The Structure of the Tutte–Grothendieck Ring of Ribbon Graphs

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Abstract. W. H. Tutte's 1947 paper on a ring generated by graphs satisfying a contraction-deletion relation is extended to ribbon graphs. This ring of ribbon graphs is a polynomial ring on an infinite set of one-vertex ribbon graphs.

1 Introduction

In 1947 Tutte [9] discovered the structure of a Grothendieck ring generated by graphs which is universal for a contraction-deletion relation. We extend this computation to an analogous ring defined using ribbon graphs.

Ribbon graphs (also called cycle graphs, fat graphs, or graphs with rotation) are combinatorial objects use to describe the embedding of graphs into surfaces. Ribbon graphs correspond to embeddings of graphs in oriented surfaces such that the complement of the graph is a disjoint union of disks. For a thorough treatment of graphs on surfaces and Grothendieck's theory of *dessins d'enfants* we refer the reader to Lando and Zvonkin [8].

Bollobás and Riordan in [2] have generalized the Tutte polynomial to ribbon graphs. A specialization of this polynomial is related to the Jones polynomial of a link (Dasbach, et al. [4]), and a duality result extending the planar duality of the Tutte polynomial has been established by Krushkal in [6]. We call this the Bollobás–Riordan–Whitney–Tutte, sharing credit with H. Whitney who earlier studied the coefficients of the *rank polynomial* which is equivalent to the Tutte polynomial under a change of variables. Lando in [7] has extended the classical work of Tutte to a larger ring of graphs and studies the resulting Hopf algebra.

The main result of this paper is a computation of a ring generated by ribbon graphs and satisfying contraction-deletion relations. As with Tutte's ring of graphs, it is a polynomial ring over $\mathbb{Z}$ in infinitely many variables.

In Section 3 we give an exposition of the construction and structure of the Tutte–Grothendieck ring of graphs, and concludes with a theorem relating the Tutte polynomial to homomorphism of the ring of graphs.

In Section 4 we extend the algebraic construction to ribbon graphs. Using ideas of Bollobas and Riordan, we construct relations in this ring and develop the technique of R–operations to show that certain elementary ribbon graphs generate the ring. We then prove that the elementary ribbon graphs are algebraically independent.

2 Background

2.1 Graphs

A *graph* $G = (V, E, I)$ is a finite set V of vertices, a set E of edges, and an incidence map I from E to the set of unordered pairs of V. Notice that our definition allows for multiple edges between two vertices, as well as loops. We call an edge a *loop* if it is incident with only one vertex and a *bridge* if its deletion increases the number of connected components of the graph. These objects are actually known to many combinatorialists as *multigraphs*. Throughout we

shall refer to these simply as graphs. For a more formal treatment of multigraphs we refer the reader to Diestel [5].

Two graphs are *isomorphic* if there is a correspondence between their vertex sets that preserves incidence.

Given a graph G and an edge e incident with vertices u and v (not necessarily distinct) the *deletion* of e is the graph $G\backslash e = (V, E - \{e\})$ and the *contraction* of e is the graph G/e in which the vertices u and v are replaced by a single vertex $w = (uv)$ and each element $f \in E - \{e\}$ that is incident with either u or v is replaced by an edge or a loop incident with w. It is clear that the order of contraction and deletion does not matter with graphs.

2.2   Ribbon Graphs and Graphs on Surfaces

Naively, an *oriented ribbon graph* $\mathbb{D}$ is a graph equipped with a cyclic ordering on the edges incident to each vertex. Ribbon graph isomorphisms are graphs isomorphisms that preserve the cyclic ordering of the edges around eacg vertex. We will follow the approach of [4] in defining ribbon graphs and their associated embeddings into oriented surfaces.

Definition 1. A *combinatorial map* M is an ordered triple $[\sigma_0, \sigma_1, \sigma_2]$ of permutations of a finite set $\mathcal{B} = \{1, \ldots, 2n\}$ such that

1. σ_1 is a fixed point free involution, i.e., $\sigma_1{}^2 = $ id and $\sigma_1(b) \neq b$ for all $b \in \mathcal{B}$,

2. $\sigma_0\sigma_1\sigma_2 = $ id,

3. the group $\langle \sigma_0, \sigma_1, \sigma_2 \rangle$ acts transitively on $\mathcal{B}$.

The elements of $\mathcal{B}$ will be called the *half-edges*. Given a combinatorial map $[\sigma_0, \sigma_1, \sigma_2]$ we obtain a connected graph G by letting the orbits of σ_0 be the vertex set, and letting the edge set be given by the orbits of σ_1, with edges connecting the vertices in whose orbits the two *half-edges* lie. From the cyclic ordering (given by σ_0) of the half-edges making up each vertex v, we get a *rotation system* π_v on the edges incident to v. We note that a loop about an edge appears twice in this rotation system, since it is incident via two half-edges. Finally, we shall call the *face set* the orbits of σ_2.

Definition 2. A *connected oriented ribbon graph* is just a pair $(G, \{\pi_v : v \in V\})$ that arises in such a way from a combinatorial map.

