Rose-Hulman Undergraduate Mathematics Journal

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Volume 13, No. 2, Fall 2012

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Abstract. In this paper, we prove that a map u between two polynomial rings, each with an associated Adem relation, is injective. We prove injectivity of u, by first finding formulas for elements within each ring polynomial, and then by computing the map with our associated formulas. After having computed the mapping of u, we then use our computations to show that the kernel of u only contains the zero vector, which proves that the map u is injective. Then having proved that the map u is injective, we then use it to find a basis for u^*, the dual map of u.

Acknowledgements: I would like to thank Mark Behrens for his mentoring and support while I took part in this research project. I would also like to thank Julie Wayzani.

1 Introduction

Morava E-theory is a cohomology theory in algebraic topology which was first introduced by Jack Morava in 1970. Morava E-theory is defined as follows: For every prime number p, there is a theory $E(n)$, for $n \geq 0$, each of which is a ring spectrum. Let k be a perfect field of characteristic p, and suppose we are given a formal group f of height n over k. The universal deformation of f is classified by the Labirinth-Tate ring $R = W(k)[[v_1, \cdots, v_{n-1}]]$. The sequence $v_0 = p, v_1, \cdots, v_{n-1}$ is regular by construction and v_n has invertible image in $R/(v_0, v_1, \cdots, v_{n-1}) \simeq k$, because of our original assumption that the formal group law has height n. We can construct an even periodic spectrum $E(n)$ with $\pi_* E(n) \simeq W(k)[[v_1, \cdots, v_{n-1}]][\beta^{\pm 1}]$, where β has degree 2. The cohomology theory $E(n)$ is called the Morava E-theory.

In [1], Charles Rezk computes the relations in the mod p Dyer-Laslof Algebra for height 2 Morava E-theory. This amounts to computing the kernel of the dual of a certain map of rings:

$$u : A_2 \to A_{1,1}$$

Rezk uses elliptic curves. Here we use an elementary computation of the map u and the kernel of u^*. Let p be an odd prime and let $k = \mathbb{Z}/p$. Let A_2 and $A_{1,1}$ be polynomial rings such that

$$A_2 = \frac{k[x_0,x_1]}{({x_0}^{p^2} - x_1)({x_0}^p - {x_1}^p)(x_0 - {x_1}^{p^2})} \quad \text{and} \quad A_{1,1} = \frac{k[x_0,x_1,x_2]}{({x_2}^{p+1} + {x_1}^{p+1} - x_1x_2 - {x_1}^p{x_2}^p = 0 \ , \ {x_1}^{p+1} + {x_0}^{p+1} - x_1x_0 - {x_1}^p{x_0}^p = 0)}$$

In section 2, we will compute the map u in terms of explicit bases for $A_{1,1}$ and A_2 (Proposition 1). In section 3, we will use these explicit formulas to rederive Rezk's relations (i.e. compute the kernel of u^*, Theorem 1).

2 Computation of the map u

Let u be the ring map

$$A_2 \overset{u}{\to} A_{1,1}$$

such that $u(x_0) = x_0$ and $u(x_1) = x_2$.

By the definition of A_2:

$$A_2 = \frac{k[x_0, x_1]}{({x_0}^{p^2} - x_1)({x_0}^p - {x_1}^p)(x_0 - {x_1}^{p^2})}$$

so then we have that

$$(x_0{}^{p^2} - x_1)(x_0{}^p - x_1{}^p)(x_0 - x_1{}^{p^2}) = 0$$

which can be expanded as

$$-x_1{}^{p^2+p+1} + x_0{}^{p^2+p+1} + x_1{}^{p^2+p}x_0{}^{p^2} - x_0{}^{p^2+p}x_1{}^{p^2} + x_1{}^{p^2+p}x_0{}^p - x_0{}^{p^2+1}x_1{}^p + x_1{}^{p+1}x_0 - x_0{}^{p+1}x_1 = 0$$

and then rewritten as

$$x_1{}^{p^2+p+1} = (-x_0{}^{p^2+p+1}) + (x_0{}^{p+1})x_1 + (x_0{}^{p^2+1})x_1{}^p - (x_0)x_1{}^{p+1} + (x_0{}^{p^2+p})x_1{}^{p^2} - (x_0{}^p - x_0{}^{p^2})x_1{}^{p^2+p}.$$

This shows that a basis as a $k[x_0]$ module of A_2 consists of $1, x_1, x_1{}^2, \ldots, x_1{}^{p^2+p}$ since every $x_1{}^q{}_{q \geq p^2+p+1}$ may be expressed (according to the above relation) into powers of x_1 smaller than $p^2 + p + 1$.

Similarly, by the definition of $A_{1,1}$:

$$A_{1,1} = \frac{k[x_0, x_1, x_2]}{x_2{}^{p+1} + x_1{}^{p+1} - x_1x_2 - x_1{}^p x_2{}^p = 0, x_1{}^{p+1} + x_0{}^{p+1} - x_1x_0 - x_1{}^p x_0{}^p = 0}$$

so then we have that

$$x_2{}^{p+1} + x_1{}^{p+1} - x_1x_2 - x_1{}^p x_2{}^p = 0 \quad , \quad x_1{}^{p+1} + x_0{}^{p+1} - x_1x_0 - x_1{}^p x_0{}^p = 0$$

which can be rewritten as

$$x_2{}^{p+1} = -x_1{}^{p+1} + x_1x_2 + x_1{}^p x_2{}^p \quad , \quad x_1{}^{p+1} = -x_0{}^{p+1} + x_1x_0 + x_1{}^p x_0{}^p. \tag{2.1}$$

If an element in $A_{1,1}$ contains an $x_2{}^q$ or $x_1{}^q$ where $q \geq p+1$, then it must be reduced into powers of x_1 and x_2 which are smaller than $p+1$, according to (2.1). This then shows that a basis for $A_{1,1}$ consists of all $x_1{}^i x_2{}^j$ where $0 \leq i, j \leq p$.

$$u(x_1{}^k) = u(x_1)^k = x_2{}^k = \begin{cases} x_2{}^k & \text{if } 0 \leq k \leq p \\ \displaystyle\sum_{i=0}^{p}\sum_{j=0}^{p} l_{ijk}(x_0)x_1{}^i x_2{}^j & \text{if } k \geq p+1. \end{cases}$$

The l_{ijk}'s are explicitly found using (2.1), the modding out relations of $A_{1,1}$.

