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An Overview of Complex Hadamard Cubes

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An Overview of Complex Hadamard Cubes

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Abstract. Hadamard matrices have been studied by many authors, but higher-dimensional generalizations of Hadamard matrices are new and still relatively unexplored. This paper will present an overview of Hadamard matrices and their generalizations. In particular we will explore Walsh functions and Hadamard matrices. We will also extend Yang's Product Construction to create complex 3-D Hadamard cubes.

1 Introduction

In 1867, French mathematician Jacques Hadamard published a paper investigating the values of determinants of square matrices with entries restricted to the set $\{-1, 1\}$ [2]. He found that the determinants of these matrices have a maximum value:

Theorem 1.1 (Hadamard's Inequality). *If M is an order n matrix with entries from the set $\{-1, 1\}$, then $|det(M)| \leq n^{n/2}$.*

Definition 1.1. A *Hadamard matrix* is an $n \times n$ matrix, with entries from the set $\{1, -1\}$, whose determinant attains an absolute value equal to the upper bound of $n^{n/2}$.

Example 1.1. Consider the matrix $H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$.
Since $det(H) = 16 = 4^{4/2}$, H is a Hadamard matrix.

Since determinants are difficult to calculate for large matrices, there is a useful theorem to verify that matrices are Hadamard. This is often presented as an alternate definition for Hadamard matrices.

Theorem 1.2. *Let H be an $n \times n$ matrix with entries from the set $\{1, -1\}$. Then $HH^T = nI_n$ if and only if H is a Hadamard matrix.*

Example 1.2. Note that the matrix $H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$ is Hadamard, since:

$$\begin{aligned} HH^T &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 4 \cdot I_4. \end{aligned}$$

For some orders, there appear to be many different Hadamard matrices. The following example will demonstrate this for order 2.

Example 1.3. Consider the matrices $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$. Finding the determinant of the matrices is simple enough:

$$|det(A)| = |det(B)| = 2 = 2^{2/2}.$$

Clearly these are both Hadamard matrices because they meet Hadamard's upper bound on determinants. Next we discuss the organization of Hadamard matrices of a given order.

Equivalence classes allow us to compare Hadamard matrices of a given order.

Definition 1.2. Let H be a Hadamard matrix. A matrix is *equivalent* to or belongs to the same *equivalence class* as H if it can be created from H using the following operations:

(i) swapping any pair of rows or columns in H.

(ii) multiplying any row or column in H by -1.

These elementary row operations partition the set of Hadamard matrices of a given order into distinct equivalence classes. In other words, this definition of equivalence class is the result of an equivalence operation on the set of Hadamard matrices of a given order [3].

Definition 1.3. A Hadamard matrix with entries in the first row and column consisting of only 1s is called a *normalized Hadamard matrix*.

It is often convenient to represent an equivalence class by a normalized Hadamard matrix.

It has been shown that the number of equivalence classes of Hadamard matrices of orders 2, 4, 8, 12, 16, 20, 24 and 28 are 1, 1, 1, 1, 5, 3, 60, and 487 respectively. The number of equivalence classes that exist for any given order is still an open question.

Notice the following interesting properties of the matrix H from Example 1.2:

1. The dot product of any distinct pair of rows or columns is always 0.

2. Compared component-wise, any pair of rows or columns have an equal number of identical and non-identical entry pairs.

3. The sum of the entries in any non-initial row or column are 0.

We will further discuss and formalize these properties in Section 2. In Section 3, we will examine the construction techniques used to create higher order Hadamard matrices. There are a variety of applications for Hadamard matrices such as error-correcting codes, radio astronomy, coding theory, signal processing, cryptography, spectroscopic analysis, variance estimation in stratified sampling, and three-dimensional memory storage [3]. We will discuss an application of Hadamard matrices to Walsh functions in Section 4. We will investigate different generalizations of Hadamard matrices in Section 5 and also expand on a generalized construction technique used for 3-dimensional Hadamard arrays. In Section 6, we will discuss open questions including a brief look into Hyperdeterminants.