An *oriented ribbon graph* $\mathbb{D}$ is the disjoint union of connected oriented ribbon graphs. We will refer to these as *ribbon graphs* for short. We may abuse terminology and speak of the half-edges of a ribbon graph without specifying the underlying combinatorial map.

Example 1. Consider the combinatorial map

$$M = [(123456)(78), (13)(24)(57)(68), (123674)(58)]$$

with half-edges $\mathcal{B} = \{1, \ldots, 8\}$. By the method outlined above, we obtain the graph G with vertices $a = \{1, 2, 3, 4, 5, 6\}$ and $b = \{7, 8\}$. The orbits $\{1, 3\}$ and $\{2, 4\}$ of σ_1 both give loops, l_1 and l_2 respectively, incident to a, while $\{5, 7\}$ and $\{6, 8\}$ each give edges, e_1 and e_2, between a and b. The rotation system on a obtained from the cycle (123456) is $\pi_a = (l_1, l_2, l_1, l_2, e_1, e_2)$ and the rotation system on b obtained from (78) is $\pi_b = (e_1, e_2)$. We draw this so that the rotation system on each vertex corresponds to a counterclockwise motion around the edges incident to it, see Figure 1.
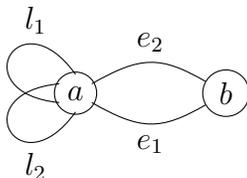

Figure 1: The connected ribbon graph obtained from M in Example 1

As with graphs, the operations of contraction and deletion are defined for ribbon graphs. The deletion of an edge from a ribbon graph is obtained by deleting the edge from the underlying graph and removing it from all the rotation systems in which it occurs. The contraction of an edge $\mathbb{D}/e$ inherits all the rotation systems except at the vertex obtained by identifying the vertices incident to e. Here the new rotation system is obtained by joining along e the rotation systems of the two vertices being identified then removing the edge e. Again the order in which two edges are contracted or deleted does not matter.

For a graph G (or a ribbon graph $\mathbb{D}$) we define

- $v(G)$ to be the number of vertices

- $e(G)$ to be the number of edges

- $k(G)$ to be the number of connected components

- $r(G) = v(G) - k(G)$ to be the *rank*

- $n(G) = e(G) - r(G)$ to be the *nullity*.

Finally, define $f(\mathbb{D})$ to be the number of faces of a ribbon graph, obtained by summing the number of faces of each connected component. We may determine the genus $g(\mathbb{D})$ of the ribbon graph using its Euler characteristic: $v(\mathbb{D}) - e(\mathbb{D}) + f(\mathbb{D}) = 2k(\mathbb{D}) - 2g(\mathbb{D})$.

2.3 Chord diagrams

When we are dealing with ribbon graphs $\mathbb{D}$ with a single vertex, we shall denote as π the only rotation system, noting that if $\mathbb{D}$ has n edges, π will have $2n$ entries since all of these will be loops.

One-vertex ribbon graphs are conveniently represented by chord diagrams. A chord diagram of degree n consists of $2n$ distinct points on the unit circle together with n chords pairing them off. The cyclic ordering of the chords read counterclockwise around the circle corresponds to the rotation system π of the edges of a one-vertex ribbon graph. We give an example of a one-vertex ribbon graph and its chord diagram representation in Figure 2.
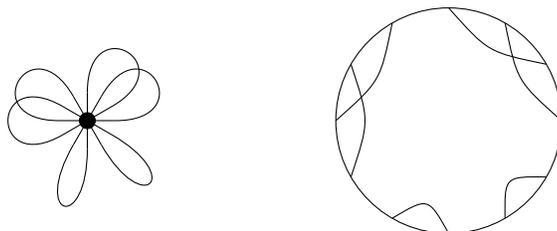

Figure 2: A one-vertex ribbon graphs given by the rotation system $\pi = (1, 2, 1, 2, \ 3, 4, 3, 4, \ 5, 5, \ 6, 6)$

2.4 Polynomial Invariants of Graphs and Ribbon graphs

We shall assume several basic results about the Tutte polynomial. These can all be found in Chapter X of Bollobás [1].

The state-sum definition of the *Tutte polynomial* of a graph G is

$$T_G(x, y) = \sum_{H \subset G} (x - 1)^{r(G) - r(H)} (y - 1)^{n(H)}$$

where H ranges over all spanning subgraphs of G, i.e, subgraphs H whose vertex sets include all the the vertices of G.

Similarly, the *Bollobás–Riordan–Whitney–Tutte polynomial* of a ribbon graph $\mathbb{D}$ is defined as

$$BRWT_{\mathbb{D}}(X, Y, Z) = \sum_{\mathbb{H} \subset \mathbb{D}} X^{r(\mathbb{D}) - r(\mathbb{H})} Y^{n(\mathbb{H})} Z^{2g(\mathbb{H})}$$

where $\mathbb{H}$ is ranges over all spanning sub-ribbon graphs. Taking $Z = 1$, it is easy to see that the BRWT polynomial of a ribbon graph $\mathbb{D}$ reduces to the Tutte polynomial of the underlying graph D:

$$BRWT_{\mathbb{D}}(X, Y, 1) = T_D(X + 1, Y + 1).$$

The Tutte has a very nice alternate definition in terms of the operations of contraction and deletion. We let $T_{E_n}(x, y) = 1$ for the empty n-graph, the graph with n vertices and zero edges, and setting

$$T_G = \begin{cases} x T_{G \setminus e} & \text{if } e \text{ is a bridge,} \\ y T_{G \setminus e} & \text{if } e \text{ is a loop,} \\ T_{G/e} + T_{G \setminus e} & \text{if } e \text{ is neither a bridge nor a loop.} \end{cases}$$

Note that the Tutte polynomial is multiplicative over disjoint unions of graphs, as well as over *one-point unions*, graphs that share only one vertex. Thus if e is a bridge or a loop, $T_{G\backslash e} = T_{G/e}$.