Proposition 1.

$$x_1{}^{np+k} = A_{n,k}(x_0) + B_{n,k}(x_0)x_1{}^k + C_{n,k}(x_0)x_1{}^p \quad \textit{if} \;\; 1 \le n \le p \;\; \textit{and} \;\; 0 \le k \le p \tag{2.2}$$

$$\textit{where} \;\; A_{n,k} = \begin{cases} -\sum_{j=1}^{n} x_0{}^{(n-j)p^2+(kp+j)} & \textit{if } k \ge 1, n \ge 2 \\ -\sum_{j=1}^{n-1} x_0{}^{(n-j)p^2+j} & \textit{if } k = 0, n \ge 2 \end{cases} \tag{2.3}$$

$$B_{n,k} = \begin{cases} x_0{}^n & \textit{if } k \ge 1, n \ge 2 \\ 0 & \textit{if } k = 0, n \ge 2. \end{cases} \tag{2.4}$$

$$C_{n,k} = \sum_{j=1}^{n} x_0{}^{(n-j)p^2+kp+(j-1)} \quad \textit{if } k \ge 0, n \ge 2 \tag{2.5}$$

$$x_2{}^{np+k} = \tilde{A}_{n,k}(x_0)x_1{}^p x_2{}^p + \sum_{j=1}^{n-1} x_0{}^{k+(j-1)p} x_1{}^{n-j} x_2{}^p + \tilde{B}_{n,k}(x_0)x_2{}^p + x_1{}^n x_2{}^k + \tilde{C}_{n,k}(x_0)x_1{}^p$$
$$- \sum_{j=1}^{n} x_0{}^{k+(j-1)p} x_1{}^{n-j+1} + \tilde{D}_{n,k}(x_0) \quad \textit{if} \;\; 1 \le n \le p \;\; \textit{and} \;\; 0 \le k \le p \tag{2.6}$$

$$\textit{where} \;\; \tilde{A}_{n,k} = \sum_{l=0}^{k-1} n \cdot x_0{}^{(n-1)p+(k-1)+l(p^2-1)} + \sum_{j=0}^{n-2} \sum_{l=0}^{p-1} (n-j-1) x_0{}^{(n-1)p+(k-1)+(k+l+jp)(p^2-1)} \quad \textit{if } k \ge 0, n \ge 2 \tag{2.7}$$

$$\tilde{B}_{n,k} = -x_0 \cdot \tilde{A}_{n,k} + x_0{}^{(n-1)p+k} \quad \textit{if } k \ge 0, n \ge 2 \tag{2.8}$$

$$\tilde{C}_{n,k} = -x_0^p \tilde{A}_{n,k} \quad \textit{if } k \ge 0, n \ge 2 \tag{2.9}$$

$$\tilde{D}_{n,k} = x_0{}^{p+1} \tilde{A}_{n,k} \quad \textit{if } k \ge 0, n \ge 2 \tag{2.10}$$

Proof:
The above formula for $x^q_{1_{q \ge p+1}}$ can be expressed as three separate equations:

$$x_1{}^{p+k}{}_{1 \le k \le p-1} = (-x_0{}^{kp+1}) + (x_0)x_1{}^k + (x_0{}^{kp})x_1{}^p \tag{2.11}$$

$$x^{np}_{1_{2 \le n \le p+1}} = A_{n,0}(x_0) + C_{n,0}(x_0)x_1{}^p \tag{2.12}$$

$$x_1{}^{np+k}{}_{2 \le n \le p, 1 \le k \le p-1} = A_{n,k}(x_0) + B_{n,k}(x_0)x_1{}^k + C_{n,k}(x_0)x_1{}^p \tag{2.13}$$

We will prove each of these formulas individually and, by doing so, we will have proven the general formula for ${x_1^q}_{q\geq p+1}$.

From (2.1), we can use the relation

$$x_1^{p+1} = -x_0^{p+1} + x_1x_0 + x_1^p x_0^p$$

in order to derive a formula for ${x_1^q}_{q\geq p+1}$. Multiply both sides of the above relation by ${x_1^{k-1}}_{1\leq k\leq p+1}$ which yields

$$x_1^{p+k} = -x_0^{p+1}x_1^{k-1} + x_0x_1^k + x_0^p x_1^{p+(k-1)}.$$

Now let $g_k := x_1^{p+k}$, for $k \geq 0$, which yields

$$g_k = -x_0^{p+1}x_1^{k-1} + x_1^k x_0 + x_0^p g_{k-1} \quad , \; k \geq 1$$

and letting $k \to k-1$ (assuming $k \geq 2$), we obtain a formula for g_{k-1}

$$g_{k-1} = -x_0^{p+1}x_1^{k-2} + x_0x_1^{k-1} + x_0^p g_{k-2}.$$

Now plugging the formula for g_{k-1} into g_k and simplifying we get

$$g_k = -x_0^{2p+1}x_1^{k-2} + x_0x_1^k + x_0^{2p} g_{k-2}.$$

Now repeating this procedure $j-1$ times of plugging our recursion into itself, we obtain

$$g_k = -x_0^{jp+1}x_1^{k-j} + x_0x_1^k + x_0^{jp} g_{k-j} \;\; , \; j \leq k \; . \tag{2.14}$$

Now setting $j = k$ in (2.14), we obtain that

$$g_k = -x_0^{kp+1}x_1^0 + x_0x_1^k + x_0^{kp} g_0.$$

Since $g_k = x_1^{p+k}$ then this implies that

$$x_1^{p+k} = -x_0^{kp+1} + x_0x_1^k + x_0^{kp}x_1^p \text{ for } 0 < k < p. \tag{2.15}$$

This proves (2.11). In order to prove (2.12), we will do a proof by induction.