2 Properties of Hadamard Matrices

Hadamard matrices have some interesting properties. They include:

1. The rows and columns of any Hadamard matrix are *pairwise orthogonal.* This means that the dot product of any two distinct rows or columns will be zero. This property follows from Theorem 1.2.

2. The rows and columns of any Hadamard matrix are *pairwise balanced.* That is, when comparing the elements in any distinct pair of rows or columns component-wise, there will be an equal number of pairs of identical entries and pairs of non-identical entries.

3. A normalized Hadamard matrix is *row balanced* and *column balanced.* This means that every row and column, except for the initial row and column, has an equal number of 1s and -1s.

The following example illustrates these properties.

Example 2.1. Consider the Hadamard matrix $H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$.

Property 1:

Taking the dot product of the 2^{nd} and 4^{th} columns from H we get:

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = 1 + 1 - 1 - 1 = 0.$$

It is left to the reader to verify that this property holds for all pairs of rows and columns of H.

Property 2:

In order to show that these columns are pairwise balanced we will use the same two columns from Example 2.3 and compare the entries in both columns like so:

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \begin{matrix} \longleftrightarrow \\ \longleftrightarrow \\ \longleftrightarrow \\ \longleftrightarrow \end{matrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

Note that two of the compared entries are the same and two are different. So these columns are pairwise balanced. It is left to the reader to verify this for all pairs of rows and columns.

Property 3:

It is obvious that H satisfies the third property.

These properties lead to the following fact:

Fact 2.1. *Every Hadamard matrix is of order* 1,2, *or* $4n$ *where* $n \in \mathbb{N}$*.*

This realization, combined with empirical evidence, led to the following conjecture, first posed by Jacques Hadamard:

Conjecture 2.1 (Hadamard's Conjecture (1867))**.** *There exists a Hadamard matrix of* every *order* 1,2, *and* $4n$ *where* $n \in \mathbb{N}$*.*

The Hadamard Conjecture is one of the longest standing named conjectures in mathematics [3].

3 Constructions of Hadamard Matrices

A large amount of research on Hadamard matrices has focused on constructing Hadamard matrices. As the reader might have guessed, finding large Hadamard matrices by hand is extremely difficult. One of the simplest constructions involves the use of the *tensor product* or *Kronecker product* .

Definition 3.1. The *tensor product* of two matrices A and B is:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1m}B \\ a_{21}B & a_{22}B & \dots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mm}B \end{bmatrix}.$$

Denote the tensor product of A with itself n times, $(...(A \otimes A) \otimes A) \otimes ...) \otimes A)$, by $A \otimes^n$, where $n \in \mathbb{N}$. This notation will be helpful in Section 4.

In 1867, Sylvester proved the following result, which allowed for the construction of arbitrarily large Hadamard matrices [5].

Theorem 3.1. *If* A *and* B *are Hadamard matrices of order* m *and* n *respectively, then* $A \otimes B$ *is a Hadamard matrix of order* mn*.*

Example 3.1. Consider the following matrices $A = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Then:

$$A \otimes B = \begin{bmatrix} -1\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} & -1\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ -1\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

It will be left to the reader to verify that this is indeed a Hadamard matrix.

In 1933, Raymond Paley, a young English mathematician, created a unique construction technique that widely expanded the number of known Hadamard matrices. The so-called Paley Construction provided the means to construct Hadamard matrices of new orders, providing further justification for the validity of Hadamard's Conjecture [3].

In order to understand the construction fully, we need to employ the Legendre symbol and the ideas of quadratic residues and non-residues.

Definition 3.2. Let $a \in \mathbb{Z}_n$. If the congruence relation $x^2 \equiv a \pmod{n}$ has a solution, then a is referred to as a *quadratic residue* modulo n. If no solution exists, a is considered a *quadratic non-residue* modulo n.