Unfortunately, no such nice expression exists for the BRWT, since we may now have loops that are homologically nontrivial (when embedded into the corresponding surface). Still, we may define it uniquely by letting $BRWT_{\mathbb{D}}(X, Y, Z) = \sum_{\mathbb{H} \subset \mathbb{D}} Y^{n(\mathbb{H})} Z^{2g(\mathbb{H})}$ for a ribbon graph $\mathbb{D}$ with a single vertex, letting

$$BRWT_{\mathbb{D}}(X, Y, Z) = (X + 1)BRWT_{\mathbb{D}\backslash e}(X, Y, Z)$$

if e is a bridge, and for a connected ribbon graph with edge e that is neither a bridge nor loop setting

$$BRWT_{\mathbb{D}}(X, Y, Z) = BRWT_{\mathbb{D}/e}(X, Y, Z) + BRWT_{\mathbb{D}\backslash e}(X, Y, Z).$$

For disconnected ribbon graphs we simply multiply the BRWT polynomials of each component. Also, though one-point union is not well-defined on isomorphism classes of ribbon graphs, the BRWT polynomial is still multiplicative over such unions. Thus we still have $BRWT_{\mathbb{D}\backslash e} = BRWT_{\mathbb{D}/e}$ for bridges e.

3 Tutte's Ring of Graphs and Universality

Let $\mathcal{G}$ denote the set of isomorphism classes of graphs (including the empty graph E_0 with no vertices). We let $[G]$ denote the isomorphism class containing the graph G. We may then define a product on $\mathcal{G}$ by letting $[G_1][G_2] = [G_1 \coprod G_2]$. Tutte defines a ring R to be the quotient of the monoid ring $\mathbb{Z}[\mathcal{G}]$ by the ideal generated by all the elements $[G] - [G/e] - [G\backslash e]$ where G is graph and e is any non-loop edge. We will refer to this as the *contraction-deletion ideal* and call any equation $[G] = [G/e] + [G\backslash e]$ in R a *contraction-deletion relation*. We will sometimes abuse notation and let G refer to the isomorphism class of the graph G, or even its equivalence class in R.

It is easy to see that for any graph, we may use the contraction-deletion relation to express any G in R as a sum and product in the elements $\{1, y_i\}$, where 1 represents (the isomorphism class of) the empty graph E_0 and y_i represents a one-vertex graph having i loops. Indeed Tutte proved that R is generated *uniquely* by this set, so that R is precisely $\mathbb{Z}[y_0, y_1, \ldots]$. We use the remainder of this section to illustrate the proof of this fact.

A *V–function* is a function $\mathcal{G} \to H$ into a commutative ring with unity H such that

1. $V(E_0) = 1$,

2. $V(G) = V(G/e) + V(G\backslash e)$ for any non-loop edge e of G,

3. $V(G_1 G_2) = V(G_1)V(G_2)$,

for any G, G_1, G_2 of $\mathcal{G}$. Tutte proved the following theorem about V–functions.

Theorem (Tutte). *A function $V : \mathcal{G} \to H$ is a V–function if and only if it factors through a homomorphism (necessarily unique) $h : R \to H$. I.e., $V = h \circ f$ where f is the canonical map $\mathcal{G} \hookrightarrow \mathbb{Z}[\mathcal{G}] \twoheadrightarrow R$.*

This is true simply by the universal property of f and since a V–function is zero on the contraction-deletion ideal.

Tutte also defines the *universal V–function* to be the function $Z : \mathcal{G} \to \mathbb{Z}[z_0, z_1, \ldots]$ given by

1. if e is not a loop then $Z(G) = Z(G/e) + Z(G \backslash e)$,

2. if G has no non-loop edges and for each vertex v we let $L(v) = \{\text{loops of } G \text{ incident on } v\}$ and $d(v) = |L(v)|$ then

$$Z(G) = \sum_{T \subset G} \left(\prod_{v \in V} z_{|T \cap L(v)|} \right)$$

over all spanning subgraphs T of G.

However, by the above theorem, it is enough to define Z by the value of the induced homomorphism $k : R \to \mathbb{Z}[z_0, z_1, \ldots]$ on the generators of R, the latter being given by $y_i \mapsto \sum_{j=0}^{i} \binom{i}{j} z_j$.

