$ we may set $\sigma_\phi = \lambda\sigma_\xi$ for some $\lambda \in \mathbb{R}$, obtaining from Eq. (18)

$$[\xi, \phi]_C = \sigma_\xi(\lambda(|\xi| - \alpha_\xi) + \frac{\check{\sigma_\phi}^2}{|\phi| + \alpha_\phi} - |\phi| + \alpha_\phi - \lambda\frac{\check{\sigma_\xi}^2}{|\xi| + \alpha_\xi});$$

using, as before, $|\sigma_\xi|^2 = |\xi|^2 - {\alpha_\xi}^2$ and therefore $\frac{\check{\sigma_\xi}^2}{|\xi|+\alpha_\xi} = |\xi| - \alpha_\xi$, together with the similar identity for ϕ, we easily conclude

$$[\xi, \phi]_C = 0.$$

□

The connection between commutativity and parallelism suggests the introduction of the following concept:

Definition 3.3. *We link with $\xi \in \mathbb{S}$ the set $\mathscr{C}(\xi) := \mathbb{R} + \mathbb{R}\sigma_\xi$, called the set of commutants of ξ, and we say that an element $\psi \in \mathscr{C}(\xi)$ is a commutant of ξ.*

Proposition 3.4. *Let $\xi \in \mathbb{S}$. Any two numbers $\psi, \psi' \in \mathscr{C}(\xi)$ commute with each other, and each of them commutes with ξ.*
If $\xi \notin \mathbb{R}$, then $\psi \in \mathbb{S}$ is an element of $\mathscr{C}(\xi)$ if and only if ψ commutes with ξ, and if in addition $\psi \notin \mathbb{R}$, then $\mathscr{C}(\psi) = \mathscr{C}(\xi)$.

The equivalences between commutativity and parallelism, eigennumbers and orthogonality establish the following property:

Proposition 3.5. *Given $\xi, \phi \in \mathbb{S}$, we may always write ϕ as a sum of a number $\phi_c \in \mathscr{C}(\xi)$, and a number $\phi_e \in \mathscr{E}(\xi) \cup \{0\}$, i.e. for any $\xi \in \mathbb{S}$, $\mathscr{C}(\xi) + \mathscr{E}(\xi) \cup \{0\} = \mathbb{S}$. If $\xi \notin \mathbb{R}$, the decomposition of ϕ into ϕ_c, ϕ_e is unique and given by*

$$\phi_c = \alpha_\phi + \frac{\check{\sigma_\xi} \cdot \check{\sigma_\phi}}{|\sigma_\xi|^2}\sigma_\xi,\ \phi_e = \phi - \phi_c. \tag{19}$$

Proof. For $\xi \in \mathbb{R}$ we may choose any pair $(\phi_c, \phi_e) \in \mathbb{R} \times \mathbb{S}$ subject to the restriction $\phi_c + \phi_e = \phi$; for $\xi \notin \mathbb{R}$, the given formulae are simply the unique orthogonal projections of ϕ onto $\mathscr{C}(\xi)$ and $\mathscr{C}(\xi)^\perp = \mathscr{E}(\xi) \cup \{0\}$ (see Thm. 2.13). □

Before we continue with the occurrences of associativity and distributivity, we shall put forward one of our most important results; its content is indeed at the heart of the entire construction of the supercomplex numbers.

Theorem 3.6. *(Weak Theorem of Isomorphism.) Let $\mathscr{S}$ be a subspace of $\mathbb{S}$ with dimension $n+1 \in \mathbb{N}$ containing $\mathbb{R}$. Then $\mathscr{S}$ is isomorphic to $\mathbb{S}_n$ under the fundamental algebraic operations (complex conjugation, addition, multiplication and their inverses).*
For any $s \in \{1, \ldots, n\}$ and purely imaginary $\sigma \in \mathscr{S}$ there exists an isomorphism $I : \mathscr{S} \to \mathbb{S}_n$ which fixes $\mathbb{R}$ and so that $I(\sigma) = |\sigma|\kappa_s$.

Proof. If $n = 0$ we must have $\mathscr{S} = \mathbb{R} = \mathbb{S}_0$, and there is nothing to show. For $n > 0$, let $(1, \sigma_1, \ldots, \sigma_n)$ constitute a basis for $\mathscr{S}$; we may choose $\sigma_1, \ldots, \sigma_n \in \mathscr{S}$ purely imaginary. Given $s \in \{1, \ldots, n\}$ and a purely imaginary $\sigma \in \mathscr{S}$, we may – using Gram-Schmidt-orthonormalisation – form an orthonormal basis for $\mathscr{S}$ (i.e. such that the vector representations of the basis elements form an orthonormal system), say $\mathscr{B} := (1, \sigma'_1, \sigma'_2, \ldots, \sigma'_n)$, with

$\sigma'_s = \sigma/|\sigma|$.
Let $m \geq n$ be such that $\mathscr{S} \subseteq \mathbb{S}_m$ (such m exists, since $\mathscr{S}$ has finite dimension and each σ'_i has finitely many non-zero components), and consider an orthonormal basis for $\mathbb{S}_m$, $\mathscr{B}_m := (1, \sigma'_1, \sigma'_2, \ldots, \sigma'_n, \ldots, \sigma'_m)$, which is an extension of $\mathscr{B}$. Now, let us construct an orthogonal $(m+1) \times (m+1)$ matrix $M := (\check{1}, \check{\sigma'_1}, \ldots, \check{\sigma'_m})^T$ and define a map $I : \mathscr{S} \to \mathbb{S}_m$ by $I(\xi)\check{} = M\check{\xi}$. I is linear and therefore a homomorphism under addition, subtraction, conjugation and inversion. We note that for any $\alpha \in \mathbb{R}$, $i \in \{1, \ldots, n\}$,

$$\begin{aligned} I(\alpha)\check{} &= (\check{1}, \check{\sigma'_1}, \ldots, \check{\sigma'_m})^T \check{\alpha} = \check{\alpha}, \\ I(\sigma'_i)\check{} &= (\check{1}, \check{\sigma'_1}, \ldots, \check{\sigma'_m})^T \check{\sigma'_i} = \check{\kappa}_i. \end{aligned} \tag{20}$$

Thus I fixes $\mathbb{R}$, I is bijective when considered as mapping to $\mathbb{S}_n$ (since $I(\mathscr{S}) = \mathbb{S}_n$ from (20), and M is invertible), and $I(\sigma) = I(|\sigma|\sigma'_s) = |\sigma| I(\sigma'_s) = |\sigma|\kappa_s$. Finally,

$$\begin{aligned} I(\xi\phi) &= \alpha_\xi\alpha_\phi - \check{\sigma_\xi} \cdot \check{\sigma_\phi} + |\xi| I(\sigma_\phi) + I(\sigma_\xi)(\alpha_\phi - \frac{\check{\sigma_\xi} \cdot \check{\sigma_\phi}}{|\xi| + \alpha_\xi}) \\ &= \alpha_\xi\alpha_\phi - I(\check{\sigma_\xi}) \cdot I(\check{\sigma_\phi}) + |\xi| I(\sigma_\phi) + I(\sigma_\xi)(\alpha_\phi - \frac{I(\check{\sigma_\xi}) \cdot I(\check{\sigma_\phi})}{|\xi| + \alpha_\xi}) \\ &= I(\xi)I(\xi), \end{aligned} \tag{21}$$

because of the preservation of inner products by orthogonal matrices and the fixing of $\mathbb{R}$. □

Corollary 3.7. *For any $\xi \in \mathbb{S} \setminus \mathbb{R}$, the subspace $\mathscr{C}(\xi)$ is isomorphic to $\mathbb{S}_1$ (which we identify as $\mathbb{C}$) under the fundamental algebraic operations.*

This, in fact, is the principle which originally motivated the definition of multiplication – that any plane through the real axis is a replica of the complex plane. Also, this provides an easy proof that parallelism implies commutativity – since all numbers in $\mathbb{C}$ commute, all numbers in $\mathscr{C}(\xi)$ must commute. Another immediate and illuminating consequence of Thm. 3.6 is the following:

Corollary 3.8. *There exists an isomorphism preserving the fundamental algebraic operations, $I : \mathbb{S}_1 = \mathscr{C}(-\kappa_1) \to \mathbb{S}_1$, so that $I(-\kappa_1) = \kappa_1$, or, in general, $I(a + b\kappa_1) = a - b\kappa_1$ for $a, b \in \mathbb{R}$.*

This isomorphism is of course ordinary *complex conjugation* – it explains the ambiguity, even for ordinary complex numbers, in defining i by $i^2 = -1$; every relation that is true in $\mathbb{C}$ preserves its validity under the replacement of i by $-i$. Hence the replacement of e.g. κ_1 by κ_2 is in this sense not much different from complex conjugation.

3.2 Associativity and Distributivity

We note the following useful facts:

Proposition 3.9. *If ξ and ϕ commute, their product is*

$$\xi\phi = (\alpha_\xi\alpha_\phi - \check{\sigma_\xi} \cdot \check{\sigma_\phi}) + \alpha_\phi\sigma_\xi + \alpha_\xi\sigma_\phi. \tag{22}$$

Proof. If either ξ or ϕ is real, the statement is trivial. For $\xi, \phi \notin \mathbb{R}$, both are elements of at least one of the sets $\mathscr{C}(\xi)$ and $\mathscr{C}(\phi)$, say $\mathscr{C}(\xi)$; for any $\xi, \phi \in \mathbb{S}_1$ the above relation holds, and using the isomorphism $\alpha_\xi + \sigma_\xi \mapsto \alpha_\xi + |\sigma_\xi|\kappa_1$ and its inverse, the desired follows. Of course the identity is also easily demonstrated algebraically from Eq. (6). □

Lemma 3.10. *Let $\xi, \phi \in \mathbb{S}$. If the product $\xi\phi$ is real, ξ and ϕ commute.*

Proof. Let $\xi\phi = \alpha \in \mathbb{R}$; if $\xi = 0$, it obviously commutes with ϕ – if $\xi \neq 0$, multiplication by ξ^{-1} yields $\phi = \xi^{-1}\alpha = \frac{\alpha}{|\xi|^2}\xi^*$. Thus ϕ is parallel to $\xi^* = \alpha_\xi - \sigma_\xi$, and thereby σ_ϕ is parallel to σ_ξ. □

Theorem 3.11. *Let $\xi, \phi, \chi \in \mathbb{S}$. For $[\xi, \phi; \chi]_A$ to vanish, it is a necessary condition that $[\xi, \phi]_C \in \mathscr{E}(\chi) \cup \{0\}$; it is sufficient that ξ, ϕ, χ commute pairwise. It is furthermore sufficient that $\chi \in \mathbb{R}$.*