Definition 3.3. Let a be an integer and p an odd prime. The *Legendre symbol*, denoted $\left(\frac{a}{p}\right)$ is defined by:

$$\left(\frac{a}{p}\right) = \left\{ \begin{array}{cr} +1 & \text{if } a \text{ is a quadratic residue modulo } p \\ -1 & \text{if } a \text{ is a quadratic non-residue modulo } p \\ 0 & \text{if } p \text{ divides } a \end{array} \right\}.$$

Theorem 3.2 (Paley's Construction)**.** *Let F be a field of order p, where p is a power of an odd prime, whose elements are $g_i = i$ for $0 \leq i \leq p-1$ and let χ be a quadratic character (assigning 1 to non-zero quadratic residues mod p, -1 to quadratic non-residues mod p and 0 to 0) for F. Consider the matrix Q defined by:*

$$Q = [\chi_{ij}] = [\chi(g_i - g_j)]_{0 \leq i,j \leq p-1} = \begin{bmatrix} \chi(g_0 - g_0) & \chi(g_0 - g_1) & \cdots & \chi(g_0 - g_{p-1}) \\ \chi(g_1 - g_0) & \chi(g_1 - g_1) & \cdots & \chi(g_1 - g_{p-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \chi(g_{p-1} - g_0) & \chi(g_{p-1} - g_1) & \cdots & \chi(g_{p-1} - g_{p-1}) \end{bmatrix}$$

Let $\vec{1}$ denote a vector with 1s for entries, and define $S = \begin{bmatrix} 0 & \vec{1}^T \\ \vec{1} & Q \end{bmatrix}$. Then:

1. *If $p \equiv 3 (mod\ 4)$, then $P_p = \begin{bmatrix} 1 & -\vec{1}^T \\ \vec{1} & Q + I_p \end{bmatrix}$ is a Hadamard matrix of order $p+1$ also known as a* Paley Type I Hadamard matrix *.*

2. *If $p \equiv 1 (mod\ 4)$, then $P_p = \begin{bmatrix} S + I_{p+1} & S - I_{p+1} \\ S - I_{p+1} & -S - I_{p+1} \end{bmatrix}$ is a Hadamard matrix of order $2(p+1)$ also known as a* Paley Type II Hadamard matrix *.*

An example, using the Legendre symbol as the quadratic character, will illustrate this construction technique.

Example 3.2. Let $p = 3$, and then let F be $\mathbb{Z}_3$. Then the quadratic character χ of F is the Legendre symbol. We first construct Q:

$$Q = [\chi_{ij}] = \begin{bmatrix} \chi(0) & \chi(2) & \chi(1) \\ \chi(1) & \chi(0) & \chi(2) \\ \chi(2) & \chi(1) & \chi(0) \end{bmatrix} = \begin{bmatrix} \left(\frac{0}{3}\right) & \left(\frac{2}{3}\right) & \left(\frac{1}{3}\right) \\ \left(\frac{1}{3}\right) & \left(\frac{0}{3}\right) & \left(\frac{2}{3}\right) \\ \left(\frac{2}{3}\right) & \left(\frac{1}{3}\right) & \left(\frac{0}{3}\right) \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Since $p \equiv 3(mod4)$ we are going to construct a Paley Type I Hadamard Matrix.

$$P_3 = \begin{bmatrix} 1 & -\vec{1}^T \\ \vec{1} & Q + I_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

It is left to the reader to verify that this is indeed a Hadamard matrix of order $p + 1 = 3 + 1 = 4$.

4 Applications of Tensor Products of Hadamard Matrices

We will show how Hadamard matrices are related to Walsh functions, which are used in a variety of disciplines. Walsh functions can be defined using the tensor product of Hadamard matrices as shown below.