The function Z is universal since the homomorphism $h : R \to \mathbb{Z}[z_0, z_1, \ldots]$ induced by Z is actually an isomorphism, mapping the element $t_i = \sum_{j=0}^{i} (-1)^{i+j} \binom{i}{j} y_j$ to z_i for each i. We may invert this formula and write $y_r = \sum_{i=0}^{r} \binom{r}{i} t_i$. This proves algebraic independence of the y_i, for if there was a nontrivial relation between them we could use this substitution to obtain a nontrivial relation between the t_i.

3.1   The Tutte polynomial in relation to $Z(G)$

We show how one can obtain the Tutte polynomial itself from the universal V–function by composing Z with the ring homomorphism $l : \mathbb{Z}[z_0, z_1, \ldots] \to \mathbb{Z}[x, y]$ taking each $z_i \mapsto (x-1)(y-1)^i$. In this way we get $T_G(x, y) = (x-1)^{-k(G)} (l \circ Z)(G)$. Since we are factoring through a ring homomorphism, this is actually an easy consequence of the following theorem.

Theorem 1. *The map $(x-1)^{k(G)} T_G(x, y) : \mathcal{G} \to \mathbb{Z}[x, y]$ is a V–function.*

Proof. Since the Tutte polynomial and the number of components k of a graph are both graph invariants, $(x-1)^{k(G)} T_G(x, y)$ clearly does give a function $\mathcal{G} \to \mathbb{Z}[x, y]$ and this map takes the empty graph to 1. The Tutte polynomial of a graph and $(x-1)^{k(G)}$ are also multiplicative over disjoint unions of graphs since k is additive over this operation, so their product will be as well.

Finally, since $k(G)$ is preserved for contraction and deletion of any edge that is not a bridge, so we have only to check that the contraction-deletion relation holds for a bridge e.

But this is true, for

$$\begin{aligned}
(x-1)^{k(G)}T_G(x,y) &= (x-1)^{k(G)}xT_{G\backslash e}(x,y) \\
&= \left((x-1)^{k(G)} + (x-1)^{k(G)+1}\right)T_{G\backslash e}(x,y) \\
&= (x-1)^{k(G)}T_{G/e}(x,y) + (x-1)^{k(G)+1}T_{G\backslash e}(x,y) \\
&= (x-1)^{k(G/e)}T_{G/e}(x,y) + (x-1)^{k(G\backslash e)}T_{G\backslash e}(x,y),
\end{aligned}$$

recalling that since e is a bridge, $T_{G\backslash e} = T_{G/e}$. $\qquad\square$

Corollary 2. $T_G(x,y) = (x-1)^{-k(G)}(l \circ Z)(G)$.

Proof. By Theorem 1 it is enough to check that $(x-1)^{k(G)}T_G(x,y) = (l \circ Z)(G)$ on the generators of R, i.e., the y_i.

But we have $(x-1)^{k(y_i)}T_{y_i}(x,y) = (x-1)y^i$, and $(k \circ Z)(y_i) = l\left(\sum_{j=0}^{i}\binom{i}{j}z_j\right) = \sum_{j=0}^{i}\binom{i}{j}l(z_j) = \sum_{j=0}^{i}\binom{i}{j}(x-1)(y-1)^j = (x-1)\sum_{j=0}^{i}\binom{i}{j}(y-1)^j(1)^{i-j} = (x-1)y^i$ by the binomial theorem. $\qquad\square$

4   The Ring of Ribbon graphs

We will produce a similar ring of ribbon graphs. Let $\mathcal{D}$ denote the set of isomorphism classes of ribbon graphs and $[\mathbb{D}]$ be the isomorphism class containing the ribbon graph $\mathbb{D}$. Again we define the operation of disjoint union on $\mathcal{D}$ by letting $[\mathbb{D}_1][\mathbb{D}_2] = [\mathbb{D}_1 \coprod \mathbb{D}_2]$. The ring of ribbon graph, which we will denote by T, is then just the quotient of the monoid ring $\mathbb{Z}[\mathcal{D}]$ by the ideal generated by all the elements $[\mathbb{D}] - [\mathbb{D}/e] - [\mathbb{D}\backslash e]$ where $\mathbb{D}$ is ribbon graph and e is any non-loop edge. Again we may abuse notation and let $\mathbb{D}$ refer to the isomorphism class of the ribbon graph $\mathbb{D}$, or even its equivalence class in T.

As with Tutte's ring of graphs, this object will be generated over $\mathbb{Z}$ by the set of all ribbon graphs with a single vertex. However, because one-vertex ribbon graphs also contain a rotation system on the edges, in general there are many different one-vertex ribbon graphs having a given number of loops.

Luckily, we may restrict our attention to a much smaller class of one-vertex ribbon graphs, which we name the *elementary ribbon graphs*, that generate the ring T.

Definition 3. For $g, n \geq 0$ let $\mathbb{D}_{g,n}$ be the one-vertex ribbon graph with $2g + n$ edges where the rotation system π on the loops labeled $1, \ldots, 2g + n$ is

$$(1, 2, 1, 2, \ \ldots, \ 2g-1, 2g, 2g-1, 2g, \ 2g+1, 2g+1, \ \ldots, \ 2g+n, 2g+n).$$

We call this the *elementary ribbon graph* of genus g with n trivial loops.