Proof. Assume associativity; if ξ, ϕ or $\xi\phi$ is real, then $[\xi, \phi]_C = 0$ (by Lem. 3.10); otherwise, we may use the standard expression for the product and simple, though a little tedious, calculation yields

$$Re([\xi, \phi; \chi]_A) = Re((\xi\phi)\chi - \xi(\phi\chi)) = -\check{[\xi, \phi]_C} \cdot \check{\sigma_\chi}, \tag{23}$$

where $Re(\cdot)$ denotes the real part of the argument. Obviously, we must have $Re([\xi, \phi; \chi]_A) = 0$, implying orthogonality between $[\xi, \phi]_C$ and σ_χ. Since however, as we have seen earlier, $Re([\xi, \phi]_C) = 0$ this is, by Thm. 2.13, equivalent to the statement that $[\xi, \phi]_C \in \mathscr{E}(\chi) \cup \{0\}$. Now, assume that all three numbers commute; then $\xi, \phi, \chi \in \mathscr{C}(\psi)$ for some $\psi \in \{\xi, \phi, \chi\}$ – $\mathscr{C}(\psi)$ is isomorphic to $\mathbb{S}_1$ (or equals $\mathbb{R}$, if $\xi, \phi, \chi \in \mathbb{R}$), whereby obviously $[\xi, \phi; \chi]_A = 0$. Finally, for $\chi \in \mathbb{R}$, clearly $[\xi, \phi; \chi]_A = 0$. □

Corollary 3.12. *Let $\xi, \phi \in \mathbb{S}, (\xi, \phi) \notin \mathbb{R}^2$. Then $[\xi, \phi; \chi]_A = 0$ for all $\chi \in \mathbb{S}$, if and only if $[\xi, \phi]_C = 0$ and $\xi, \phi, \xi\phi$ are all non-negative.*

Proof. If $[\xi, \phi; \chi]_A = 0$ for all $\chi \in \mathbb{S}$, Eq. (23) (or the statement of Thm. 3.11) implies $[\xi, \phi]_C = 0$. We cannot have $\xi \in \mathbb{R}^-$; for then we would have $\phi \notin \mathbb{R}$ (because $(\xi, \phi) \notin \mathbb{R}^2$) and we could take $\chi \in \mathscr{E}(\phi)$, and, writing $\xi = -\alpha$ for some $\alpha \in \mathbb{R}^+$, noting $\chi \in \mathscr{E}(-\alpha\phi)$, we would have $[\xi, \phi; \chi]_A = (-\alpha\phi)\chi - (-\alpha)(\phi\chi) = 2\alpha|\phi|\chi \neq 0$. Similarly we cannot have $\phi \in \mathbb{R}^-$. Also $\xi\phi \notin \mathbb{R}^-$ – for if not, note first that $\xi, \phi \notin \mathbb{R}$, since $\xi\phi \in \mathbb{R}^-$, but $(\xi, \phi) \notin \mathbb{R}^2$;

then we could take $\chi \in \mathscr{E}(\xi) \cap \mathscr{E}(\phi)$, which is clearly non-empty (in fact $\mathscr{E}(\xi) = \mathscr{E}(\phi)$, since $\mathscr{C}(\xi) = \mathscr{C}(\phi)$), and find $(\xi\phi)\chi - \xi(\phi\chi) = -2|\xi||\phi|\chi \neq 0$ (since $\xi\phi = -|\xi||\phi|$).

Conversely, let $\xi, \phi, \xi\phi$ be non-negative and $[\xi, \phi]_C = 0$. If either ξ or ϕ is real (necessarily non-negative), the statement is obvious; if not, let $\xi, \phi \in \mathscr{C}(\psi)$ for some ψ. For $\chi \in \mathscr{C}(\psi)$ there is nothing to show, since then ξ, ϕ, χ all commute. For $\chi \notin \mathscr{C}(\psi)$, we have $\xi, \phi, \chi \in \mathscr{S} := \mathscr{C}(\psi) + \mathscr{C}(\chi)$, and there exists an isomorphism $I : \mathscr{S} \to \mathbb{S}_2$ so that $I(\xi) = \alpha_\xi + |\sigma_\xi|\kappa_1, I(\phi) = \alpha_\phi \pm |\sigma_\phi|\kappa_1$. As is easily verified, we have $(I(\xi)I(\phi))\breve{} = \tilde{I(\xi)}\tilde{I(\phi)}$ (since $\xi, \phi, \xi\phi$ are non-negative, Eq. (7) holds); this however, is exactly the statement, that $[I(\xi)I(\phi)]I(\chi) = I(\xi)[I(\phi)I(\chi)]$, and the desired follows from an application of I^{-1}.

□

For distributivity we may state a result of the same sort;

Theorem 3.13. *Let $\xi, \phi, \chi \in \mathbb{S} \setminus \{0\}$. For $[\xi, \phi; \chi]_{D\pm}$ to vanish, it is necessary that either ξ and ϕ are parallel or that $\sigma_\chi \in \mathscr{C}(\sigma_\xi) + \mathscr{C}(\sigma_\phi)$; it is sufficient that all three numbers commute pairwise. It is furthermore sufficient that $\chi \in \mathbb{R}$.*

Proof. Assume distributivity; it is clear that the imaginary part of the distributor is a linear combination of $\sigma_\xi, \sigma_\phi, \sigma_\chi$. In fact we find

$$[\xi, \phi; \chi]_{D\pm} - Re([\xi, \phi; \chi]_{D\pm}) = (|\xi \pm \phi| - (|\xi| \pm |\phi|))\sigma_\chi + \lambda_\xi\sigma_\xi + \lambda_\phi\sigma_\phi = 0,$$

for some $\lambda_\xi, \lambda_\phi \in \mathbb{R}$, which depend on ξ and ϕ; therefore either $|\xi \pm \phi| = |\xi| \pm |\phi|$ which means that ξ and ϕ are parallel (squaring each side shows that $\check{\xi} \cdot \check{\phi} = |\xi||\phi|$), or else σ_χ can be written as a linear combination of σ_ξ and σ_ϕ, i.e. $\sigma_\chi \in \mathscr{C}(\sigma_\xi) + \mathscr{C}(\sigma_\phi)$.
It is clearly sufficient that $\chi \in \mathbb{R}$, and commutativity between all three numbers is also sufficient, since then $\xi, \phi, \chi \in \mathscr{C}(\psi)$ for some $\psi \in \{\xi, \phi, \chi\}$, and $\mathscr{C}(\psi)$ is isomorphic to $\mathbb{S}_1$ (or equals $\mathbb{R}$) under the fundamental algebraic operations. □

Corollary 3.14. $[\xi, \phi; \chi]_{D\pm} = 0$ *for all χ, only if ξ and ϕ are parallel.*

Proof. Given $\xi, \phi \in \mathbb{S}$, assume $[\xi, \phi; \chi]_{D\pm} = 0$ for all $\chi \in \mathbb{S}$; choose a $\chi \in \mathbb{S}, \chi \notin \mathscr{C}(\sigma_\xi) + \mathscr{C}(\sigma_\phi)$. The desired follows from Thm. 3.13. □

4 Exponentials, deMoivre's Formula and the Strong Theorem of Isomorphism

Now, having explored some fundamental algebraic operations and properties of supercomplex numbers, we may take the theory further; let us first introduce the following notation:

Definition 4.1. *Let $n \in \mathbb{Z}, \xi \in \mathbb{S}$. We define*[9]

[9] Note that for clarity a '$\cdot$' is used here to indicate multiplication; moreover, indeed $\xi \cdot \xi^n = \xi^n \cdot \xi$ since $\xi^n \in \mathscr{C}(\xi)$, by an inductive argument.

$$\xi^0 := 1,$$
$$\xi^{n+1} := \xi \cdot \xi^n = \xi^n \cdot \xi, \text{ for } n \geq 0, \text{ and}$$
$$\xi^n := (\xi^{-1})^{-n} \text{ for } n < 0 \text{ (and } \xi \neq 0\text{)}.$$

More interesting, perhaps, is the idea of supercomplex exponents; for ordinary complex numbers, the core of exponentiation is Euler's famous identity,

$$e^{\alpha+\beta i} = e^{\alpha}(\cos\beta + i\sin\beta). \tag{24}$$

Appealing to the principle we set up to motivate the construction of multiplication by considering replicas of the complex plane, we define the exponential of a supercomplex number as follows:

Definition 4.2. *Let $\xi \in \mathbb{S} \setminus \mathbb{R}$. We define*

$$e^{\xi} := e^{\alpha_\xi}(\cos|\sigma_\xi| + \frac{\sigma_\xi}{|\sigma_\xi|}\sin|\sigma_\xi|). \tag{25}$$

(For $\sigma_\xi = 0$, i.e. $\xi \in \mathbb{R}$, we use the existing definition, $e^\xi = e^{\alpha_\xi}$.)

This definition agrees with the Taylor expansion of e^ξ as an infinite power series, and, if we were to define for $x \in \mathbb{R}$, the derivative $\frac{d}{dx}(e^{\xi x})$ as the supercomplex number corresponding to $\frac{d}{dx}(e^{\xi x})\check{}$ (the derivative of the vector representation), then the identity

$$\frac{d}{dx}(e^{\xi x}) = \xi e^{\xi x} \tag{26}$$

would be true.[10] Eq. (25) of course reduces to (24) when $\xi \in \mathbb{S}_1$. For any $\xi \in \mathbb{S}$, we have $e^\xi \in \mathscr{C}(\xi)$; also we realise that the rule of adding exponents when multiplying exponentials still holds within $\mathscr{C}(\xi)$:

Theorem 4.3. *Let $\xi, \phi \in \mathbb{S}$. If $[\xi, \phi]_C = 0$ we have the identity*

$$e^\xi e^\phi = e^{\xi+\phi}. \tag{27}$$

Proof. $\xi, \phi \in \mathscr{C}(\psi)$ for some $\psi \in \{\xi, \phi\}$; $\mathscr{C}(\psi)$ is isomorphic to $\mathbb{S}_1$ (or equals $\mathbb{R}$) under fundamental algebraic operations, and here the identity is known to be true. Given the isomorphism $\xi \mapsto \alpha_\xi + |\sigma_\xi|\kappa_1$, we have $I(e^\xi) = e^{I(\xi)}$ from Eq. (25), and hence $I(e^\xi e^\phi) = e^{I(\xi)}e^{I(\phi)} = e^{I(\xi)+I(\phi)} = I(e^{\xi+\phi})$, or $e^\xi e^\phi = e^{\xi+\phi}$. □

[10]The power series of e^ξ would be the limit of $\sum_{n=0}^{N} \frac{1}{n!}\xi^n$, as $N \to \infty$ and because of the isomorphism between $\mathscr{C}(\xi)$ and $\mathbb{C}$, we would – as we do for $\sum_{n=0}^{N} \frac{1}{n!}z^n$, $z \in \mathbb{C}$ – recognise the Taylor expansion for sine, cosine and the real exponential. Thus, accepting the Taylor expansion (after realising its convergence) as definition of e^ξ would *imply* Eq. (25). Since that definition is actually completely equivalent to the statement of Eq. (26), the latter could as well have been used as a definition.