Base Case: $n = 2$

First, we must prove the base case true in which

$$x_1^{np}|_{n=2} = x_1^{2p} = A_{2,0} + C_{2,0}x_1^p.$$

where

$$A_{2,0} = -x_0^{p^2+1} \quad , \quad C_{2,0} = x_0 + x_0^{p^2}.$$

Using (2.15) and evaluating this formula at $k = p - 1$, then we obtain

$$x_1{}^{p+k}|_{k=p-1} = x_1{}^{p+p-1} = -x_0{}^{(p-1)p+1} + x_0 x_1{}^{p-1} + x_0{}^{(p-1)p} x_1{}^p.$$

Now mulitplying both sides by x_1 of the above equation, then we obtain

$$x_1{}^{2p} = -x_0{}^{(p-1)p+1} x_1 + x_0 x_1{}^p + x_0{}^{(p-1)p} x_1{}^{p+1}. \quad (2.16)$$

Now substituting (2.1), our relation for $x_1{}^{p+1}$, into the right-hand side of (2.16) yields

$$\begin{aligned} x_1{}^{2p} &= -x_0{}^{(p-1)p+1} x_1 + x_0 x_1{}^p + x_0{}^{(p-1)p}(-x_0{}^{p+1} + x_1 x_0 + x_1{}^p x_0{}^p) && (2.17)\\ &= -x_0{}^{p^2+1} + (x_0 + x_0{}^{p^2}) x_1{}^p && (2.18)\\ &= A_{2,0} + C_{2,0} x_1{}^p && (2.19)\end{aligned}$$

This proves the base case.

Inductive hypothesis:

Lets assume

$$x_1{}^{np} = A_{n,0} + C_{n,0} x_1{}^p \quad (2.20)$$

holds for some n, in which $n \geq 2$.

Inductive step:

Lets see if this equation holds for $n + 1$. Multiply both sides of the previous equation by $x_1{}^p$, then we obtain

$$x_1{}^{(n+1)p} = A_{n,0} x_1{}^p + C_{n,0} x_1{}^{2p}. \quad (2.21)$$

Now plugging in (2.18), the formula for $x_1{}^{2p}$, into the right-hand side of (2.21), we obtain

$$x_1{}^{(n+1)p} = A_{n,0} x_1{}^p + C_{n,0}(-x_0{}^{p^2+1} + (x_0 + x_0{}^{p^2}) x_1{}^p). \quad (2.22)$$

(2.22) can be rewritten as

$$x_1{}^{(n+1)p} = -x_0{}^{p^2+1} C_{n,0} + (A_{n,0} + (x_0 + x_0{}^{p^2}) C_{n,0}) x_1{}^p$$

and it suffices to prove that

$$A_{n+1,0} = -x_0{}^{p^2+1} C_{n,0} \text{ and } C_{n+1,0} = A_{n,0} + (x_0 + x_0{}^{p^2}) C_{n,0}.$$

if

$$x_1{}^{(n+1)p} = A_{n+1,0} + C_{n+1,0} x_1{}^p.$$

To prove the first relation:

$$\begin{aligned}
-{x_0}^{p^2+1}C_{n,0} &= -{x_0}^{p^2+1}\cdot\left(\sum_{j=1}^{n} {x_0}^{(n-j)p^2+(j-1)}\right) && \text{from (2.5)}\\
&= -\sum_{j=1}^{n} {x_0}^{(n-j+1)p^2+j}\\
&= -\sum_{j=1}^{n} {x_0}^{((n+1)-j)p^2+j}\\
&= A_{n+1,0}
\end{aligned}$$

To prove the second relation:

$$\begin{aligned}
A_{n,0}+(x_0+{x_0}^{p^2})C_{n,0} &= A_{n,0}+x_0\cdot C_{n,0}+{x_0}^{p^2}\cdot C_{n,0}\\
&= {x_0}^{n}+{x_0}^{p^2}\cdot C_{n,0} && \text{from(2.3) and (2.5)}\\
&= {x_0}^{n}+{x_0}^{p^2}\cdot\left(\sum_{j=1}^{n} {x_0}^{(n-j)p^2+(j-1)}\right) && \text{from (2.5)}\\
&= {x_0}^{n}+\sum_{j=1}^{n} {x_0}^{(n-j+1)p^2+(j-1)}\\
&= {x_0}^{n}+\sum_{j=1}^{n} {x_0}^{((n+1)-j)p^2+(j-1)}\\
&= \sum_{j=1}^{n+1} {x_0}^{((n+1)-j)p^2+(j-1)}\\
&= C_{n+1,0}
\end{aligned}$$

This proves the formula for (2.12). In order to prove (2.13), we will do a proof by induction over k.