See Figure 2 for an example of $\mathbb{D}_{2,2}$.

4.1   The four term relation

Suppose a ribbon graph $\mathbb{D}$ has at least two vertices u, v and at least two edges e, f between them. Then in the ring T, $[\mathbb{D}] = [\mathbb{D}/e] + [\mathbb{D}\backslash e] = [\mathbb{D}/e] + [\mathbb{D}\backslash e/f] + [\mathbb{D}\backslash e \backslash f]$ by performing the contraction-deletion reduction on edge e. Similarly, we could have written $[\mathbb{D}] = [\mathbb{D}/f] + [\mathbb{D}\backslash f/e] + [\mathbb{D}\backslash f \backslash e]$. Since contraction and deletion behave well with respect to contraction and deletion, this gives us a four term equality

$$[\mathbb{D}/e] - [\mathbb{D}/f] = [\mathbb{D}/e\backslash f] - [\mathbb{D}/f\backslash e]. \tag{1}$$

We now describe a method of obtaining four term relations between one-vertex ribbon graphs by performing operations on their respective chord diagrams. Two chord diagrams are related by an R–*operation* if one can be obtained from the other by fixing a chord e and that half of the diagram on one side of e, and on the other side "dragging" the end of a chord adjacent to e down to the other side of the diagram.

Definition 4. More formally, a *counterclockwise R–operation of a fixing e* on a one-vertex ribbon graph given by the rotation system $\pi = (e, \dots, A, \dots, e, a, \dots, B)$ given the one-vertex ribbon graph given by $\pi = (e, \dots, A, \dots, e, \dots, B, \dots, a)$. The reverse operation,

$$(e, \dots, A, \dots, e, \dots, B, \dots, a) \mapsto (e, \dots, A, \dots, e, a, \dots, B)$$

will also be called a *clockwise R–operation*. Both types we shall refer to simply as R–*operations*.

We will say two chord diagrams are R–*equivalent* if they are related by a sequence of R–operations. Figure 3 gives an example of an R–operation of the first type.
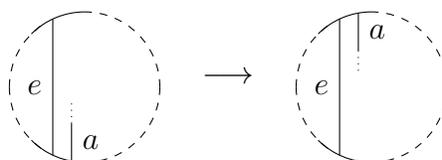

Figure 3: An example of an R–operation of a fixing e

If two chord diagrams are related by an R–operation, as are the two diagrams in Figure 4, we show how we may obtain them from the same two-vertex ribbon graph by contraction of different edges between them. This gives the relation between chord diagrams shown in Figure 5.

4.2   The elementary ribbon graphs generate T

First we give an example of how to use R–operations to express a chord diagram in terms of elementary ribbon graphs.
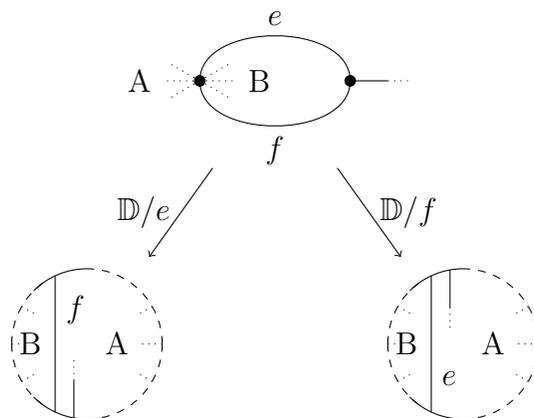
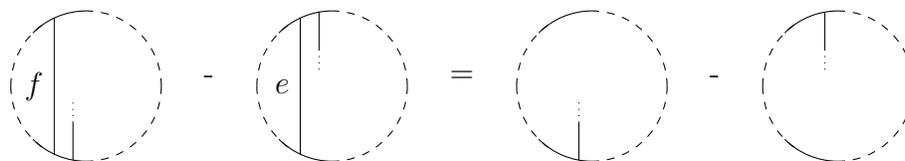Figure 4: R–operations yield relations between ribbon graphs

Figure 5: The four term relation (1) for chord diagrams

Example 2. Consider the one-vertex ribbon graph $\mathbb{D}$ given by the rotation system $\pi = (e, f, g, e, f, g)$. In Figure 6 we show a series of R–operations from $\mathbb{D}$ to the elementary ribbon graph $\mathbb{D}_{1,1}$. The first is an R–operation fixing e, the second an R–operation fixing g. Each of these R–operations give four term relations as in Figure 5, which when added together give the equation $\mathbb{D} - \mathbb{D}_{1,1} = 2(\mathbb{D}_{1,0} - \mathbb{D}_{0,2})$ in T.
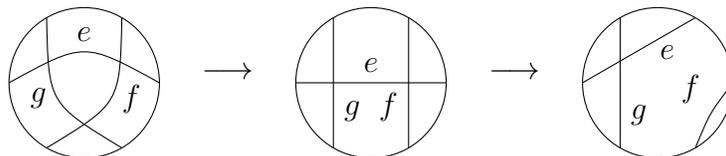

Figure 6: A sequence of R–operations from $\mathbb{D}$ of Example 2 to $\mathbb{D}_{1,1}$

Lemma 5 of [2] shows that every chord diagram is R–equivalent to some $\mathbb{D}_{g,n}$. We use this fact for the proof of the following theorem.