This ensures that when ξ and ϕ commute, we may write $e^{\xi}e^{\phi} = e^{\xi+\phi} = e^{\phi+\xi} = e^{\phi}e^{\xi}$ consistently (since then also e^{ξ} and e^{ϕ} commute). Also, it implies $(e^{\xi})^n = e^{n\xi}$ for any $n \in \mathbb{N}$; in fact, we find from the given definitions that also $(e^{\xi})^n = e^{n\xi}$ for $n = 0$ and $n = -1$, hence any $n \in \mathbb{Z}^-$. This may be stated as a generalised version of the deMoivre formula;

Proposition 4.4. *For any $\xi \in \mathbb{S} \setminus \mathbb{R}, n \in \mathbb{Z}$, we have deMoivre's identity:*

$$(e^{\alpha_\xi} \cos|\sigma_\xi| + e^{\alpha_\xi} \frac{\sigma_\xi}{|\sigma_\xi|} \sin|\sigma_\xi|)^n = e^{n\alpha_\xi} \cos(n|\sigma_\xi|) + e^{n\alpha_\xi} \frac{\sigma_\xi}{|\sigma_\xi|} \sin(n|\sigma_\xi|). \tag{28}$$

We may now define $\xi^{\frac{p}{q}}, p \in \mathbb{Z}, q \in \mathbb{N}$ as a number satisfying $(\xi^{\frac{p}{q}})^q = \xi^p$, thus extending the validity of Eq. (28) to all $n \in \mathbb{Q}$, and finally, by considering the limit of real numbers as rational numbers and insisting on continuity, consistently define that the n^{th} power(s)[11] of e^{ξ} are found by invoking the validity of the deMoivre formula for all $n \in \mathbb{R}$.

Definition 4.5. *For any $\xi \in \mathbb{S} \setminus \{0\}, p \in \mathbb{R}$, we define ξ^p as a number of the form $e^{p\xi'}$, where $e^{\xi'} = \xi$. (For $\xi = 0$ we let $\xi^p = 0^p$.) For $p \in \mathbb{Z}$, this agrees with the earlier definition, since then $e^{p\xi'} = (e^{\xi'})^p = \xi^p$, by Eq. (28).*

This offers an elegant way to determine powers of a supercomplex number; for every ξ has a representation $e^{\xi'} = \xi$, as we shall see shortly – this equivalates the polar form of an ordinary complex number. As for ordinary complex numbers, however, ξ' is not unique, and in complete equivalence, this ambiguity allows for the computation of *different powers* or *roots* of a given $\xi \in \mathbb{S}$.

Theorem 4.6. *Every $\xi \in \mathbb{S} \setminus \{0\}$ has a polar decomposition, i.e. there exists a $\xi' \in \mathbb{S}$ so that $\xi = e^{\xi'}$. In fact, $\xi' \in \mathscr{C}(\xi)$ (if $\xi \notin \mathbb{R}$), and $\alpha_{\xi'} = \ln|\xi|$ whereas simultaneous solution of the equations*

$$\cos|\sigma_{\xi'}| = \frac{\alpha_\xi}{|\xi|},$$
$$\sin|\sigma_{\xi'}| = \pm\frac{|\sigma_\xi|}{|\xi|},$$

determines $|\sigma_{\xi'}| \geq 0$ up to a multiple of 2π; then $\sigma_{\xi'} = \mp\sigma_\xi \frac{|\sigma_{\xi'}|}{|\sigma_\xi|}$, unless $|\sigma_\xi| = 0$, in which case any purely imaginary $\sigma_{\xi'} \in \mathbb{S}$ satisfying the above equations may be chosen.

Proof. Writing out explicitly and solving the equation $e^{\alpha_{\xi'}+\sigma_{\xi'}} = |\xi|e^{\sigma_{\xi'}} = \xi$ yields the given determining equations, and since these always have solutions, the existence of a $\xi' \in \mathbb{S}$ with the desired property follows. □

Since we have now established the computation of both real powers of supercomplex numbers as well as the supercomplex exponential function, we extend Thm. 3.6:

[11] There may be several, as shall shortly be illustrated; for instance there are (at least) two 'square roots' of any $\xi \in \mathbb{S}$.

Theorem 4.7. *(Strong Theorem of Isomorphism.) Let $\mathscr{S}$ be a subspace of $\mathbb{S}$ with dimension $n+1 \in \mathbb{N}$ containing $\mathbb{R}$. Then $\mathscr{S}$ is isomorphic to $\mathbb{S}_n$ under all algebraic operations (the fundamental as well as exponentiation).*
For any $s \in \{1, \dots, n\}$ and purely imaginary $\sigma \in \mathscr{S}$ there exists an isomorphism $I : \mathscr{S} \to \mathbb{S}_n$ which fixes $\mathbb{R}$ and so that $I(\sigma) = |\sigma|\kappa_s$.

Proof. The proof of this is similar to that of the Weak Theorem – now, however, we furthermore need to check that $I(e^{\xi}) = e^{I(\xi)}$ and $I(\xi^p) = I(\xi)^p$; the former is clear from Eq. (25), and as for the latter, it is first of all worth noticing, that since ξ^p may be a collection of numbers rather than one unique number, the statement should be interpreted as follows: Each number of the form ξ^p has an image, $I(\xi^p)$, under I, which is a number of the form $I(\xi)^p$; conversely, each number of the form $I(\xi)^p$ is the image of some ξ^p under I. This however, is quite obvious; letting $e^{\xi'} = \xi$ (for $\xi \neq 0$), we have, using Def. 4.2, $I(\xi^p) = I(e^{p\xi'}) = e^{pI(\xi')} = (e^{I(\xi')})^p = I(\xi)^p$, since $e^{pI(\xi')}$ qualifies as $(e^{I(\xi')})^p$. Clearly any number of the form $I(\xi)^p$ can be written as $e^{pI(\xi')}$ for some ξ' (since I is bijective), and is thus the image of $e^{p\xi'}$.

□

5 Equations

5.1 The equation $\xi\phi = \chi$

In what is to follow, we shall need a concept similar to the one of eigennumbers.

Definition 5.1. *A number $\phi \in \mathbb{S}, \phi \neq 0$ is called an eigenconjugate of $\xi \in \mathbb{S}$ if for some $\tau_* \in \mathbb{R}$,*

$$\xi\phi = \tau_*\phi^*. \tag{29}$$

τ_* is called the corresponding eigenvalue.

Proposition 5.2. *For $\xi \in \mathbb{R} \setminus \{0\}$, the eigenconjugates of ξ are purely real and purely imaginary; the eigenvalues are ξ and $-\xi$, respectively.*

Theorem 5.3. *A number $\phi \in \mathbb{S}, \phi \neq 0$ is an eigenconjugate of $\xi \in \mathbb{S} \setminus \mathbb{R}$ with eigenvalue $|\xi|$ if and only if $\sigma_\phi = -\frac{\alpha_\phi}{|\xi|+\alpha_\xi}\sigma_\xi$, and an eigennumber with eigenvalue $-|\xi|$ if and only if $\alpha_\phi = \frac{\check{\sigma}_\xi \cdot \check{\sigma}_\phi}{|\xi|+\alpha_\xi}$.*

Proof. Assume ϕ is an eigenconjugate of $\xi \notin \mathbb{R}$; since $|\xi\phi| = |\tau_*\phi|$, obviously $\tau_* = \pm|\xi|$. Hence

$$\alpha_\xi\alpha_\phi - \check{\sigma}_\xi \cdot \check{\sigma}_\phi + |\xi|\sigma_\phi + \sigma_\xi(\alpha_\phi - \frac{\check{\sigma}_\xi \cdot \check{\sigma}_\phi}{|\xi| + \alpha_\xi}) = \pm|\xi|\alpha_\phi \mp |\xi|\sigma_\phi. \tag{30}$$

For $\tau_* = +|\xi|$ the equation for the imaginary part implies, that σ_ξ and σ_ϕ are parallel; therefore ξ and ϕ commute and the product $\xi\phi$ may be written as in Prop. 3.9. The imaginary part of Eq. (30) then becomes $\alpha_\xi\sigma_\phi + \alpha_\phi\sigma_\xi = -|\xi|\sigma_\phi$, or $\sigma_\phi = -\frac{\alpha_\phi}{|\xi|+\alpha_\xi}\sigma_\xi$ ($|\xi| + \alpha_\xi \neq 0$, since $\xi \notin \mathbb{R}$). On substitution into (30), this is confirmed to be also a sufficient condition for the equality $\xi\phi = |\xi|\phi^*$.
For $\tau_* = -|\xi|$, the equation for the real part gives $\alpha_\phi = \frac{\check{\sigma_\xi}\cdot\check{\sigma_\phi}}{|\xi|+\alpha_\xi}$, which is also easily realised to be sufficient. □

Definition 5.4. *By $\mathscr{E}^*_+(\xi), \mathscr{E}^*_-(\xi)$ we denote the sets of eigenconjugates of ξ, with eigenvalues $\pm|\xi|$, respectively; i.e. $\mathscr{E}^*_\pm(\xi) := \{\phi \in \mathbb{S} \setminus \{0\} \,|\, \xi\phi = \pm|\xi|\phi^*\}$.*

We speak of *two* different sets of eigenconjugates, whereas there is only *one* set of eigennumbers. From Thms. 2.13 and 5.3 we see that

Corollary 5.5. *For any $\xi \in \mathbb{S} \setminus \mathbb{R}$:*

$$\mathscr{E}(\xi) \subset \mathscr{E}^*_-(\xi). \tag{31}$$

We shall now try to solve for ξ the equation

$$\xi\phi = \chi, \tag{32}$$

given any $\phi, \chi \in \mathbb{S}, \phi, \chi \neq 0$; this equation is in principle very simple, and its treatment therefore essential – its solution is connected with the operation of right-division. We shall discover that for some ϕ, χ only one solution exists, for some there will be infinitely many, and for yet others there is *no* solution.
First, if the equation has a real solution, clearly it is given by $\xi = \phi^{-1}\chi$. Since any other (or further) solutions are non-real, we assume $\xi \notin \mathbb{R}$ from now on. Writing out each side of Eq. (32) yields

$$(\alpha_\xi\alpha_\phi - \check{\sigma_\xi}\cdot\check{\sigma_\phi})\check{1} + |\xi|\check{\sigma_\phi} + \check{\sigma_\xi}(\alpha_\phi - \frac{\check{\sigma_\xi}\cdot\check{\sigma_\phi}}{|\xi| + \alpha_\xi}) = \alpha_\chi\check{1} + \check{\sigma_\chi}; \tag{33}$$

computing the inner product of Eq. (33) with $\check{\sigma_\phi}$, using from the real part the equality $\check{\sigma_\xi}\cdot\check{\sigma_\phi} = \alpha_\xi\alpha_\phi - \alpha_\chi$, yields

$$|\xi||\sigma_\phi|^2 + (\alpha_\xi\alpha_\phi - \alpha_\chi)(\alpha_\phi - \frac{\alpha_\xi\alpha_\phi - \alpha_\chi}{|\xi| + \alpha_\xi}) = \check{\sigma_\phi}\cdot\check{\sigma_\chi}.$$