Base case: $k=1$

$$\begin{aligned}
{x_1}^{np+1} &= x_1{x_1}^{np}\\
&= x_1(A_{n,0}+C_{n,0}{x_1}^{p}) && \text{from (2.20)}\\
&= x_1A_{n,0}+C_{n,0}{x_1}^{p+1}\\
&= x_1A_{n,0}+C_{n,0}(-{x_0}^{p+1}+x_0x_1+{x_1}^{p}{x_0}^{p}) && \text{from (2.1)}\\
&= -{x_0}^{p+1}C_{n,0}+(A_{n,0}+x_0C_{n,0})x_1+C_{n,0}{x_0}^{p}{x_1}^{p}\\
&= -{x_0}^{p+1}C_{n,0}+{x_0}^{n}x_1+C_{n,0}{x_0}^{p}{x_1}^{p} && \text{from (2.5) and (2.3)}
\end{aligned}$$

and it suffices to prove that

$$-{x_0}^{p+1}C_{n,0} = A_{n,1} \;\; , \;\; {x_0}^n = B_{n,1} \;\; , \;\; {x_0}^p C_{n,0} = C_{n,1}$$

to show that

$${x_1}^{np+1} = A_{n,1} + B_{n,1}x_1 + C_{n,1}{x_1}^p.$$

To prove the first relation:

$$\begin{aligned} -{x_0}^{p+1}C_{n,0} &= -{x_0}^{p+1} \cdot \sum_{j=1}^{n} {x_0}^{(n-j)p^2+(j-1)} \qquad \text{from (2.5)} \\ &= -\sum_{j=1}^{n} {x_0}^{(n-j)p^2+p+j} \\ &= A_{n+1,0}. \end{aligned}$$

The second relation holds true because by definition

$$B_{n,1} = {x_0}^n. \qquad \text{from (2.4)}$$

To prove the third relation:

$$\begin{aligned} {x_0}^p \cdot C_{n,0} &= {x_0}^p \cdot \left(\sum_{j=1}^{n} {x_0}^{(n-j)p^2+(j-1)} \right) \qquad \text{from (2.5)} \\ &= \sum_{j=1}^{n} {x_0}^{(n-j)p^2+p+(j-1)} \\ &= C_{n,1}. \end{aligned}$$

Having proved these three relations, then this proves our base case.

Inductive Hypothesis:
Lets assume that

$${x_1}^{np+k} = A_{n,k} + B_{n,k}{x_1}^k + C_{n,k}{x_1}^p$$

holds true for some k, where $1 \leq k \leq p-2$.

Inductive Step:
Lets see if the formula holds true for $k+1$. Multiply both sides of our equation by x_1 then

$${x_1}^{np+(k+1)} = A_{n,k}x_1 + B_{n,k}{x_1}^{k+1} + C_{n,k}{x_1}^{p+1}$$

Now plugging (2.1), the relation for ${x_1}^{p+1}$, into the right-hand side of ${x_1}^{np+(k+1)}$ and grouping terms we obtain

$${x_1}^{np+(k+1)} = -{x_0}^{p+1}C_{n,k} + (A_{n,k} + x_0 C_{n,k})\, x_1 + B_{n,k}{x_1}^{k+1} + {x_0}^{p}C_{n,k}{x_1}^{p}.$$

It suffices to show that

$$-{x_0}^{p+1}C_{n,k} = A_{n,k+1} \quad , \quad A_{n,k} + x_0 \cdot C_{n,k} = 0 \quad , \quad B_{n,k} = B_{n,k+1} \quad , \quad {x_0}^{p}C_{n,k} = C_{n,k+1}$$

to prove that

$${x_1}^{np+(k+1)} = A_{n,k+1} + B_{n,k+1}x_1^{k+1} + C_{n,k+1}x_1^{p}.$$

Proving the first relation:

$$\begin{aligned} -{x_0}^{p+1}C_{n,k} &= -{x_0}^{p+1} \cdot \left(\sum_{j=1}^{n} {x_0}^{(n-j)p^2+(kp+j-1)} \right) \qquad \text{from (2.5)} \\ &= -\sum_{j=1}^{n} {x_0}^{(n-j)p^2+(k+1)p+j} \\ &= A_{n,k+1} \end{aligned}$$

Proving the second relation:

$$\begin{aligned} A_{n,k} + x_0 \cdot C_{n,k} &= -\sum_{j=1}^{n} {x_0}^{(n-j)p^2+kp+j} + x_0 \cdot \left(\sum_{j=1}^{n} {x_0}^{(n-j)p^2+kp+(j-1)} \right) \qquad \text{from (2.3) and (2.5)} \\ &= -\sum_{j=1}^{n} {x_0}^{(n-j)p^2+kp+j} + \sum_{j=1}^{n} {x_0}^{(n-j)p^2+kp+j} \\ &= 0. \end{aligned}$$

By definition of (2.4):

$$B_{n,k+1} = B_{n,k} = {x_0}^{n} \qquad \text{for } 0 < k < p-1$$

which proves the third relation.

Proving the fourth relation:

$$\begin{aligned} {x_0}^p C_{n,k} &= {x_0}^p \cdot \left(\sum_{j=1}^{n} {x_0}^{(n-j)p^2+kp+(j-1)} \right) \quad \text{from (2.5)} \\ &= \sum_{j=1}^{n} {x_0}^{(n-j)p^2+(k+1)p+(j-1)} \\ &= C_{n,k+1}. \end{aligned}$$

Having proved all four relations, then this proves (2.13).

Now to prove the formulas for $x_{2q\geq p+1}^{q}$.The formula for $x_{2q\geq p+1}^{q}$ can be expressed as three separate equations:

$$ {x_2}^{p+k} = C_{k,0}(x_0){x_1}^p{x_2}^p + A_{k,0}(x_0){x_2}^p + x_1{x_2}^k - C_{k,1}(x_0){x_1}^p - B_{k,1}(x_0)x_1 - A_{k,1}(x_0) \quad (2.23)$$

$$ {x_2}^{np} = \tilde{A}_{n,0}(x_0){x_1}^p{x_2}^p + {x_1}^{n-1}{x_2}^p + \sum_{j=1}^{n-2} {x_0}^{jp}{x_1}^{n-j-1}{x_2}^p + \tilde{B}_{n,0}(x_0){x_2}^p + \tilde{C}_{n,0}(x_0){x_1}^p - \sum_{j=1}^{n-1} {x_0}^{jp}{x_1}^{n-j} + \tilde{D}_{n,0}(x_0) \quad (2.24)$$

$$ {x_2}^{np+k} = \tilde{A}_{n,k}(x_0){x_1}^p{x_2}^p + \sum_{j=1}^{n-1} {x_0}^{k+(j-1)p}{x_1}^{n-j}{x_2}^p + \tilde{B}_{n,k}(x_0){x_2}^p + {x_1}^n{x_2}^k + \tilde{C}_{n,k}(x_0){x_1}^p - \sum_{j=1}^{n} {x_0}^{k+(j-1)p}{x_1}^{n-j+1} + \tilde{D}_{n,k}(x_0) \quad (2.25)$$

We will prove each of these formulas individually and, by doing so, we will have proven the general formula for ${x_2}^q{}_{q\geq p+1}$.