Theorem 3. *Any ribbon graph $\mathbb{D}$ can be expressed in T as a polynomial over $\mathbb{Z}$ in the elementary ribbon graphs $\mathbb{D}_{g,n}$.*

Proof. We first prove the result for one-vertex ribbon graph by induction on on the number of edges. Note that the only one-vertex ribbon graphs with zero or one edge are $\mathbb{D}_{0,0}$ or $\mathbb{D}_{0,1}$ respectively. Now suppose the result holds for one-vertex ribbon graphs having fewer than k edges and let $\mathbb{D}$ be a one-vertex ribbon graph with k edges. We know there is a sequence of R–operations taking $\mathbb{D}$ to some $\mathbb{D}_{g,n}$ with $2g + n = k$. Let $\mathbb{D} = \mathbb{D}_0, \mathbb{D}_1, \ldots, \mathbb{D}_l = \mathbb{D}_{g,n}$ be this sequence. By the four term relation in equation (1), each of the differences $[\mathbb{D}_i] - [\mathbb{D}_{i+1}]$ is equal to a difference of one-vertex ribbon graph with fewer than k edges so by inductive hypothesis is expressible as a polynomial as desired. Summing all these expressions for $[\mathbb{D}_i] - [\mathbb{D}_{i+1}]$, for $0 \le i < l$, and adding $\mathbb{D}_{g,n}$ we then have an expression for $[\mathbb{D}]$ in terms of the elementary ribbon graphs.

For the general result we use induction on the number of edges to show that we may express $[\mathbb{D}]$ as a polynomial over $\mathbb{Z}$ in one-vertex ribbon graphs. We first observe that if $e(\mathbb{D}) = 0$ then $\mathbb{D}$ is the empty ribbon graph on $v(\mathbb{D})$ vertices, and so $[\mathbb{D}] = [\mathbb{D}_{0,0}]^{v(\mathbb{D})}$.

Now suppose the result is true for all ribbon graphs with fewer than k edges and let $\mathbb{D}$ be a ribbon graph with k edges. If $\mathbb{D}$ is connected, either it has only a single vertex in which case we are done, or it has a non-loop edge e so that $[\mathbb{D}] = [\mathbb{D}/e] + [\mathbb{D}\backslash e]$. In the latter case, we have expressed $[\mathbb{D}]$ as the sum of ribbon graphs both having fewer than k edges, so by the inductive hypothesis we are done. In general for possibly disconnected $\mathbb{D}$, we may express $[\mathbb{D}]$ as a product of its disjoint components, each of which having at most k edges, so we may express $[\mathbb{D}]$ as a polynomial in the one-vertex ribbon graphs as desired.

Since each of these one-vertex ribbon graphs we can express as a polynomial over $\mathbb{Z}$ in the $[\mathbb{D}_{g,n}]$, this suffices to prove the theorem. □

Corollary 4. *Any element of T can be expressed as a polynomial over $\mathbb{Z}$ in the elementary ribbon graphs $\mathbb{D}_{g,n}$.*

4.3   Algebraic independence of the $\mathbb{D}_{g,n}$

Now we turn our attention to proving that the polynomial given in Corollary 4 is actually unique.

We define V–*functions* for ribbon graphs $\mathcal{D} \to H$ in analogue to those for graphs and note that as in the previous case a map is a V–function if and only if it factors through a ring homomorphism $h : T \to H$.

Define the map $W : T \to \mathbb{Z}[\{x_{i,j}\}_{i,j=0}^{\infty}]$ by

$$W(\mathbb{D}) = \sum_{\mathbb{H} \subset \mathbb{D}} \prod_{\mathbb{H}' \subset \mathbb{H}} x_{g(\mathbb{H}'),n(\mathbb{H}')-2g(\mathbb{H}')}$$

where $\mathbb{H}$ runs over all spanning sub-ribbon graphs of $\mathbb{D}$ and $\mathbb{H}'$ runs over all connected components of $\mathbb{H}$.

Theorem 5. *W is a V–function of ribbon graphs, and is given by*

1. *if e is a non-loop edge of $\mathbb{D}$ then $W(\mathbb{D}) = W(\mathbb{D}/e) + W(\mathbb{D}\backslash e)$,*

2. *if $\mathbb{D}$ has no non-loop edges then*

$$W(\mathbb{D}) = \prod_{\mathbb{D}' \subset \mathbb{D}} \left(\sum_{\mathbb{H} \subset \mathbb{D}'} x_{g(\mathbb{H}),e(\mathbb{H})-2g(\mathbb{H})} \right)$$

where $\mathbb{D}'$ runs over all connected components of $\mathbb{D}$ and $\mathbb{H}$ over all spanning sub-ribbon graphs of $\mathbb{D}'$.

Proof. W is certainly a ribbon graph invariant, as genus and nullity are both graph invariants.