Multiplying by $|\xi| + \alpha_\xi$ and rearranging, we obtain

$$\alpha_\xi(|\phi||\chi| + Re(\phi\chi)) = {\alpha_\chi}^2 + |\xi|\check{\phi}\cdot\check{\chi} - |\xi|^2|\sigma_\phi|^2, \tag{34}$$

where we have used $\alpha_\phi\alpha_\chi - \check{\sigma_\phi}\cdot\check{\sigma_\chi} = Re(\phi\chi), \alpha_\phi\alpha_\chi + \check{\sigma_\phi}\cdot\check{\sigma_\chi} = \check{\phi}\cdot\check{\chi}$ and $|\chi| = |\xi||\phi|$.
Now, first consider the case $\phi\chi \in \mathbb{R}^-$; then, by Lem. 3.10, ϕ and χ commute. For $\alpha_\phi - \frac{\check{\sigma_\xi}\cdot\check{\sigma_\phi}}{|\xi|+\alpha_\xi} \neq 0$, Eq. (33) implies that σ_ξ is a linear combination of σ_ϕ, σ_χ, and since these are parallel, ξ

commutes with ϕ, demanding that the only possible solution to $\xi\phi(=\phi\xi)=\chi$ is $\xi=\phi^{-1}\chi$ – this is also sufficient, upon substitution into Eq. (32), since $\phi,\phi^{-1},\chi\in\mathscr{C}(\phi)$ (or $\mathscr{C}(\chi)$) and thereby obey the commutative and associative laws for multiplication. On the other hand, for $\alpha_\phi-\frac{\check{\sigma_\xi}\cdot\check{\sigma_\phi}}{|\xi|+\alpha_\xi}=0$, by Thm. 5.3, ϕ is an eigenconjugate of ξ. This, of course, can only be if $-|\chi|/|\phi|\,\phi^*=-|\xi|\phi^*=\chi$, but if that holds, indeed any ξ with $\phi\in\mathscr{E}^*_-(\xi)$ solves the equation.

Next, consider the case $\phi\chi\notin\mathbb{R}^-$; then $Re(\phi\chi)+|\phi||\chi|\neq 0$, and we obtain from Eq. (34) an explicit formula for α_ξ;[12]

$$\alpha_\xi=\frac{{\alpha_\chi}^2+\frac{|\chi|}{|\phi|}(\check{\phi}\cdot\check{\chi}-\frac{|\chi|}{|\phi|}|\sigma_\phi|^2)}{|\phi||\chi|+Re(\phi\chi)}. \tag{35}$$

From the imaginary part of (33), for $\alpha_\phi-\frac{\check{\sigma_\xi}\cdot\check{\sigma_\phi}}{|\xi|+\alpha_\xi}=\alpha_\phi-\frac{\alpha_\xi\alpha_\phi-\alpha_\chi}{|\xi|+\alpha_\xi}\neq 0$, we obtain

$$\check{\sigma_\xi}=\frac{\check{\sigma_\chi}-\frac{|\chi|}{|\phi|}\check{\sigma_\phi}}{\alpha_\phi-\frac{\alpha_\xi\alpha_\phi-\alpha_\chi}{|\xi|+\alpha_\xi}} \tag{36}$$

(using $|\xi|=|\chi|/|\phi|$), and it is seen that ξ as given by Eqs. (35) and (36) uniquely solves the equation $\xi\phi=\chi$. On the other hand, for $\alpha_\phi-\frac{\check{\sigma_\xi}\cdot\check{\sigma_\phi}}{|\xi|+\alpha_\xi}=0$, it is easily observed, as before, that we must have $\chi=\xi\phi=-|\xi|\phi^*$. Thus, if the condition $\chi=-|\chi|/|\phi|\,\phi^*$ is met, any $\xi\in\mathbb{S}$ with $\phi\in\mathscr{E}^*_-(\xi)$, solves the equation. However, if the condition is *not* met, the equation has no solutions (with $\alpha_\phi-\frac{\check{\sigma_\xi}\cdot\check{\sigma_\phi}}{|\xi|+\alpha_\xi}=0$). We sum up in a recipe:

Theorem 5.6. *The equation $\xi\phi=\chi$, $\phi,\chi\neq 0$ is solved as follows:*
(i) If a real solution exists (which is easily seen by inspection), it is $\xi=\phi^{-1}\chi$.
(ii) If $\phi\chi$ is real and negative, a solution is $\xi=\phi^{-1}\chi$. If furthermore $\chi=-\frac{|\chi|}{|\phi|}\phi^*$, any $\xi\in\mathbb{S}$ such that $\phi\in\mathscr{E}^*_-(\xi)$ solves the equation. Else;
(iii) Consider

$$\alpha_\xi:=\frac{{\alpha_\chi}^2+\frac{|\chi|}{|\phi|}(\check{\phi}\cdot\check{\chi}-\frac{|\chi|}{|\phi|}|\sigma_\phi|^2)}{|\phi||\chi|+Re(\phi\chi)};$$

if $\alpha_\phi-\frac{\alpha_\xi\alpha_\phi-\alpha_\chi}{|\xi|+\alpha_\xi}\neq 0$, a unique solution exists and its real part is α_ξ, whereas its imaginary part is

$$\sigma_\xi:=\frac{\sigma_\chi-\frac{|\chi|}{|\phi|}\sigma_\phi}{\alpha_\phi-\frac{\alpha_\xi\alpha_\phi-\alpha_\chi}{|\xi|+\alpha_\xi}}.$$

(iv) If on the other hand $\alpha_\phi-\frac{\alpha_\xi\alpha_\phi-\alpha_\chi}{|\xi|+\alpha_\xi}=0$, observe whether the condition $\chi=-\frac{|\chi|}{|\phi|}\phi^*$ is satisfied – if so, any $\xi\in\mathbb{S}$ with $\phi\in\mathscr{E}^*_-(\xi)$ solves the equation. Else, if $\chi\neq-\frac{|\chi|}{|\phi|}\phi^*$, there are no solutions to the equation.

[12]Note, of course, that this is only a *necessary* equality; it may be that α_ξ is computable, although no ξ exists, which solves the equation.

We should note, that if the equation has a real solution, ξ_0, then $\chi = \pm|\xi_0|\phi$ (since $\xi_0 = \pm|\xi_0|$) , and therefore all further (i.e. non-real) solutions must satisfy $\xi\phi = \pm|\xi_0|\phi = \pm|\xi|\phi$. If the real solution is positive, the equation therefore has, in addition, infinitely many non-real solutions, all captured in (*iv*) (remember $\mathscr{E}(\xi) \subset \mathscr{E}^*_-(\xi)$) – if the real solution is negative, it is the only one (by Thm. 2.13, $-|\xi|$ is not a possible eigenvalue for non-real ξ). Hence, from this and Thm. 5.6, the equation $\xi\phi = \chi$ always has either *one*, *none* or *infinitely many* solutions. This allows the following observation, which is a modified 'cancellation law' for supercomplex numbers:

Corollary 5.7. *Let $\mu, \nu, \xi \in \mathbb{S}, \xi \neq 0$, and let the relation $\mu\xi = \nu\xi$ be given. We may conclude that either*
(i) $\mu = \nu$ or
(ii) ξ is a simultaneous eigenconjugate of μ and ν with eigenvalue $-|\mu| = -|\nu|$.

Proof. Define $\chi := \mu\xi = \nu\xi$; the equation $\psi\xi = \chi$, for unknown ψ, has one, none or infinitely many solutions (if $\chi = 0$, so that Thm. 5.6 does not apply, we must have $\mu = \nu = 0$, since $\xi \neq 0$) – it cannot have none, since both μ and ν solve the equation. If there is only one solution it must be $\mu = \nu$, and if there are infinitely many they all necessarily satisfy $\mu\xi = -|\mu|\xi^* = \nu\xi = -|\nu|\xi^*$ (from Thm. 5.6, an infinitude of solutions could only occur from (*ii*) or (*iv*)), which is to say that ξ is simultaneously an eigenconjugate of μ and ν. □

5.1.1 The Binomial and Quadratic Equations

We shall now investigate other types of equations in supercomplex numbers; primarily the binomial equation, partly because it has some quite interesting features, partly because of its simplicity. It reads

$$\xi^n = p, \tag{37}$$

for some fixed $n \in \mathbb{N}, p \in \mathbb{S}$. We shall soon restrict p to being real, since then in fact the solutions are more interesting than the ones obtained for a general $p \in \mathbb{S}$. To exemplify this, we consider the case $n = 2$ and write $\xi = \alpha_\xi + \sigma_\xi$; since $\xi \mapsto \alpha_\xi + |\sigma_\xi|\kappa_1 \in \mathbb{S}_1$ is an isomorphism, we have the binomial expansion formula,

$$\xi^n = (\alpha_\xi + \sigma_\xi)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} {\alpha_\xi}^{n-k} {\sigma_\xi}^k. \tag{38}$$

It splits up into a real and imaginary part, noting the following expressions which easily follow from the isomorphism with $\mathbb{S}_1$:

$${\sigma_\xi}^k = \begin{cases} (-1)^{\frac{k-1}{2}} |\sigma_\xi|^{k-1}\sigma_\xi & \text{if } k \text{ is odd} \\ (-1)^{\frac{k}{2}} |\sigma_\xi|^k & \text{if } k \text{ is even} \end{cases}. \tag{39}$$

We shall not need the full generality of Eq. (38) in the following; for $n = 2$, though, we obtain $\xi^2 = {\alpha_\xi}^2 + 2\alpha_\xi\sigma_\xi + {\sigma_\xi}^2 = {\alpha_\xi}^2 - |\sigma_\xi|^2 + 2\alpha_\xi\sigma_\xi$, and the equation $\xi^2 = p$ reads

$$ {\alpha_\xi}^2 - |\sigma_\xi|^2 = \alpha_p, $$

$$ 2\alpha_\xi\sigma_\xi = \sigma_p; $$

for $p \notin \mathbb{R}$, this system is uniquely solved (as the reader may verify) by $\xi = \pm(\sqrt{\frac{\alpha_p+|p|}{2}} + \frac{\sigma_p}{\sqrt{2(\alpha_p+|p|)}})$. Not much unlike for real numbers, this quadratic equation has *two* solutions – those solutions might be called the square-roots of the number p. If, however, p is real, something quite different happens; solving the above equations for $\sigma_p = 0$ yields the solution $\xi = \pm\sqrt{p}$, for $p \geq 0$ and the solution $\alpha_\xi = 0, |\sigma_\xi| = \sqrt{-p}$ for $p < 0$. Again the former case is, as before, endowed with two solutions; the latter, however, is indeed distinct, being the entire collection of numbers satisfying $|\sigma_\xi|^2 = -p$. In $\mathbb{S}_2$, this collection forms a *circle* in the (κ_1, κ_2)-plane, centered at 0 with radius $\sqrt{-p}$, in $\mathbb{S}_3$ a *sphere* in the imaginary space, and so on; the appearance of the locus of solutions in $\mathbb{S}_n$ is simply the intersection of the locus of the solutions in $\mathbb{S}_{n+1}$ with $\mathbb{S}_n$ itself.