We can use (2.1), our modding out relations, in order to proof the formulas for (2.23), (2.24), and (2.25). Since the modding out relation

$$ {x_1}^{p+1} = -{x_0}^{p+1} + x_0x_1 + {x_0}^p{x_1}^p \quad \text{is symmetric to} \quad {x_2}^{p+1} = -{x_1}^{p+1} + x_1x_2 + {x_1}^p{x_2}^p $$

we can let

$$(x_0, x_1) \to (x_1, x_2)$$

and use the formulas for ${x_1}^q{}_{q\geq p+1}$ to find the formulas for ${x_2}^q{}_{q\geq p+1}$. By plugging in x_1 for x_0 and x_2 for x_1 into (2.11), (2.12), and (2.13), we obtain

$$ {x_2}^{p+k} = (-{x_1}^{kp+1}) + (x_1){x_2}^k + ({x_1}^{kp}){x_2}^p \tag{2.26}$$

$$ {x_2}^{np} = A_{n,0}(x_1) + C_{n,0}(x_1){x_2}^p \tag{2.27}$$

$$ {x_2}^{np+k} = A_{n,k}(x_1) + B_{n,k}(x_1){x_2}^k + C_{n,k}(x_1){x_2}^p \tag{2.28}$$

Now reducing (2.26) in x_1 according to (2.11), (2.12), and (2.13) yields

$$\begin{aligned} {x_2}^{p+k} &= (-{x_1}^{kp+1}) + (x_1){x_2}^k + ({x_1}^{kp}){x_2}^p \\ &= -(A_{k,1}(x_0) + B_{k,1}(x_0)x_1 + C_{k,1}(x_0){x_1}^p) + (x_1){x_2}^k + (A_{k,0}(x_0) + C_{k,0}(x_0){x_1}^p){x_2}^p. \end{aligned}$$

The previous relation is exactly (2.6) evaluated at $n = 1$, so this proves (2.23). We will now prove (2.24) and (2.25) together, namely the formula for $x_2^{np+k}{}_{2\leq n\leq p, 0\leq k\leq p}$.

$$ {x_2}^{np+k} = A_{n,k}(x_1) + B_{n,k}(x_1){x_2}^k + C_{n,k}(x_1){x_2}^p \tag{2.29}$$

Lets now reduce the coefficients $A_{n,k}(x_1), B_{n,k}(x_1), C_{n,k}(x_1)$ according to the relations derived for ${x_1}^q{}_{q\geq p+1}$:

$$\begin{aligned} A_{n,k}(x_1) &= -\sum_{j=1}^{n} {x_1}^{[(n-j)p+k]\cdot p+j} \\ &= -\sum_{j=1}^{n} \left(A_{(n-j)p+k,j} + B_{(n-j)p+k,j}{x_1}^j + C_{(n-j)p+k,j}{x_1}^p\right) \\ &= -\left(\sum_{j=1}^{n} A_{(n-j)p+k,j}\right) - \sum_{j=1}^{n} B_{(n-j)p+k,j}{x_1}^j - \left(\sum_{j=1}^{n} C_{(n-j)p+k,j}\right){x_1}^p \end{aligned}$$

The relation for $B_{n,k}$ is simple since there is no reduction

$$B_{n,k}(x_1) = {x_1}^n.$$

Now for the last coefficient $C_{n,k}$:

$$\begin{aligned}
C_{n,k}(x_1) &= \sum_{j=1}^{n} {x_1}^{(n-j)p^2+kp+(j-1)} \\
&= \sum_{j=1}^{n} {x_1}^{[(n-j)p+k]p+(j-1)} \\
&= \sum_{j=1}^{n} \left(A_{(n-j)p+k,j-1} + B_{(n-j)p+k,j-1}{x_1}^{j-1} + C_{(n-j)p+k,j-1}{x_1}^{p}\right) \\
&= \sum_{j=1}^{n} \left(A_{(n-j)p+k,j-1} + B_{(n-j)p+k,j-1}{x_1}^{j-1} + C_{(n-j)p+k,j-1}{x_1}^{p}\right) \\
&= \left(\sum_{j=1}^{n} A_{(n-j)p+k,j-1}\right) + \sum_{j=1}^{n} B_{(n-j)p+k,j-1}{x_1}^{j-1} + \left(\sum_{j=1}^{n} C_{(n-j)p+k,j-1}\right){x_1}^{p}
\end{aligned}$$

Now plugging in the relations for $A_{n,k}(x_1), B_{n,k}(x_1), C_{n,k}(x_1)$ into the right-hand side of (2.29), the formula for ${x_2}^{np+k}$, yields

$$\begin{aligned}
{x_2}^{np+k} = \left(\sum_{j=1}^{n} C_{(n-j)p+k,j-1}\right){x_1}^{p}{x_2}^{p} + \sum_{j=1}^{n} B_{(n-j)p+k,j-1}{x_1}^{j-1}{x_2}^{p} + \left(\sum_{j=1}^{n} A_{(n-j)p+k,j-1}\right){x_2}^{p} + {x_1}^{n}{x_2}^{k} \\
- \left(\sum_{j=1}^{n} C_{(n-j)p+k,j}\right){x_1}^{p} - \sum_{j=1}^{n} B_{(n-j)p+k,j}{x_1}^{j} - \left(\sum_{j=1}^{n} A_{(n-j)p+k,j}\right). \quad (2.30)
\end{aligned}$$