If e is a non-loop edge of $\mathbb{D}$, then the spanning sub-ribbon graphs of $\mathbb{D}$ which do not contain e are simply the spanning sub-ribbon graphs of $\mathbb{D}\backslash e$, and the spanning sub-ribbon graphs $\mathbb{H}$ of $\mathbb{D}$ which do contain e are in one-to-one correspondence with the spanning sub-ribbon graphs $\mathbb{H}/e$ of $\mathbb{D}/e$. To show that $W(\mathbb{D}) = W(\mathbb{D}/e) + W(\mathbb{D}\backslash e)$ it will suffice to show that each of the spanning sub-ribbon graphs $\mathbb{H}$ of $\mathbb{D}$ containing e has the same number of components with each genus and nullity as its corresponding spanning sub-ribbon graph $\mathbb{H}/e$ of $\mathbb{D}/e$.

But $\mathbb{H}/e$ differs from $\mathbb{H}$ only in that a component $\mathbb{H}'$ of $\mathbb{H}$ corresponds to $\mathbb{H}'/e$ and this is connected and has nullity and genus the same as $\mathbb{H}'$ since we have the equations:

$$v(\mathbb{D}) = v(\mathbb{D}/e) + 1, \quad e(\mathbb{D}) = e(\mathbb{D}/e) + 1, \quad k(\mathbb{D}) = k(\mathbb{D}/e), \quad f(\mathbb{D}) = (\mathbb{D}/e)$$

for any ribbon graph $\mathbb{D}$ with non-loop edge e. These equations all follow easily from the definitions.

Finally, for any product $\mathbb{D}_1\mathbb{D}_2$, the spanning sub-ribbon graphs of $\mathbb{D}_1\mathbb{D}_2$ are just all the products of spanning sub-ribbon graphs $\mathbb{H}_1$ of $\mathbb{D}_1$ with spanning sub-ribbon graphs $\mathbb{H}_2$ of

$\mathbb{D}_2$. The set of connected components then of $\mathbb{H}_1\mathbb{H}_2$ is just the union of the sets of connected components of $\mathbb{H}_1$ and $\mathbb{H}_2$. Thus

$$
\begin{aligned}
W(\mathbb{D}_1\mathbb{D}_2) &= \sum_{\mathbb{H}_1\mathbb{H}_2} \prod_{\mathbb{H}'\subset\mathbb{H}_1, \mathbb{H}'\subset\mathbb{H}_2} x_{g(\mathbb{H}'),n(\mathbb{H}')-2g(\mathbb{H}')} \\
&= \sum_{\mathbb{H}_1\mathbb{H}_2} \left(\prod_{\mathbb{H}'_1\subset\mathbb{H}_1} x_{g(\mathbb{H}'_1),n(\mathbb{H}'_1)-2g(\mathbb{H}'_1)} \right) \left(\prod_{\mathbb{H}'_2\subset\mathbb{H}_2} x_{g(\mathbb{H}'_2),n(\mathbb{H}'_2)-2g(\mathbb{H}'_2)} \right) \\
&= \left(\sum_{\mathbb{H}_1} \prod_{\mathbb{H}'_1\subset\mathbb{H}_1} x_{g(\mathbb{H}'_1),n(\mathbb{H}'_1)-2g(\mathbb{H}'_1)} \right) \left(\sum_{\mathbb{H}_2} \prod_{\mathbb{H}'_2\subset\mathbb{H}_2} x_{g(\mathbb{H}'_2),n(\mathbb{H}'_2)-2g(\mathbb{H}'_2)} \right) \\
&= W(\mathbb{D}_1)W(\mathbb{D}_2).
\end{aligned}
$$

Thus W is a V–function.

The expression given for W is obvious from the fact that W is a V–function of ribbon graphs. $\square$

Lemma 6. *The function W induces a homomorphism $h : T \to \mathbb{Z}[\{x_{i,j}\}_{i,j=0}^{\infty}]$ given by*

$$
\mathbb{D}_{g,n} \mapsto \sum_{i=0}^{g}\sum_{j=0}^{n} \binom{g}{i} \left(\sum_{k=0}^{g-i} 2^k \binom{g}{k}\binom{n}{j} x_{i,j+k} \right).
$$

Proof. For elementary ribbon graphs $\mathbb{D}_{0,n}$, the number of subgraphs $\mathbb{H}$ with j edges is $\binom{n}{j}$ and all of these have genus 0 so $\mathbb{D}_{0,n} \mapsto \sum_{j=0}^{n} \binom{n}{j}x_{0,j}$.