It is indeed clear why the solutions have this appearance – in $\mathbb{C}$ the solutions are simply $\xi = \pm\sqrt{-p}i$, but any plane through the real axis is isomorphic to $\mathbb{C}$; therefore, when p is real, there exist infinitely many planes containing solutions. Also, understanding this, we realise that the infinitude of solutions is not only characteristic for $n = 2$ but in fact for any n, as long as $p \in \mathbb{R}$.

Considering thus only $p \in \mathbb{R}$ we could proceed, by solving through the same procedure as above, the binomial equation for $n = 3, 4, \ldots$; in stead, however, we shall use a much more efficient method: Letting $\xi = |\xi|e^\sigma$ for some purely imaginary $\sigma \in \mathbb{S}$ (which is possible by Thm. 4.6 for $\xi \neq 0$, i.e. $p \neq 0$) the binomial equation becomes

$$ |\xi|^n (e^\sigma)^n = p; \tag{40} $$

equating the moduli of each side, this obviously implies $|\xi| = \sqrt[n]{|p|}$ and referring to Eq. (28), for $p > 0$, we must have $|\sigma| = \frac{2\pi m}{n}, m \in \mathbb{N}_0$, whereas $p < 0$ requires $|\sigma| = \frac{\pi+2\pi m}{n}, m \in \mathbb{N}_0$. We see that only the modulus of σ, and not σ itself, is restricted, and thus conclude that for $p > 0$ the solutions are summarised by

$$ \xi = \sqrt[n]{p}(\cos\frac{2\pi m}{n} + \sigma' \sin\frac{2\pi m}{n}), $$

where $m \in \{1, 2, \ldots, n\}, \sigma' \in \{\sigma \in \mathbb{S} \,|\, \alpha_\sigma = 0, |\sigma| = 1\}$, i.e. σ' runs over all purely imaginary numbers with unit modulus – we might of course restrict σ' to $\mathbb{S}_k$ for some $k \in \mathbb{N}$, if we wish. This exactly gives rise to circles, spheres, and so on. Similarly, using the same notation, for $p < 0$, the solutions are

$$ \xi = \sqrt[n]{-p}(\cos\frac{\pi + 2\pi m}{n} + \sigma' \sin\frac{\pi + 2\pi m}{n}). $$

We shall also briefly touch on the solution of the quadratic equation in ξ,

$$\xi^2 + \mu\xi + \nu = 0, \tag{41}$$

where $\xi, \mu, \nu \in \mathbb{S}$. In fact, this is only one of several possible distinct quadratic equations (the distinctions are due to the missing commutativity; another one would be $\xi^2\lambda + \xi\mu + \nu = 0$). We shall simply mention here the cases $\mu \in \mathbb{R}$ and $\mu \notin \mathbb{R}, \nu \in \mathbb{R}$. For the former case, $\mu \in \mathbb{R}$, the imaginary part of the equation reads

$$(2\alpha_\xi + \mu)\sigma_\xi + \sigma_\nu = 0,$$

and therefore $\mu, \nu, \xi \in \mathscr{C}(\xi)$; there exists an isomorphism $\mathscr{C}(\xi) \to \mathbb{S}_1$ and thereby we easily conclude (invoking the algebraic properties of ordinary complex numbers), that the solution is given by the well-known formula

$$\xi = -\frac{\mu}{2} \pm (\frac{\mu^2}{4} - \nu)^{\frac{1}{2}}, \tag{42}$$

following the convention implied by Def. 4.5 that $\psi^{\frac{1}{2}}$ is a number satisfying $(\psi^{\frac{1}{2}})^2 = \psi$; we solved this equation above (actually, this makes the $\pm$-sign redundant, but for good measure it has been included). Hence if $\mu^2/4 - \nu$ is real and negative there are infinitely many roots to be considered, otherwise only two.
Next, if $\mu \notin \mathbb{R}, \nu \in \mathbb{R}$, the imaginary part of the equation,

$$(2\alpha_\xi + |\mu|)\sigma_\xi + \sigma_\mu(\alpha_\xi - \frac{\check{\sigma_\xi} \cdot \check{\sigma_\mu}}{|\mu| + \alpha_\mu}) = 0, \tag{43}$$

implies that either σ_μ and σ_ξ are parallel, or else $\alpha_\xi - \frac{\check{\sigma_\xi} \cdot \check{\sigma_\mu}}{|\mu|+\alpha_\mu} = 0$. In case of the former, once again there exists an isomorphism with $\mathbb{S}_1$, and therefore some solutions of the equation are as in Eq. (42); if the latter is true, additional solutions could only be contained in $\mathscr{E}^*_-(\mu)$, by Thm. 5.3, and Eq. (43) implies $\alpha_\xi = -\frac{|\mu|}{2}$ (unless $\sigma_\xi = 0$ in which case $\xi = 0$, which is only a solution for $\nu = 0$). Furthermore, using $\mu\xi = -|\mu|\xi^*$, the real part of (41) now reads $(\frac{-|\mu|}{2})^2 - |\sigma_\xi|^2 - |\mu|(-\frac{|\mu|}{2}) + \alpha_\nu = 0$ or $|\sigma_\xi| = \sqrt{\frac{3}{4}|\mu|^2 + \alpha_\nu}$ – these additional solutions thus only exist for $\frac{3}{4}|\mu|^2 + \alpha_\nu \geq 0$.

6 Applications to Physics

Physics is deeply supported by mathematics and in its description of nature often makes use of vectorial relationships, because of their *independence of coordinates*. As we have seen, this property of coordinate-independence is also exhibited by the imaginary parts of supercomplex numbers – equations only depend explicitly on the vectors $\sigma_\xi, \sigma_\phi, \ldots$, not their components as referred to a given basis of imaginary units;[13] this mimics rotational invariance of ordinary

[13]The various bases are (of course) related by orthogonal transformations; this, as well as the coordinate-independence, is exactly the content of the Theorems of Isomorphism.

space. Since we (at least as far as everyday experience is concerned) live in a 3-dimensional world, it is plausible that $\mathbb{S}_3$ may be used to describe some physical relations;[14] we shall present here two different applications. The first has to do with a connection to 'double' vector cross products – the second is a deeper relation to special relativity.
Let us first note that two purely imaginary numbers $\sigma, \zeta \in \mathbb{S}$ (naturally identified with ordinary physical vectors) have a particularly simple product:

$$\sigma\zeta = -\check{\sigma} \cdot \check{\zeta} + |\sigma|\zeta - \sigma \frac{\check{\sigma} \cdot \check{\zeta}}{|\sigma|}. \tag{44}$$

For vectors with three components, we have the identity $|\sigma|^2\check{\zeta} - \check{\sigma}(\check{\sigma} \cdot \check{\zeta}) = \check{\sigma} \times (\check{\zeta} \times \check{\sigma})$; if $\sigma, \zeta \in \mathbb{S}_3$, they have *four* components, but since we take both to be purely imaginary, we may as well use the $\times$-symbol to indicate the same operation performed only on the three lower components, whereas the first, 0, is simply carried along. Therefore, for $\sigma, \zeta \in \mathbb{S}_3$, we may write

$$(\sigma\zeta)\check{} = -(\check{\sigma} \cdot \check{\zeta})\check{1} + \frac{1}{|\sigma|}\check{\sigma} \times (\check{\zeta} \times \check{\sigma}). \tag{45}$$

Such a double cross product occurs when considering the angular momentum, $\vec{L}$, of a continuous mass distribution rotating about an axis through a fixed point, P; we have

$$\vec{L} = \int \vec{r} \times (\vec{\omega} \times \vec{r})\rho \, \mathrm{d}V,$$

where $\vec{\omega}$ is the angular velocity, $\vec{r}$ the position vector (relative to P) and ρ the density of mass; the integral is to be performed over the entire volume of the body. The expression suggests that we identify $\vec{r} \times (\vec{\omega} \times \vec{r})\rho$ with a 'density of angular momentum', $\vec{\mathscr{L}}$. Writing $r = x\kappa_1 + y\kappa_2 + z\kappa_3$, $\omega = \omega_x\kappa_1 + \omega_y\kappa_2 + \omega_z\kappa_3$ we may define

$$\mathscr{L} := \rho|r|r\omega; \tag{46}$$

the imaginary part of this quantity is now $\mathscr{L}_x\kappa_1 + \mathscr{L}_y\kappa_2 + \mathscr{L}_z\kappa_3$, using Eq. (45). Thus, using supercomplex numbers allows for a compact notation – but perhaps nothing more. It is natural to investigate whether the *real* part of $\mathscr{L}$ carries any physical significance; it is $-\rho|r|\vec{r} \cdot \vec{\omega}$. This quantity might (based on inspection) represent some sort of (a)symmetry in the rotation, but the apparent lack of $\int Re(\mathscr{L})\mathrm{d}V$ to exhibit interpretable transformations under changes of coordinates (e.g. spatial translations) suggests that it might be more artificial than genuine.

The question arises, however, what physical meaning we should generally ascribe to the real part of a supercomplex number, if the imaginary part could be identified with a vector in ordinary space; this brings to our main and much more elucidating application. In view

[14]Of course, for theories proposing the existence of e.g. 10 spatial dimensions, one might instead consider $\mathbb{S}_{10}$.

of Einstein's theory of relativity a qualified proposal is to identify the real part with a time-like part of the vector in consideration; in fact, if the supercomplex numbers are to describe *anything* physical, an obvious candidate is relativistic phenomena. And indeed, an examination of the core of special relativity – the Lorentz transformations – reveals an utterly exciting connection. Letting π denote a generic four-vector, it transforms under a Lorentz transformation according to $\pi' = \Lambda\pi$, where

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+\gamma^2\frac{{\beta_x}^2}{1+\gamma} & \gamma^2\frac{\beta_x\beta_y}{1+\gamma} & \gamma^2\frac{\beta_x\beta_z}{1+\gamma} \\ -\gamma\beta_y & \gamma^2\frac{\beta_y\beta_x}{1+\gamma} & 1+\gamma^2\frac{{\beta_y}^2}{1+\gamma} & \gamma^2\frac{\beta_y\beta_z}{1+\gamma} \\ -\gamma\beta_z & \gamma^2\frac{\beta_z\beta_x}{1+\gamma} & \gamma^2\frac{\beta_z\beta_y}{1+\gamma} & 1+\gamma^2\frac{{\beta_z}^2}{1+\gamma} \end{pmatrix}. \tag{47}$$