(2.30) must be equivalent to our formula for x_2^{np+k}

$$\begin{aligned}
{x_2}^{np+k} = \tilde{A}_{n,k}(x_0){x_1}^{p}{x_2}^{p} + \sum_{j=1}^{n-1} {x_0}^{k+(j-1)p}{x_1}^{n-j}{x_2}^{p} + \tilde{B}_{n,k}(x_0){x_2}^{p} + {x_1}^{n}{x_2}^{k} + \tilde{C}_{n,k}(x_0){x_1}^{p} \\
- \sum_{j=1}^{n} {x_0}^{k+(j-1)p}{x_1}^{n-j+1} + \tilde{D}_{n,k}(x_0)
\end{aligned}$$

so then these relations must hold if (2.25) is true:

$$\tilde{A}_{n,k} = \sum_{j=1}^{n} C_{(n-j)p+k,j-1}$$

$$\tilde{B}_{n,k} = \sum_{j=1}^{n} A_{(n-j)p+k,j-1}$$

$$\tilde{C}_{n,k} = -\sum_{j=1}^{n} C_{(n-j)p+k,j}$$

$$\tilde{D}_{n,k} = -\sum_{j=1}^{n} A_{(n-j)p+k,j}$$

Now we will prove that the above relations hold:

Verification of $\tilde{A}_{n,k}$

$$\begin{aligned}
\tilde{A}_{n,k} &= \sum_{j=1}^{n} C_{(n-j)p+k,j-1} \\
&= \sum_{j=0}^{n-1} C_{jp+k,n-j-1} \\
&= \sum_{j=0}^{n-1} \sum_{q=1}^{jp+k} {x_0}^{(jp+k-q)p^2+(n-j-1)p+(q-1)} \\
&= \sum_{l=0}^{k-1} n \cdot {x_0}^{(n-1)p+(k-1)+l(p^2-1)} + \sum_{j=0}^{n-2} \sum_{l=0}^{p-1} (n-j-1){x_0}^{(n-1)p+(k-1)+(k+l+jp)(p^2-1)}
\end{aligned}$$

Verification of $\tilde{B}_{n,k}$

$$\begin{aligned}
\tilde{B}_{n,k} &= \sum_{j=1}^{n} A_{(n-j)p+k,j-1} \\
&= \sum_{j=0}^{n-1} A_{jp+k,n-j-1} \\
&= A_{(n-1)p+k,0} + \sum_{j=0}^{n-2} A_{jp+k,n-j-1} \\
&= -x_0 C_{(n-1)p+k,0} + {x_0}^{(n-1)p+k} + \sum_{j=0}^{n-2} (-x_0) \cdot C_{jp+k,n-j-1} \\
&= {x_0}^{(n-1)p+k} + \sum_{j=0}^{n-1} (-x_0) \cdot C_{jp+k,n-j-1} \\
&= {x_0}^{(n-1)p+k} - x_0 \cdot \sum_{j=0}^{n-1} C_{jp+k,n-j-1} \\
&= {x_0}^{(n-1)p+k} - x_0 \cdot \tilde{A}_{n,k}
\end{aligned}$$

Verification of $\tilde{C}_{n,k}$

$$\begin{aligned}
\tilde{C}_{n,k} &= -\sum_{j=1}^{n} C_{(n-j)p+k,j} \\
&= -\sum_{j=0}^{n-1} C_{jp+k,n-j} \\
&= -\sum_{j=0}^{n-1}\sum_{q=1}^{jp+k} {x_0}^{(jp+k-q)p^2+(n-j)p+(q-1)} \\
&= -{x_0}^p \cdot \sum_{j=0}^{n-1} C_{jp+k,n-j-1} \\
&= -{x_0}^p \cdot \tilde{A}_{n,k}
\end{aligned}$$

Verification of $\tilde{D}_{n,k}$

$$\begin{aligned}
\tilde{D}_{n,k} &= -\sum_{j=1}^{n} A_{(n-j)p+k,j} \\
&= -\sum_{j=1}^{n} (-x_0) \cdot C_{(n-j)p+k,j} \\
&= x_0 \cdot \sum_{j=1}^{n} C_{(n-j)p+k,j} \\
&= x_0 \cdot \left({x_0}^p \tilde{A}_{n,k}\right) \\
&= {x_0}^{p+1} \tilde{A}_{n,k}
\end{aligned}$$

Since we have verified the coefficient relations for $\tilde{A}_{n,k}, \tilde{B}_{n,k}, \tilde{C}_{n,k}, \tilde{D}_{n,k}$, this then proves (2.25). Having verified the formulas for ${x_1}^q{}_{q\geq p+1}$ and ${x_2}^q{}_{q\geq p+1}$, then we have completely proven Proposition 1.