For the elementary ribbon graphs with nontrivial genus $\mathbb{D}_{g,n}$, the number of ways we can chose spanning sub-ribbon graphs with genus i is $\binom{g}{i}$ since we have g pairs of edges which affect the genus. For each of these choices, not only do we have $\binom{n}{j}$ choices for sub-ribbon graphs with j trivial loops, but for each $k \in \{1, \ldots, g-i\}$ corresponding to the $g-i$ unused paired loops we have $\binom{g}{k}$ choices of pairs from which to add trivial loops. Of each of these choices of k pairs, we may only chose one loop for each pair lest we increase the genus of the sub-ribbon graph, so finally we have 2^k choices of these. Thus we get the desired formula. $\square$

Theorem 7. *The function W induces an isomorphism $h : T \to \mathbb{Z}[\{x_{i,j}\}_{i,j=0}^{\infty}]$ as in the previous lemma. Thus W is a universal V–function, i.e., every V–function $V : T \to H$ factors through W and some homomorphism $l : \mathbb{Z}[\{x_{i,j}\}_{i,j=0}^{\infty}] \to H$.*

Proof. We prove that the images of the $\mathbb{D}_{g,n}$ are algebraically independent, showing $\ker h = 0$, and that they generate all of $\mathbb{Z}[\{x_{i,j}\}_{i,j=0}^{\infty}]$, showing h is onto.

Let

$$
t_{g,n} = h(\mathbb{D}_{g,n}) = \sum_{i=0}^{g}\sum_{j=0}^{n}\sum_{k=0}^{g-i} 2^k \binom{g}{i}\binom{g}{k}\binom{n}{j} x_{i,j+k}. \tag{2}
$$

Notice first that $t_{g,n}$ is of the form

$$x_{g,n} + \sum_{i,j} c_{i,j} x_{i,j} \tag{3}$$

where the $x_{i,j}$ in the sum have $2i + j < 2g + n$ and $i \leq g$. Suppose there was a polynomial relation among the $t_{g,n}$, say, $p(\{t_{g,n}\}) = 0$ for a nonzero polynomial p. Of the non-zero terms of p take M_1 the subset of them having the largest power of factors t_{g_1,n_1}, where $2g_1 + n_1$ is the highest of all the $t_{g,n}$ occurring in p. Of these, take M_1' the subset with the largest g_1. Let M_2 the subset thereof having the largest power of factors t_{g_2,n_2} for the second highest $2g_2 + n_2$, and M_2' the subset of this with the largest g_2. Continue this process until we are left with one term $c t_{g_1,n_1}^{a_1} \cdots t_{g_k,n_k}^{a_k}$. Substituting equation (2) into $p(\{t_{g,n}\})$, we get a non-zero term $c x_{g_1,n_1}^{a_1} \cdots x_{g_k,n_k}^{a_k}$ contradicting algebraic independence of the $x_{i,j}$. Thus $p = 0$ the $t_{g,n}$ are algebraically independent.

To show the $t_{g,n}$ generate $\mathbb{Z}[\{x_{i,j}\}_{i,j=0}^{\infty}]$ we claim that for all e, $\{t_{g,n} : 2g+n \leq e\}$ generates $\mathbb{Z}[\{x_{i,j} : 2i + j \leq e\}]$ and proceed by induction on e. First, $t_{0,0} = x_{0,0}$ so $\{t_{0,0}\}$ generates $\mathbb{Z}[x_{0,0}]$. Now suppose $\{t_{g,n} : 2g + n \leq e\}$ generates $\mathbb{Z}[\{x_{i,j} : 2i + j \leq e\}]$. Now by equation (3), $\{t_{g,n} : 2g+n \leq e\} \cup \{t_{g',n'}\}$ for $2g'+n' = e+1$ generates $\mathbb{Z}[\{x_{i,j} : 2i+j \leq e\} \cup \{x_{g',n'}\}]$ so including all such $t_{g',n'}$, we have that $\{t_{g,n} : 2g+n \leq e+1\}$ generates $\mathbb{Z}[\{x_{i,j} : 2i+j \leq e+1\}]$.

The second statement follows since W is isomorphic to the canonical map $\mathcal{D} \hookrightarrow \mathbb{Z}[\mathcal{D}] \twoheadrightarrow T$ which we already know to be universal. $\square$

Remark. We have noted that in the change of basis (2) the leading coefficient of the $\{x_{ij}\}$ graded by $2i + j$ is one. It is this fact that allows us to invert the map h and show that the images of the elementary ribbon graphs generate the whole ring $\mathbb{Z}[\{x_{i,j}\}_{i,j=0}^{\infty}]$.

Since we have shown that the set of images under W of the elementary ribbon graphs is algebraically independent over $\mathbb{Z}$, we obtain the following corollary, for if we had a nontrivial polynomial relation among the $\mathbb{D}_{g,n}$, by mapping under W we would obtain a nontrivial relation among their images.

Corollary 8. *The elementary ribbon graphs $\mathbb{D}_{g,n}$ are algebraically independent.*

Thus the elementary ribbon graphs $\mathbb{D}_{g,n}$ generate the ring of ribbon graphs T uniquely.

Just as for graphs, the BRWT polynomial gives rise to a V–function of ribbon graphs. In fact, precisely the same argument as in Theorem 1 shows that $X^{k(\mathbb{D})} BRWT_{\mathbb{D}}(X, Y, Z)$ is a V–function of ribbon graphs so it factors through the function W.

5   Further Directions

Bollobás and Riordan in [3] have studied graphs embedded into non-oriented surfaces and have developed a non-oriented version of the BRWT polynomial used here. A natural extension would be to characterize the ring of these graphs, for in addition to the elementary ribbon graphs we will also have non-oriented generators.

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