Here $\beta_i := \frac{v_i}{c}, \beta := \sqrt{{\beta_x}^2 + {\beta_y}^2 + {\beta_z}^2}, \gamma := \frac{1}{\sqrt{1-\beta^2}}$, as is common notation.[15] The resemblance to Eq. (8) is striking. The only significant difference seems to be signs; Λ is completely symmetric whereas $\tilde{\xi}$ is not – there is, however, a way in which we can go about this: If we let the imaginary components of ξ in Eq. (8) *themselves* be imaginary, and equally so the imaginary components of the multiplicand (the four-vector π), the problem of the missing symmetry is resolved. More formally, we modify $\mathbb{S}$ so that it becomes a vector space over $\mathbb{C}$, cf. Def. 2.1; observe namely, that if

$$\tilde{\Lambda} = \begin{pmatrix} \gamma & i\gamma\beta_x & i\gamma\beta_y & i\gamma\beta_z \\ -i\gamma\beta_x & 1+\gamma^2\frac{{\beta_x}^2}{1+\gamma} & \gamma^2\frac{\beta_x\beta_y}{1+\gamma} & \gamma^2\frac{\beta_x\beta_z}{1+\gamma} \\ -i\gamma\beta_y & \gamma^2\frac{\beta_y\beta_x}{1+\gamma} & 1+\gamma^2\frac{{\beta_y}^2}{1+\gamma} & \gamma^2\frac{\beta_y\beta_z}{1+\gamma} \\ -i\gamma\beta_z & \gamma^2\frac{\beta_z\beta_x}{1+\gamma} & \gamma^2\frac{\beta_z\beta_y}{1+\gamma} & 1+\gamma^2\frac{{\beta_z}^2}{1+\gamma} \end{pmatrix},$$

and $\breve{\pi} = (\pi_0, i\pi_1, i\pi_2, i\pi_3)$, then its components change under a Lorentz transformation according to $\breve{\pi}' = \tilde{\Lambda}\breve{\pi}$. The imaginary unit, i, does *not* interact with the imaginary units $\kappa_n, n \geq 1$; it simply goes along throughout the calculations as for instance would a physical unit. Writing $\pi = \pi_0 + (i\pi_1)\kappa_1 + (i\pi_2)\kappa_2 + (i\pi_3)\kappa_3$ and $\Lambda = \gamma - (i\gamma\beta_x)\kappa_1 - (i\gamma\beta_y)\kappa_2 - (i\gamma\beta_z)\kappa_3$, the transformed four-vector is found from the product $\Lambda\pi$; note that $|\Lambda| = \sqrt{\gamma^2 - \gamma^2({\beta_x}^2 + {\beta_y}^2 + {\beta_z}^2)} = 1$, and therefore the modulus $|\pi| = \sqrt{{\pi_0}^2 - {\pi_1}^2 - {\pi_2}^2 - {\pi_3}^2}$ is Lorentz invariant, by Prop. 2.9.[16] It should be clear that the extension over $\mathbb{C}$ has no longer much to do with supercomplex numbers; these 'double-complex' numbers seem (inherently) even less real than imaginary numbers. But they allow for neat notation – and, perhaps more importantly, they reveal an intimate connection between the product of supercomplex numbers and the Lorentz transformations. The statement that the product of

[15]The reader not acquainted with the Lorentz transformations and the ideas of special relativity, may find it interesting to consult a text on the subject; an all-round overview of concepts and formulae in this section is given in [3].

[16]One must be aware, that after letting the components themselves be ordinarily complex, the obtained results may not be unaffected; most of them are, however, and this is one of them.

supercomplex numbers does not distribute over addition, relates to the statement that the sum of two Lorentz transformations is not itself a Lorentz transformation – in fact it is not really anything physically sensible. Similarly, the statement that the product is not associative, relates to the fact that the composition of two Lorentz transformations is generally not a single Lorentz transformation – or equivalently, that the Lorentz transformations do not form a *group* under composition.[17]

Even though the connection therefore rather enlightens us about structural similarities than provides efficient methods of computation, there is one last step in making the expression $\pi' = (\gamma - (i\gamma\beta_x)\kappa_1 - (i\gamma\beta_y)\kappa_2 - (i\gamma\beta_z)\kappa_3)\pi$ even more elegant. We may write $\gamma = \cosh(\phi), \gamma\beta = \sinh(\phi)$, where $\phi \geq 0$ (defined through the relations just given) is the so-called rapidity. If we furthermore let $\rho = (in_x)\kappa_1 + (in_y)\kappa_2 + (in_z)\kappa_3$ where (n_x, n_y, n_z) is a unit vector in the direction of motion, we have, since $\beta_i = \beta n_i$,

$$\Lambda = \gamma - (i\gamma\beta_x)\kappa_1 - (i\gamma\beta_y)\kappa_2 - (i\gamma\beta_z)\kappa_3 = \cosh(\phi) - \rho\sinh(\phi).$$

Now we observe that $|\rho|^2 = -1$, or $|\rho| = \varepsilon i$, where $\varepsilon = \pm 1$; furthermore, invoking

$$\cosh(\phi) = \cos(i\phi) = \cos(\varepsilon i\phi),$$
$$\sinh(\phi) = -i\sin(i\phi) = -\varepsilon i\sin(\varepsilon i\phi),$$

we have $\Lambda = \cos(\varepsilon i\phi) + \rho\varepsilon i\sin(\varepsilon i\phi) = \cos(|\rho|\phi) - \frac{\rho}{|\rho|}\sin(|\rho|\phi) = \cos(|\rho\phi|) - \frac{\rho\phi}{|\rho\phi|}\sin(|\rho\phi|)$, for $\phi \neq 0$. This finally, through Eq. (25), allows for the 'identification' $\Lambda = e^{-\rho\phi}$ (which also holds for $\phi = 0$), and so we conclude, that given ρ, ϕ as quantities describing the relative motion of one inertial system to another, a four-vector, π, transforms according to the simple law

$$\pi' = e^{-\rho\phi}\pi.$$

7 Generalising and Abstracting

The algebra of the supercomplex numbers is more or less based on the idea of vectorlike addition and subtraction together with the principle, that the product of two supercomplex numbers is a purely geometric construction, i.e. 'coordinate-independent'; as mentioned earlier, the latter is captured in the Theorems of Isomorphism. We might ask, therefore, if there are other algebraic structures exhibiting the same properties, and thus seek the most general extension of the complex numbers in consistence with these thoughts. Since all further algebraic operations stem originally from the simple ones of adding (subtracting) and multiplying, only the latter shall concern us in the following.

[17] What is 'missing', is ordinary spatial rotations, corresponding – in the language of supercomplex numbers – to transformations changing from one basis of imaginary units to another.

It seems apparent that addition (and subtraction) as already defined on $\mathbb{S}$ is a natural generalisation; in fact, we shall want to maintain the entire vector space structure of $\mathbb{S}$ and any subspaces under consideration.[18] (Thus, for scalars $\lambda_1, \lambda_2 \in \mathbb{R}$ and ξ, ϕ elements of a subspace $\mathscr{S}$ of $\mathbb{S}$, we let $\lambda_1\xi + \lambda_2\phi \in \mathscr{S}$ denote the quantity defined by $(\lambda_1\xi + \lambda_2\phi)\check{} = \lambda_1\check{\xi} + \lambda_2\check{\phi}$.) The identification of $\mathbb{R} + \mathbb{R}\kappa_1$ with the complex plane is a necessity if we are to consider an algebraic system as an extension of the complex numbers; furthermore it is desirable that a product distributes over sums, or even better, is linear in the multiplicand – hence we shall impose that condition too. We did however make a debatable choice in specifically defining a product on $\mathbb{S}$ by Eq. (6); therefore, we shall now endow our system with a 'generalised product', $\circ$, replacing the old, as given by (6). [19] In order to properly formalise the above ideas, we introduce the following terminology:

Definition 7.1. *Let $\mathscr{S}$ be a subspace of $\mathbb{S}$ equipped with a binary operation, $(\xi, \phi) \mapsto \xi \circ \phi \in \mathscr{S}$; a subspace $\mathscr{A} \subseteq \mathscr{S}$ is said to be closed under $\circ$, if $\xi \circ \phi \in \mathscr{A}$ for all $\xi, \phi \in \mathscr{A}$. If in addition $T : \mathscr{S} \to \mathscr{S}$ is a linear mapping, we say that $\mathscr{A}$ is stable under T, if T is a homomorphism from $(\mathscr{A}, \circ)$ to $(T(\mathscr{A}), \circ)$, i.e. if*

$$T(\xi \circ \phi) = T(\xi) \circ T(\phi),$$

for all $\xi, \phi \in \mathscr{A}$.

Definition 7.2. *Given $\mathbb{S}_m, m \in \mathbb{N}$, a special orthogonal transformation that fixes $\mathbb{R}(= \mathbb{S}_0)$, is a mapping $T : \mathbb{S}_m \to \mathbb{S}_m$ which is linear, i.e.*

$$T(\lambda_1\xi + \lambda_2\phi) = \lambda_1 T(\xi) + \lambda_2 T(\phi),$$

for all $\lambda_1, \lambda_2 \in \mathbb{R}, \xi, \phi \in \mathbb{S}_m$, with $T(\alpha) = \alpha$ for all $\alpha \in \mathbb{R}(= \mathbb{S}_0)$, and such that the $(m+1) \times (m+1)$ matrix with elements $M_{i,j} := T(\kappa_i)\check{} \cdot \check{\kappa}_j$, $i, j \in \{0, \dots, m\}$ is orthogonal with determinant $+1$.

If $T : \mathbb{S} \to \mathbb{S}$ is a mapping we call it a special orthogonal transformation that fixes $\mathbb{R}$, if it is linear and if there exists $m \in \mathbb{N}$ such that the restriction of T to $\mathbb{S}_m$, $T|_{\mathbb{S}_m}$, is an $\mathbb{R}$-fixing special orthogonal transformation in the above sense, whereas the restriction to the orthogonal complement, $\mathbb{S}_m^\perp$ (in the sense of Def. 2.4), is the identity map, i.e. $T|_{\mathbb{S}_m^\perp} = \mathrm{id}_{\mathbb{S}_m^\perp}$.

Note that $M_{i,j}$ is the matrix representing $T|_{\mathbb{S}_m}$, and by invertibility of orthogonal matrices, a special orthogonal transformation fixing $\mathbb{R}$ is bijective.

Now we can express explicitly what we mean by the statement about the product of supercomplex numbers as a geometric construction: $\mathbb{S}$ is stable under any special orthogonal

[18] This allows us to speak of subspaces at all.

[19] For $\lambda \in \mathbb{R}$, we shall still want $\lambda \circ \xi$ (thinking of λ as an element of $\mathbb{S}_0$) to coincide with $\lambda\xi$ (thinking of λ as a scalar); that this indeed turns out to follow from the presented definitions, shall be remarked soon enough. Until then we will distinguish between the two expressions.

transformation $T : \mathbb{S} \to \mathbb{S}$ that fixes $\mathbb{R}$ – this is an immediate consequence of the linearity of T and Eq. (21), which is seen to hold for arbitrary orthogonal $\mathbb{R}$-fixing transformations.[20] Hence we can now state the definition of an algebraic system with the desired properties:

Definition 7.3. *(Generalised Supercomplex Algebra.) Let $\mathscr{S} = \mathbb{S}$ or $\mathscr{S} = \mathbb{S}_k, k \in \mathbb{N}$, and let $\mathscr{S}$ be equipped with a binary operation, $(\xi, \phi) \mapsto \xi \circ \phi \in \mathscr{S}$, called a generalised product. We say that $(\mathscr{S}, \pm, \circ)$ forms a generalised supercomplex algebra (GSA from now on), if for all $\xi, \phi, \chi \in \mathscr{S}$, $a_i, b_i \in \mathbb{R}$, the following holds:*

(i) $(a_1 + b_1\kappa_1) \circ (a_2 + b_2\kappa_1) = a_1a_2 - b_1b_2 + (a_1b_2 + a_2b_1)\kappa_1$.
(ii) $\xi \circ (a_1\phi + a_2\chi) = a_1(\xi \circ \phi) + a_2(\xi \circ \chi)$.
(iii) $\mathscr{S}$ is stable under any special orthogonal transformation $T : \mathscr{S} \to \mathscr{S}$ that fixes $\mathbb{R}$.