Definition 4.1. Let A be the Hadamard matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and H be the Hadamard matrix $A\otimes^n$ for some $n \in \mathbb{N}$. Note that H is of order 2^{n+1}. Let $j \in \{0, 1, ..., 2^{n+1} - 1\}$. The *Walsh function* defined by the i^{th} row of H, denoted $Wal(k, t)$, where k is the number of sign changes in the i^{th} row, is:

$$Wal(k,t) = \left\{ \begin{array}{ll} H(i, j+1) & \text{if } t \in [\frac{j}{2^{n+1}}, \frac{j+1}{2^{n+1}}) \\ H(i, j+1) & \text{if } t = 1 \end{array} \right\}$$

Example 4.1. Consider the matrix $H = A\otimes^1$ constructed like so:

$$H = A\otimes^1 = \begin{bmatrix} 1\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & 1\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ 1\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & -1\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

By examining the number of sign changes in each row we can see that each row in H corresponds to a Walsh function like so:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{array}{l} Wal(0,t) \\ Wal(3,t) \\ Wal(1,t) \\ Wal(2,t) \end{array}$$

The graphs of these four Walsh functions are shown below in Figure 1.

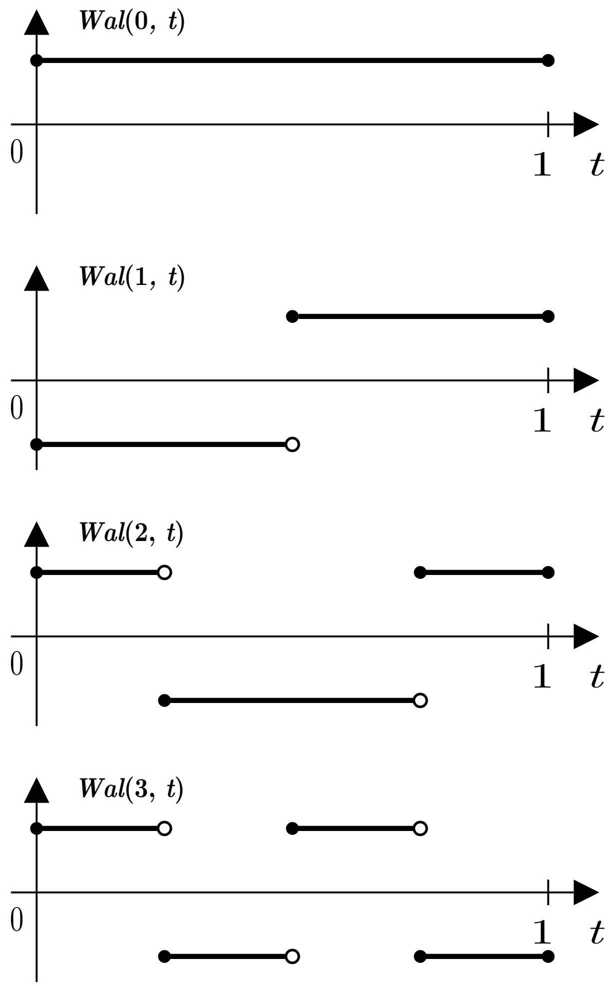

Figure 1

Walsh functions are the basis for Hadamard transforms, which have a wide variety of applications in spectroscopy, data encryption, and data compression. The interested reader can consult [3] and the extensive reference list of [4].

5 *Generalizations of Hadamard Matrices*

Hadamard matrices have been generalized in several ways.

Definition 5.1 leads to the complex generalization of Hadamard matrices. The generalization stems from the fact that the entries of Hadamard matrices are roots of the polynomial $x^2 - 1$. We will denote the complex conjugate of H as $\overline{H}$ throughout.

Definition 5.1. Let H be a square matrix of order n whose entries are complex m^{th} roots of unity where $m \in \mathbb{N}$. Then H is a *Butson matrix* if $H \cdot \overline{H}^T = n \cdot I_n$.

Example 5.1. Consider $H = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$. Note that $\overline{H}^T = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$ So:

$$H \cdot \overline{H}^T = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} = \begin{bmatrix} 1 - i^2 & 1 + i^2 \\ 1 + 1^2 & 1 - i^2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2 \cdot I_2.$$

Therefore, H is a Butson matrix of order 2 with 4^{th} roots of unity.