□

3 Computation of the kernel of u^*

Proposition 2. *The map u is injective.*

Proof:
Now we will give a proof by contradiction that u is injective by using the general formula for ${x_2}^q_{q\geq p+1}$. Let $z \in A_2$ then

$$z = a_0(x_0) + a_1(x_0)x_1 + \ldots + a_{p^2+p}(x_0){x_1}^{p^2+p}$$

$$\text{Suppose } \ u(z) = 0 \ \text{ and } \ z \neq 0.$$

Take m maximal s.t.

$$z = {x_0}^m z' \quad \text{where} \quad z' = \sum {a_i}'(x_0){x_1}^i.$$

There exists a j where $j = np + k_{1 \leq k \leq p,\ 1 \leq n \leq p}$ s.t.

$${a_j}' = C + (\text{terms involving } x_0) \quad , \quad \text{where } \ C \neq 0.$$

Since

$$0 = u(z) = u({x_0}^m z') = {x_0}^m u(z') \quad \Rightarrow \quad u(z') = 0.$$

If $u(z') = 0$, then equivalently ${a_j}' = 0$. If $a_j' = 0$, then

$$\begin{gathered} {a_j}' = 0 \\ C + (\text{terms involving } x_0) = 0 \\ C = -\,(\text{terms involving } x_0) \end{gathered}$$

This is a contradiction since the left-hand side is a non-zero constant and the right-hand side is terms involving x_0. So then $u(z) = 0$ if and only if $z = 0$, which is an equivalent statement as u being injective.

□

Theorem 1. *Consider the dual map*

$${A_{1,1}}^* \xrightarrow{u^*} {A_2}^*$$

The kernel of u^* has basis:

$$P_iP_0 + x_0P_iP_1 + \ldots + {x_0}^pP_iP_p \quad \textit{for } 1 \leq i \leq p$$

where

$$P_iP_j \in A^*_{1,1} \quad \textit{and} \quad P_iP_j\,({x_1}^m{x_2}^n) = \begin{cases} 1 & \textit{if } i = m \textit{ and } j = n \\ 0 & \textit{otherwise} \end{cases}$$

Proof:
As was shown previously:

$$u({x_1}^k) = u(x_1)^k = x_2^k = \begin{cases} {x_2}^k & \text{for } 0 \le k \le p \\ \sum_{i=0}^{p}\sum_{j=0}^{p} l_{ijk}{x_1}^i{x_2}^j & \text{for } k \ge p+1 \end{cases}$$

Since $P_iP_j({x_1}^m{x_2}^n) = 1$ if and only if $i = m$ and $j = n$ and 0 otherwise, then

$$(P_iP_0 + x_0P_iP_1 + \ldots + {x_0}^pP_iP_p)\sum_{i=0}^{p}\sum_{j=0}^{p} l_{ijk}{x_1}^i{x_2}^j = \sum_{j=0}^{p}{x_0}^j l_{ijk}(x_0) = 0$$

This must hold true for any k where $0 \le k \le p^2 + p$ and also for all i s.t. $1 \le i \le p$. We will prove this true by using case-work analysis.

Case ${x_2}^k{}_{0\le k\le p}$ for $1 \le i \le p$:

$$\sum_{j=0}^{p}{x_0}^j l_{ijk}(x_0) = \sum_{j=0}^{p}{x_0}^j \cdot 0 = 0$$

This is true because $l_{ijk} = 0$ for $0 \le k \le p$ and $1 \le i \le p$.

Case ${x_2}^{p+k}{}_{0<k<p}$ for $1 \le i \le p$:

Subcase $i = 1$:

$$\begin{aligned}\sum_{j=0}^{p}{x_0}^j l_{1jk}(x_0) &= -B_{k,1} + x_0 \cdot 0 + \ldots + {x_0}^{k-1} \cdot 0 + {x_0}^k \cdot 1 + {x_0}^{k+1} \cdot 0 + \ldots + {x_0}^p \cdot 0 \\ &= -{x_0}^k + 0 + {x_0}^k + 0 = 0\end{aligned}$$

Subcase $2 \le i \le p-1$:

$$\sum_{j=0}^{p}{x_0}^j l_{ijk}(x_0) = \sum_{j=0}^{p}{x_0}^j \cdot 0 = 0$$

Subcase $i = p$:

$$\begin{aligned}\sum_{j=0}^{p}{x_0}^j l_{pjk}(x_0) &= -C_{k,1} + x_0 \cdot 0 + \ldots + {x_0}^{p-1} \cdot 0 + {x_0}^p \cdot C_{k,0} \\ &= -{x_0}^p C_{k,0} + 0 + {x_0}^p \cdot C_{k,0} = 0 \text{ from (2.5)}\end{aligned}$$

Case ${x_2^{np}}_{2\leq n\leq p}$ for $1 \leq i \leq p$:

Subcase $i = 1$:

$$\begin{aligned}\sum_{j=0}^{p} {x_0}^j l_{1jk}(x_0) &= -{x_0}^{(n-1)p} + x_0 \cdot 0 + \ldots + {x_0}^{p-1} \cdot 0 + {x_0}^p \cdot {x_0}^{(n-2)p} \\ &= -{x_0}^{(n-1)p} + 0 + {x_0}^{(n-1)p} = 0\end{aligned}$$

Subcase $2 \leq i \leq n-2$:

$$\begin{aligned}\sum_{j=0}^{p} {x_0}^j l_{ijk}(x_0) &= -{x_0}^{(n-k)p} + x_0 \cdot 0 + \ldots + {x_0}^{p-1} \cdot 0 + {x_0}^p \cdot {x_0}^{(n-k-1)p} \\ &= -{x_0}^{(n-k)p} + 0 + {x_0}^{(n-k)p} = 0\end{aligned}$$

Subcase $i = n-1$:

$$\begin{aligned}\sum_{j=0}^{p} {x_0}^j l_{(n-1)jk}(x_0) &= -{x_0}^p + x_0 \cdot 0 + \ldots + {x_0}^{p-1} \cdot 0 + {x_0}^p \cdot 1 \\ &= -{x_0}^p + 0 + {x_0}^p = 0\end{aligned}$$

Subcase $n \leq i \leq p-1$:

$$\begin{aligned}\sum_{j=0}^{p} {x_0}^j l_{ijk}(x_0) &= -{x_0}^p + x_0 \cdot 0 + \ldots + {x_0}^{p-1} \cdot 0 + {x_0}^p \cdot 1 \\ &= -{x_0}^p + 0 + {x_0}^p = 0\end{aligned}$$