We have the following generalisation of Thm. 3.6:

Theorem 7.4. *Let $(\mathscr{S}, \pm, \circ)$ be a GSA and let $\mathscr{A}$ be a subspace of $\mathscr{S}$ with dimension $n + 1 \in \mathbb{N}$ containing $\mathbb{R}$. Then there exists a linear automorphism $I : (\mathscr{S}, \pm, \circ) \to (\mathscr{S}, \pm, \circ)$ which fixes $\mathbb{R}$ and with $I(\mathscr{A}) = \mathbb{S}_n$. In particular, $\mathscr{A}$ is closed if and only if $\mathbb{S}_n$ is closed under $\circ$, and in that case, I regarded as a mapping $\mathscr{A} \to \mathbb{S}_n$ is an isomorphism.*
For any $s \in \{1, \dots, n\}$ and purely imaginary $\sigma \in \mathscr{A}$, I may be chosen so that $I(\sigma) = |\sigma|\kappa_s$.

Proof. Since $\mathscr{A}$ contains $\mathbb{R}$, a basis may be written as $(1, \sigma_1, \dots, \sigma_n)$ for some purely imaginary $\sigma_1, \dots, \sigma_n$ (if $n = 0$, we have $\mathscr{A} = \mathbb{R}$ and choose $I = \mathrm{id}_{\mathbb{R}}$). For $n \geq 1$ we now proceed almost as in the proof of Thm. 3.6; given $s \in \{1, \dots, n\}$ and a purely imaginary $\sigma \in \mathscr{A}$, we may form an orthonormal basis for $\mathscr{A}$, say $\mathscr{B} := (1, \sigma'_1, \sigma'_2, \dots, \sigma'_n)$, with $\sigma'_s = \sigma/|\sigma|$. We choose $m \geq n$ such that $\mathscr{A} \subseteq \mathbb{S}_m \subseteq \mathscr{S}$ (this is possible, since either $\mathscr{S} = \mathbb{S}$ or $\mathscr{S} = \mathbb{S}_k$ for some $k \geq n$); then we consider an orthonormal basis for $\mathbb{S}_m$, $\mathscr{B}_m := (1, \sigma'_1, \sigma'_2, \dots, \sigma'_n, \dots, \sigma'_m)$, which is an extension of $\mathscr{B}$. For $m = 1$ we have $\mathscr{A} \subseteq \mathbb{S}_1$, which by closure of $\mathscr{A}$ means $\mathscr{A} = \mathbb{S}_1$ (since $\mathscr{A} \neq \mathbb{R}$, since $n \geq 1$); $\sigma \in \mathscr{A}$ must be of the form $\lambda\kappa_1$, and we can choose for I the bijection $\alpha + \beta\kappa_1 \mapsto \alpha + \mathrm{sign}(\lambda)\beta\kappa_1$. For $m > 1$, we can choose an $i \in \{1, \dots, m\}, i \neq s$, construct the orthogonal $(m + 1) \times (m + 1)$ matrix $M := (\check{1}, \check{\sigma'_1}, \dots, \pm\check{\sigma'_i}, \dots, \check{\sigma'_m})^T$, and choose the $\pm$-sign so as to make $\det M = +1$. Then we define a special orthogonal transformation $I : \mathscr{S} \to \mathscr{S}$ by $I(\xi)\check{} = M\check{\xi}$ for $\xi \in \mathbb{S}_m$ and $I(\xi) = \xi$ for $\xi \in \mathbb{S}_m^\perp \subseteq \mathscr{S}$ (this defines I completely, since it is linear and any $\xi \in \mathscr{S}$ has a decomposition into $\mathbb{S}_m, \mathbb{S}_m^\perp$); we note that I fixes $\mathbb{R}$. Hence by *(iii)*, I is a linear homomorphism under $\circ$ (and $\pm$ because of linearity); moreover, I is bijective and $I(\sigma) = I(|\sigma|\sigma'_s) = |\sigma|I(\sigma'_s) = |\sigma|\kappa_s$. Therefore I is in fact an *automorphism* with the desired properties. We easily confirm that $I(\mathscr{A}) = \mathbb{S}_n$, since $I(\sigma'_i) = \kappa_i$.
Any two elements of $\mathbb{S}_n$ can be written as $I(\xi), I(\phi)$ for some $\xi, \phi \in \mathscr{A}$, and if $\mathscr{A}$ is closed under $\circ$, we have $I(\xi) \circ I(\phi) = I(\xi \circ \phi) \in I(\mathscr{A}) = \mathbb{S}_n$, i.e. $\mathbb{S}_n$ is closed too. To see that closure

[20] Note, that it need not be special (i.e. with $\det M = +1$ as opposed to -1); generally, however, the less restricting requirement that the space be stable only under special transformations captures more properly the idea of a product as a geometric construction.

of $\mathbb{S}_n$ implies closure of $\mathscr{A}$, we simply note that I^{-1} is also a special orthogonal transformation that fixes $\mathbb{R}$, and use a similar argument. Obviously I is then an isomorphism between $\mathscr{A}$ and $\mathbb{S}_n$.

□

In particular, from the case $n = 1$, we learn that any 2-dimensional subspace of $\mathscr{S}$ containing $\mathbb{R}$ is closed under $\circ$ and isomorphic to $\mathbb{S}_1$ (since $\mathbb{S}_1$ is closed by property (i)) and thus an algebraic replica of the complex plane. This ensures coincidence between $\lambda \circ \xi$ and $\lambda\xi$ for all $\lambda \in \mathbb{R}, \xi \in \mathscr{S}$ (note that, by (i), $\lambda \circ \kappa_1 = \lambda\kappa_1$ for all $\lambda \in \mathbb{R}$), and we need therefore no more distinguish between these expressions. Moreover, we can always endow a GSA with the operation of complex conjugation, $(\alpha_\xi + \sigma_\xi)^* := \alpha_\xi - \sigma_\xi$, and be sure that $\xi\xi^* = \xi^*\xi = |\xi|^2$ for all $\xi \in \mathscr{S}$.

We obviously have the following:

Proposition 7.5. *The only GSA with exactly one imaginary unit (i.e. with $\mathscr{S} \subseteq \mathbb{S}_1$), is the complex numbers (under the identification $\kappa_1 = i$).*

We easily confirm directly from the definition, that $(\mathbb{S}, \pm, \cdot)$, $\cdot$ being the product defined by Eq. (6), does indeed form a GSA; what is interesting, however, is which other constructions we can find – the following result is quite helpful.

Theorem 7.6. *Let $(\mathscr{S}, \pm, \circ)$ constitute a GSA. If it contains two or more imaginary units (i.e. if $\mathbb{S}_2 \subseteq \mathscr{S}$), it is completely specified once the product $(\alpha + \beta\kappa_1) \circ \kappa_2$ is given for all $\alpha \in \mathbb{R}, \beta \in \mathbb{R}^+$. Its most general form is*

$$(\alpha + \beta\kappa_1) \circ \kappa_2 = \gamma_0 + \gamma_1\kappa_1 + \gamma_2\kappa_2 + \gamma_3\kappa_3,$$

where of course, implicitly, γ_i are functions of α and β, and we must have $\gamma_2(\alpha, 0) = \alpha$ and $\gamma_i(\alpha, 0) = 0$ for $i \neq 2$. If $\mathscr{S} = \mathbb{S}_2$ we must have $\gamma_3 \equiv 0$, if $\mathscr{S} = \mathbb{S}_3$ it is necessary that $\gamma_0 = \gamma_1 \equiv 0$, and if $\mathbb{S}_4 \subseteq \mathscr{S}$, $\gamma_0 = \gamma_1 = \gamma_3 \equiv 0$.[21]

If $\mathbb{S}_3 \subseteq \mathscr{S}$, the GSA is normed, i.e. $|\xi \circ \phi| = |\xi||\phi|$ (in the sense of Def. 2.3) for all $\xi, \phi \in \mathscr{S}$, if and only if γ_0, γ_1 vanish identically and ${\gamma_2}^2 + {\gamma_3}^2 \equiv \alpha^2 + \beta^2$.

Proof. To specify $\xi \circ \phi$ for general $\xi, \phi \in \mathscr{S}$, we need only, by (ii), specify $\xi \circ \kappa_i$ for all $\kappa_i \in \mathscr{S}$; given κ_i, let $\mathscr{A}$ be a subspace of dimension $n + 1 \in \mathbb{N}$ containing $\mathbb{R}$, with $\xi, \kappa_i, \xi \circ \kappa_i \in \mathscr{A}$ (e.g. the span of these three entities and the real axis). By Thm. 7.4 there exists a linear $\mathbb{R}$-fixing automorphism $I : \mathscr{S} \to \mathscr{S}$ with $I(\mathscr{A}) = \mathbb{S}_n$ and $I(\xi) = I(\alpha_\xi + \sigma_\xi) = I(\alpha_\xi) + I(\sigma_\xi) = \alpha_\xi + |\sigma_\xi|\kappa_1$; considering I as a mapping $\mathscr{A} \to \mathbb{S}_n$ and denoting its inverse by I^{-1}, we have

$$\xi \circ \kappa_i = I^{-1}(I(\xi) \circ I(\kappa_i))) = I^{-1}((\alpha_\xi + |\sigma_\xi|\kappa_1) \circ I(\kappa_i)).$$

[21] We take the $\equiv$-symbol to mean ‘identically (i.e. for all α, β) equal to’.

Now, $I(\kappa_i) = \eta_0\kappa_0 + \eta_1\kappa_1 + \ldots + \eta_m\kappa_m$, so (by (ii), again) this product is determined by specifying $(\alpha_\xi + |\sigma_\xi|\kappa_1) \circ \kappa_j$, for all j; for $j = 0, 1$ it is given by (i), and if specified for $j = 2$, it is follows for any other possible $j > 2$, since the special orthogonal transformation defined by $T(\kappa_2) = \kappa_j, T(\kappa_j) = -\kappa_2, T(\kappa_s) = \kappa_s, s \neq 2, j$, is an automorphism $\mathscr{S} \to \mathscr{S}$ (by (iii) and bijectivity of T), and since then

$$(\alpha_\xi + |\sigma_\xi|\kappa_1) \circ \kappa_j = T((\alpha_\xi + |\sigma_\xi|\kappa_1) \circ \kappa_2).$$

Obviously $|\sigma_\xi| \in \mathbb{R}^+ \cup \{0\}$, showing the sufficiency of specifying $(\alpha + \beta\kappa_1) \circ \kappa_2$ only for $\alpha \in \mathbb{R}, \beta \in \mathbb{R}^+$ (for $\beta = 0$, $\circ$ coincides with scalar multiplication).