The following is a specific type of Butson matrix:

Definition 5.2. A *complex Hadamard matrix* C is a matrix whose entries are roots of the polynomial $x^4 - 1$ (from the set $\{1, -1, i, -i\}$) and that has maximal determinant.

The following theorem is an extension of Theorem 1.2 to complex Hadamard matrices.

Theorem 5.1. *Let C be a $n \times n$ matrix with entries from the set $\{1, -1, i, -i\}$. Then $C\overline{C}^T = nI_n$ if and only if C is a complex Hadamard matrix.*

Complex Hadamard matrices have similar properties to Hadamard matrices such as orthogonality and summation of elements to zero in non-initial rows and columns.

We would also like to examine three dimensional generalizations of Hadamard matrices. These are relatively new and unexplored.

Definition 5.3. A *proper 3-D Hadamard array* or a *3-D Hadamard cube* of order n is an $n \times n \times n$ array with entries from the set $\{1, -1\}$ and in which each planar slice (an $n \times n$ array resulting from fixing one of the three dimensions of the cube) is a Hadamard matrix.

In 1986, Yang developed what is now known as his "product construction." We can use this construction to create a proper $n \times n \times n$ Hadamard array from a $n \times n$ Hadamard matrix [3,6].

Theorem 5.2 (Yang's Product Construction for Three Dimensions)**.** *Let H be a Hadamard matrix of order n. Then the 3-D array C defined as:*

$$C = [c_{ijk}] = [h_{ij} \cdot h_{ik} \cdot h_{jk}] \text{ where } i, j, k \in \{1, 2, ..., n\}$$

is a proper 3-D Hadamard cube of order n.

Example 5.2. Let $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. There are 8 entries of C (representing the 8 vertices of the cube). Evaluating for the first couple entries gives us:

$$c_{111} = h_{11}h_{11}h_{11} = 1 \cdot 1 \cdot 1 = 1$$

$$c_{122} = h_{12}h_{12}h_{22} = 1 \cdot 1 \cdot -1 = -1$$

Continue for every entry of C which will then represent a cube composed of front and back slices:

$$C_1 = c_{ij1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \text{ and } C_2 = c_{ij2} = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

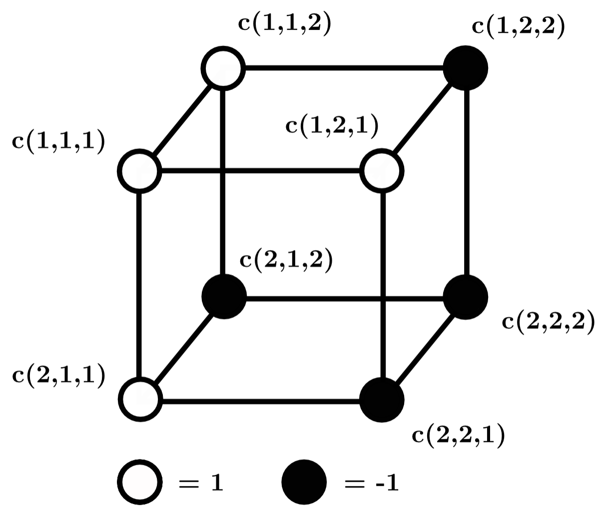

Figure 2

Figure 2 shows the labelled vertices of our Hadamard cube. The fact that each slice is a Hadamard matrix can easily be verified.

Next we would like to look at these three dimensional arrays with complex entries.

Definition 5.4. A *proper unimodular complex 3-D Hadamard array* or a *unimodular complex 3-D Hadamard cube* of order n is an $n \times n \times n$ array with entries that lie on the complex unit circle and in which each planar slice is a complex Hadamard matrix.

Theorem 5.3. *Yang's Product Construction can be extended to create complex 3-D Hadamard cubes using complex Hadamard matrices.*

Proof. Let H be a complex Hadamard matrix of order n and C be the $n \times n \times n$ cube constructed using Yang's Product Construction.