Subcase $i = p$:

$$\begin{aligned}\sum_{j=0}^{p} {x_0}^j l_{pjk}(x_0) &= \tilde{C}_{n,0} + x_0 \cdot 0 + \ldots + {x_0}^{p-1} \cdot 0 + {x_0}^p \cdot \tilde{A}_{n,0} \\ &= -{x_0}^p \cdot \tilde{A}_{n,0} + 0 + {x_0}^p \cdot \tilde{A}_{n,0} = 0 \quad \text{from (2.9)}\end{aligned}$$

Case ${x_2^{np+k}}_{2\leq n\leq p, 0<k<p}$ for $1 \leq i \leq p$:

Subcase $i = 1$:

$$\begin{aligned}\sum_{j=0}^{p} {x_0}^j l_{1jk}(x_0) &= -{x_0}^{k+(n-1)p} + x_0 \cdot 0 + \ldots + {x_0}^p \cdot {x_0}^{k+(n-2)p} \\ &= -{x_0}^{k+(n-1)p} + 0 + {x_0}^{k+(n-1)p} = 0\end{aligned}$$

Subcase $2 \leq i \leq n-1$:

$$\begin{aligned}\sum_{j=0}^{p} {x_0}^j l_{ijk}(x_0) &= -{x_0}^{k+(n-k)p} + x_0 \cdot 0 + \ldots + {x_0}^{p-1} \cdot 0 + {x_0}^p \cdot {x_0}^{k+(n-k-1)p} \\ &= -{x_0}^{k+(n-k)p} + 0 + {x_0}^{k+(n-k)p} = 0\end{aligned}$$

Subcase $i = n$:

$$\begin{aligned}\sum_{j=0}^{p} {x_0}^j l_{njk}(x_0) &= -{x_0}^k + x_0 \cdot 0 + \ldots + {x_0}^{k-1} \cdot 0 + {x_0}^k \cdot 1 + {x_0}^{k+1} \cdot 0 + \ldots + {x_0}^p \cdot 0 \\ &= -{x_0}^k + 0 + {x_0}^k + 0 = 0\end{aligned}$$

Subcase $n+1 \leq i \leq p-1$:

$$\sum_{j=0}^{p} {x_0}^j l_{ijk}(x_0) = \sum_{j=0}^{p} {x_0}^j \cdot 0 = 0$$

Subcase $i = p$:

$$\begin{aligned}\sum_{j=0}^{p} {x_0}^j l_{pjk}(x_0) &= \tilde{C}_{n,k} + x_0 \cdot 0 + \ldots + {x_0}^{p-1} \cdot 0 + {x_0}^p \cdot \tilde{A}_{n,k} \\ &= -{x_0}^p \cdot \tilde{A}_{n,k} + 0 + {x_0}^p \cdot \tilde{A}_{n,k} = 0 \text{ from (2.9)}\end{aligned}$$

Case: ${x_2}^{(p+1)p}$ for $1 \leq i \leq p$:

Subcase $i = 1$:

$$\begin{aligned}\sum_{j=0}^{p} {x_0}^j l_{1jk}(x_0) &= -{x_0}^{p^2} + x_0 \cdot 0 + \ldots + {x_0}^{p-1} \cdot 0 + {x_0}^p \cdot \left({x_0}^{p^2-p}\right) \\ &= -{x_0}^{p^2} + 0 + {x_0}^{p^2} = 0\end{aligned}$$

Subcase $2 \leq i \leq p-1$:

$$\begin{aligned}\sum_{j=0}^{p} {x_0}^j l_{ijk}(x_0) &= -{x_0}^{(p+1-k)p} + x_0 \cdot 0 + \ldots + {x_0}^{p-1} \cdot 0 + {x_0}^p \cdot \left({x_0}^{(p-k)p}\right) \\ &= -{x_0}^{(p+1-k)p} + 0 + {x_0}^{(p+1-k)p} = 0\end{aligned}$$

Subcase $i = p$:

$$\begin{aligned}\sum_{j=0}^{p} {x_0}^j l_{pjk}(x_0) &= \left(\tilde{C}_{p+1,0} - {x_0}^p\right) + x_0 \cdot 0 + \ldots + {x_0}^{p-1} \cdot 0 + {x_0}^p \cdot \left(\tilde{A}_{p+1,0} + 1\right) \\ &= \tilde{C}_{p+1,0} - {x_0}^p + 0 + {x_0}^p \cdot \tilde{A}_{p+1,0} + {x_0}^p \\ &= -{x_0}^p \cdot \tilde{A}_{p+1,0} - {x_0}^p + {x_0}^p \cdot \tilde{A}_{p+1,0} + {x_0}^p = 0 \text{ from (2.9)}\end{aligned}$$

□

This then shows that

$$P_i P_0 + x_0 P_i P_1 + \ldots + {x_0}^p P_i P_p \quad \text{for } 1 \leq i \leq p$$

are contained in the basis of the kernel of u^*, but we do not know if there are any other elements which could be in the basis. Since u is injective, this implies that u^* is surjective. The basis for ${A_{1,1}}^*$ consists of all $P_i P_j$ such that $0 \leq i, j \leq p$. ${A_{1,1}}^*$ then contains $(p+1)^2$ elements in its basis.The basis for ${A_2}^*$ consists of all P_i such that $0 \leq i \leq p^2 + p$. ${A_2}^*$ then contains $p^2 + p + 1$ elements in its basis. So then the basis for the kernel of u^* has $(p+1)^2 - (p^2 + p + 1) = p$ elements in it. Since we have found p elements which compose a basis for the kernel of u^* namely:

$$P_i P_0 + x_0 P_i P_1 + \ldots + {x_0}^p P_i P_p \quad \text{for } 1 \leq i \leq p$$

then these basis elements form a complete basis for the kernel of u^*.

References

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