Now we examine how general $(\alpha + \beta\kappa_1) \circ \kappa_2$ can be. For $\mathscr{S} = \mathbb{S}_2$, the most general form must be $\gamma_0 + \gamma_1\kappa_1 + \gamma_2\kappa_2$; for $\mathbb{S}_3 \subseteq \mathscr{S}$, consider the special orthogonal transformation $\mathscr{S} \to \mathscr{S}$ defined by $T'(\kappa_2) = -\kappa_2, T'(\kappa_3) = -\kappa_3, T'(\kappa_s) = \kappa_s, s \neq 2, 3$; this is again an automorphism, and therefore

$$(\alpha + \beta\kappa_1) \circ \kappa_2 = -T'((\alpha + \beta\kappa_1) \circ \kappa_2).$$

Letting $\gamma_s, s \in \mathbb{N}_0$ denote the coefficient in front of κ_s in the expression for $(\alpha + \beta\kappa_1) \circ \kappa_2$, this implies $\gamma_s \equiv -\gamma_s$ for $s \neq 2, 3$, i.e. $\gamma_s \equiv 0$. For three imaginary components, the product thus takes the form $\gamma_2\kappa_2 + \gamma_3\kappa_3$; for strictly more than three imaginary components we can use the same technique (with e.g. κ_2 and κ_4) to show that also $\gamma_3 \equiv 0$, leaving the most general form as $\gamma_2\kappa_2$. Clearly $\gamma_2(\alpha, 0) = \alpha$, and $\gamma_i(\alpha, 0) = 0$ for $i \neq 2$, by coincidence of $\circ$ with scalar multiplication.

Finally, let the GSA be normed and equipped with at least three imaginary units; choosing $\xi := \alpha + \beta\kappa_1 \in \mathscr{S}$ we then have $|\xi \circ \kappa_2 + \xi \circ \kappa_3| = |\xi \circ (\kappa_2 + \kappa_3)| = \sqrt{\alpha^2 + \beta^2}\sqrt{2}$, whereas $|\xi \circ \kappa_2| = |\xi \circ \kappa_3| = \sqrt{\alpha^2 + \beta^2}$ and thus $(\xi \circ \kappa_2)\check{} \cdot (\xi \circ \kappa_3)\check{} = 0$ (inverse Pythagorean Theorem). The linear map $T'' : \mathscr{S} \to \mathscr{S}$ defined by $T''(\kappa_2) = \kappa_3, T''(\kappa_3) = -\kappa_2, T''(\kappa_s) = \kappa_s, s \neq 2, 3$ is an automorphism, and we deduce that

$$(\alpha + \beta\kappa_1) \circ \kappa_3 = \gamma_0 + \gamma_1\kappa_1 - \gamma_3\kappa_2 + \gamma_2\kappa_3,$$

(the terms $\gamma_1, \gamma_2, \gamma_3$ may be present or not), and computing $(\xi \circ \kappa_2)\check{} \cdot (\xi \circ \kappa_3)\check{}$ the statement becomes ${\gamma_0}^2 + {\gamma_1}^2 \equiv 0$, or $\gamma_0 = \gamma_1 \equiv 0$. Hence we also obtain ${\gamma_2}^2 + {\gamma_3}^2 \equiv |(\alpha + \beta\kappa_1) \circ \kappa_2|^2 = \alpha^2 + \beta^2$.
Conversely, let $\gamma_0 = \gamma_1 \equiv 0$ and ${\gamma_2}^2 + {\gamma_3}^2 \equiv \alpha^2 + \beta^2$; the first condition implies, using automorphisms similar to the above, that $\xi \circ \kappa_i$ is orthogonal to $\xi \circ \kappa_j$, for $i \neq j$. Likewise, the second one implies $|(\alpha + \beta\kappa_1) \circ (\eta_i\kappa_i)| = |\alpha + \beta\kappa_1||\eta_i\kappa_i|$ for any i; combining these observations yields (Pythagorean Theorem) $|(\alpha + \beta\kappa_1) \circ \phi| = |\alpha + \beta\kappa_1||\phi|$ for all $\phi := \sum_{n=0}^{\infty} \eta_n\kappa_n \in \mathscr{S}$, whereby $|\xi \circ \phi| = |\xi||\phi|$ for all $\xi, \phi \in \mathscr{S}$, by use of the transformation I from above (with $I(\xi) = \alpha_\xi + |\sigma_\xi|\kappa_1$), its inverse I^{-1}, and the preservation of moduli by orthogonal transformations. □

Now we realise that the supercomplex numbers (i.e. $\mathbb{S}$ with the product from Eq. (6)) form essentially the only normed GSA with more than three imaginary units; there are others, most particularly perhaps the one generated by choosing $\gamma_2(\alpha,\beta) = -\sqrt{\alpha^2+\beta^2}$ (for $\beta \neq 0$) in stead of $+\sqrt{\alpha^2+\beta^2}$ (geometrically this corresponds to also reflecting orthogonal directions in the plane spanned by the multiplicator and $\mathbb{R}$)– but the difference is almost only one of convention. We might construct very discontinuous products by switching between the signs, depending on the value of α and β, but the possible advantage is not apparent. However, we now understand that it is *impossible* to avoid a discontinuity (in the case of supercomplex numbers, the one occurring when the multiplicator becomes negative), since we must have $|\gamma_2(\alpha,\beta)| = \sqrt{\alpha^2+\beta^2}$ and $\gamma_2(\alpha,0) = \alpha$; running through the unit circle $\alpha^2+\beta^2 = 1$ we hence have $|\gamma_2| \equiv 1$ whereas $\gamma_2(1,0) = 1$ and $\gamma_2(-1,0) = -1$.

With three imaginary units, though, we are only restricted by $\gamma_2(\alpha,\beta)^2 + \gamma_3(\alpha,\beta)^2 = \alpha^2+\beta^2$. Here we may try to impose right-distributivity of the product; then the functions γ_i are linear[22], and we easily find that there exist only two systems satisfying this, namely the ones with $\gamma_2(\alpha,\beta) = \alpha, \gamma_3(\alpha,\beta) = \pm\beta$. The one obtained through the +-sign is more commonly known as the *quaternions* and the generalised product $\circ$ is then the 'quaternion product' defined by Hamilton;[23] the other, of course, corresponds to setting $ij = -k$ in stead of $ij = k$, and is completely isomorphic to the quaternions (in fact, it is simply a permuted identification of $\kappa_1, \kappa_2, \kappa_3$ with i, j, k). It is evident that there are infinitely many *intermediate* normed algebras between the quaternions and $\mathbb{S}_3$ endowed with the ordinary supercomplex product; the notion of a GSA builds a bridge between these quite different structures.

As we imposed right-distributivity, we can try to impose associativity; this however, turns out to be impossible with strictly more than three imaginary units, even when the GSA is not normed: Consider the expressions $(\kappa_1 \circ \kappa_2) \circ \kappa_2$ and $\kappa_1 \circ (\kappa_2 \circ \kappa_2)$; the latter attains the value $-\kappa_1$ (since $\kappa_2 \circ \kappa_2 = -1$), and by the same token, the former must be purely real, since, by Thm. 7.6, $\kappa_1 \circ \kappa_2 = \gamma_2(0,1)\kappa_2$ – they could never coincide.

Finally we might ask, whether also the octonions with their product form a GSA; they do not. Consider for instance the ('octonian') identity[24] $\kappa_1\kappa_2 = \kappa_4$ – from Thm. 7.6, this is not an allowed product. We see what goes wrong when performing the special orthogonal $\mathbb{R}$-fixing transformation $\mathbb{S}_7 \to \mathbb{S}_7$ defined by $T(\kappa_3) = \kappa_4, T(\kappa_4) = -\kappa_3, T(\kappa_i) = \kappa_i, i \neq 3, 4$; it results in a *false* equation, $\kappa_1\kappa_2 = \kappa_3$, and so the octonions do not respect (iii). The ultimate interpretation of this is that the octonion product is not a geometric construction; even if we could visualise eight dimensions, it would be possible to 'turn the coordinate system' in

[22]Strictly speaking, in order to conclude linearity, we need to impose also, either that the product varies continuously, or (equivalently, in fact) that $(\lambda\xi) \circ \phi = \lambda(\xi \circ \phi)$, for $\lambda \in \mathbb{R}$.

[23]It is interesting to confirm directly through the criteria $(i) - (iii)$, that the quaternions do really form a GSA.

[24]This, of course, is arbitrary naming – however, it mimics the one used by [1].

such a way as to obtain false relationships – we are able to *distinguish* the imaginary units by the missing symmetry of the space.

Remark (Generalised products in general): When describing the completion of $\mathbb{S}$, we introduced the view of multiplication as the identification of elements of $\mathbb{S}$ (or $\ell^2_{\mathbb{R}}$) with certain automorphisms of $\mathbb{S}$ itself (endomorphisms, in fact, if we include 0) – that is, under addition and multiplication by reals. Abstractly, we had a set $\mathbb{S}$ endowed with some primordial structure ($\mathbb{S}$ was a vector space); we then constructed a map Π which ascribed to each element an endomorphism $\mathbb{S} \to \mathbb{S}$ preserving this initial structure (corresponding, roughly, to distributivity). It pays to consider what we have done in this last section of the paper, formulating it in the same language: We let the specific choice of the map Π fall, and required instead certain properties of it; in particular, we considered a subgroup, Σ (namely the group of special orthogonal $\mathbb{R}$-fixing transformations), of the automorphism group of $\mathbb{S}$ with respect to its initial structure (i.e. bijective linear mappings), and we demanded that $T(\Pi(\xi)(\phi)) = \Pi(T(\xi))(T(\phi))$ for all $\xi, \phi \in \mathbb{S}, T \in \Sigma$.
This, however, is obviously a procedure which can be performed under much more general circumstances. Starting with any space X endowed with a primordial structure (e.g. a monoid, a ring, a vector space etc.), we may consider the set of endomorphisms of X, $\mathrm{End}(X)$, and define a generalised product, $\circ$, by choosing a map $\Pi : X \to \mathrm{End}(X)$ and then letting $x \circ y := \Pi(x)\,(y)$; the fact that $\Pi(x)$ is an endomorphism will ensure 'distributivity' over the primordial structure on X. If Π maps to automorphisms, we furthermore have the possibility to map certain elements of X to 'inverses' of others. Moreover, as we have done above, we may choose a certain subgroup Σ of the group of automorphisms of X, $\mathrm{Aut}(X)$, which couples the space to itself, promoting the elements of Σ to symmetries of X reflected in the generalised product. Formally, we would require, as before, $\sigma(\Pi(x)(y)) = \Pi(\sigma(x))(\sigma(y))$, or equivalently, $\sigma(x \circ y) = \sigma(x) \circ \sigma(y)$ for all $x, y \in X, \sigma \in \Sigma$, and say that $\circ$ respects the symmetries in Σ.

Even though the generality of the above construction extends beyond the aim of this paper, it captures very precisely the underlying thoughts on which the presented material has been developed. One might in fact say, that the ultimate gain of our results is not as much a specific extension of the complex numbers, as it is a scheme of *how to extend*.

References

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