We need to show that every planar slice of C is a complex Hadamard matrix. To that end, consider the k^{th} planar slice resulting from freezing the third dimension. Call it C_k.

Then the dot product of the i^{th} row of C_k with the t^{th} column of $\overline{C}_k^T$ is:

$$\begin{aligned}\sum_{a=1}^{n} c_{iak}\overline{c}_{atk}^T &= \sum_{a=1}^{n} c_{iak}\overline{c}_{tak} \\ &= \sum_{a=1}^{n} h_{ia}h_{ik}h_{ak}\overline{h}_{ta}\overline{h}_{tk}\overline{h}_{ak} \\ &= \sum_{a=1}^{n}(h_{ak}\overline{h}_{ak})(h_{ik}\overline{h}_{tk})(h_{ia}\overline{h}_{ta}) \\ &= h_{ik}\overline{h}_{tk}\sum_{a=1}^{n}(h_{ak}\overline{h}_{ak})(h_{ia}\overline{h}_{ta}) \\ &= h_{ik}\overline{h}_{tk}\sum_{a=1}^{n} h_{ia}\overline{h}_{ta}\end{aligned}$$

Note that $h_{ak}\overline{h}_{ak} = 1$ because any complex number multiplied by its complex conjugate is 1. Also, recall that H is a complex Hadamard matrix and satisfies the equation $H\overline{H}^T = n \cdot I_n$. Then consider the following two cases:

1. If $i \neq t$, then $\sum_{a=1}^{n} h_{ia}\overline{h}_{ta} = 0$.

2. If $i = t$, then $h_{ik}\overline{h}_{tk} = 1$ and $\sum_{a=1}^{n} h_{ia}\overline{h}_{ta} = n$.

Therefore, $C_k\overline{C}_k^T = nI_n$. Thus all planar slices C_k are complex Hadamard matrices. It is left to the reader to verify this when freezing the first and second dimensions of C.

□

6 Conclusion and Open Questions

Theorem 1.1 links the classic definition of Hadamard matrices to determinants. In the context of Hadamard and complex Hadamard cubes, this link requires the use of hyperdeterminants. However, there is no clear (or agreed upon) definition of a hyperdeterminant, and the most useful forms deal only with $2 \times 2 \times 2$ cubes. Arthur Cayley, an 18^{th} century French mathematician, found 2 different formulas for the computation of hyperdeterminants [1].

Definition 6.1 (Cayley's 2^{nd} Hyperdeterminant Formula)**.** Let $A = [a_{ijk}]$ be a $2 \times 2 \times 2$

array, where $i, j, k \in \{0, 1\}$. Then the hyperdeterminant of A is:

$$\begin{aligned} det(A) =& a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2 - 2a_{000}a_{001}a_{110}a_{111} \\ & - 2a_{000}a_{010}a_{101}a_{111} - 2a_{000}a_{011}a_{100}a_{111} - 2a_{001}a_{010}a_{101}a_{110} \\ & - 2a_{001}a_{011}a_{110}a_{100} - 2a_{010}a_{011}a_{101}a_{100} + 4a_{000}a_{011}a_{101}a_{110} \\ & + 4a_{001}a_{010}a_{100}a_{111} \end{aligned}$$

We composed a Matlab program to find the determinants of all possible $2 \times 2 \times 2$ arrays with entries from the set $\{-1, 1, i, -i\}$. We found that the largest hyperdeterminant is 16. We also found that all 3-D Hadamard cubes of order 2 created using Yang's Construction satisfy this maximal hyperdeterminant value.

This formula is widely arrepted because it possesses some of the same properties of two-dimensional matrix determinants. Cayley's formula, however, applies only to $2 \times 2 \times 2$ cubes. There are at least three other hyperdeterminant definitions that appear in the literature. We remain hopeful that one of these hyperdeterminant formulas will verify that these complex 3-D Hadamard cubes have maximal hyperdeterminant.

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