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ESTABLISHING A METRIC IN MAX-PLUS GEOMETRY

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Abstract. Using the characterization of the segments in the max-plus semimodule \mathbb{R}_{\max}^n , provided by Nitica and Singer in [5], we find a class of metrics on the finite part of \mathbb{R}_{\max}^n . One of them is the Euclidean length of the max-plus segment connecting two points. This metric is not quasi-convex. There is exactly one other metric in our class that does possess this property. Each metric in our class is associated with a weighting function, which is concave and non-decreasing.

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1 Introduction

Max-plus algebra is the semiring consisting of the set $\mathbb{R}_{\max} := \mathbb{R} \cup \{-\infty\}$ endowed with the binary operations $\oplus = \max$ and $\otimes = +$. A natural structure of semimodule can be imposed on the Cartesian product \mathbb{R}_{\max}^n using pointwise addition and scalar multiplication. Semimodules over max-plus algebra have many practical applications in optimization, game theory and scheduling. For a basic textbook on the subject that includes real life applications, see [3].

A natural question to ask is which properties transfer from the usual Euclidean space \mathbb{R}^n to the max-plus semimodule \mathbb{R}_{\max}^n . Specifically, one may wonder if there exists a metric on this space. The Hilbert projective metric, which is only a semi metric, has already been considered (see [2], [1]). Nevertheless, no standard metric has been introduced so far. In this paper, we find a class of metrics for the finite part of the max-plus semimodule, exactly one metric being quasi-convex.

The structure of the paper is as follows. In Section 2, we look at the geometry of max-plus segments and build upon their characterization already found in [5]. We define polygonal paths that are composed of Euclidean line segments, with specific directions described by linear combinations of the standard basis vectors. A length is defined for these polygonal paths involving a weighting depending on the direction of the Euclidean segments. Once these definitions are in place, Proposition 1 presents a characterization of the max-plus segment that is unique up to a specific equivalence relation. Our main result is found in Theorem 2, in which the max-plus segments are characterized as the polygonal paths that minimize the lengths over the domain of all polygonal paths connecting two points in the max-plus semimodule. The proof of this theorem is found in the next section. It immediately follows from this theorem that the length of a max-plus segment is a metric.

In Section 3, we proceed to prove Theorem 2 using two lemmas that are central in understanding how the shortest paths are constructed. The first lemma proves that shortest paths cannot cross themselves when projected down to any coordinate axis; the second lemma shows that these paths are made up of ordered polygonal paths. The proofs of both lemmas are done by contradiction. Then, we invoke Proposition 1 to complete the proof that one of the shortest paths is indeed the max-plus segment.

In Section 4, we discuss the fact that in order to generate a metric, the weighting function of the length must be concave and non-decreasing.

In Section 5, we look for metrics in our class of metrics that satisfy the quasi-convexity property. The motivation is the furthering of the comparison between the Euclidean space and the max-plus semimodule. Moreover, since quasi-convexity is a geometric property associated with the field of convexity, possessing this property enables us to use some nice results in this field, a major one being that of Martínón (see [4]). We show that when the weighting function is given by certain square-roots, related to the natural Euclidean length, then the metric is not quasi-convex. We provide a counter example in \mathbb{R}_{\max}^2 . After that, we introduce a second metric, inspired by the counter-example, where the weighting function is uniform. We show that this is the unique quasi-convex metric in our class of metrics.

We conclude this paper with Section 6, in which we return to the quasi-convexity of the metric induced by the Euclidean length. Although this metric is not quasi-convex, the question remains as to what is the smallest bounding coefficient for which a natural weaker inequality holds. We prove that the constant 2 serves as a bound in any dimension, and provide a conjecture for a better coefficient, that is dependent on the dimension.

2 Characterization of Segments

Consider the space \mathbb{R}_{\max}^n with the following operations of addition and scalar multiplication:

$$(x_1, x_2, \dots, x_n) \oplus (y_1, y_2, \dots, y_n) = (x_1 \oplus y_1, x_2 \oplus y_2, \dots, x_n \oplus y_n),$$

$$\lambda \otimes (x_1, x_2, \dots, x_n) = (\lambda \otimes x_1, \lambda \otimes x_2, \dots, \lambda \otimes x_n),$$

where $x_i, y_i, \lambda \in \mathbb{R}_{\max}$.

Definition 1. The max-plus segment connecting the points $x, y \in \mathbb{R}_{\max}^n$ is defined as the set of all convex combinations of the endpoints:

$$[x, y] = \{z \in \mathbb{R}_{\max}^n \mid z = (\alpha \otimes x) \oplus (\beta \otimes y), \alpha \oplus \beta = 0\}.$$

The segments are already characterized in [5]:

Theorem 1. Let $x, y \in \mathbb{R}_{\max}^n$, and (i_1, \dots, i_n) be an ordering on the set of indices $\{1, \dots, n\}$. Then, σ is defined to be the one-to-one mapping of \mathbb{R}_{\max}^n onto itself defined by

$$\sigma(x_{i_1}, \dots, x_{i_n}) = (x_1, \dots, x_n), \quad (x_{i_1}, \dots, x_{i_n}) \in \mathbb{R}_{\max}^n.$$

For $x \leq y$ (commensurable case), with $x_{i_1} - y_{i_1} \leq \dots \leq x_{i_n} - y_{i_n}$, we have

$$[x, y] = [x, \sigma(y_{i_1} + x_{i_2} - y_{i_2}, x_{i_2}, \dots, x_{i_n})] \cup [\sigma(y_{i_1} + x_{i_2} - y_{i_2}, x_{i_2}, \dots, x_{i_n}),$$

$\sigma(y_{i_1} + x_{i_3} - y_{i_3}, y_{i_2} + x_{i_3} - y_{i_3}, x_{i_3}, \dots, x_{i_n})] \cup \dots \cup [\sigma(y_{i_1} + x_{i_n} - y_{i_n}, \dots, y_{i_{n-1}} + x_{i_n} - y_{i_n}, x_{i_n}), y]$,

where each of the segments on the right hand side are the usual Euclidean segments.

For $x \not\leq y$ and $x \not\geq y$ (incommensurable case), there exists some p with $1 \leq p \leq n - 1$ such that

$$x_{i_1} - y_{i_1} \leq \dots \leq x_{i_p} - y_{i_p} \leq 0 \leq x_{i_{p+1}} - y_{i_{p+1}} \leq \dots \leq x_{i_n} - y_{i_n},$$

with at least one of the inequalities $x_{i_j} - y_{i_j} \leq 0$ ($j \leq p$) and at least one of the inequalities $0 \leq x_{i_k} - y_{i_k}$ ($p + 1 \leq k \leq n$) being strict. Then,

$$[x, y] = [x, \sigma(y_{i_1}, \dots, y_{i_p}, x_{i_{p+1}}, \dots, x_{i_n})] \cup [\sigma(y_{i_1}, \dots, y_{i_p}, x_{i_{p+1}}, \dots, x_{i_n}), y],$$

where

$$[x, \sigma(y_{i_1}, \dots, y_{i_p}, x_{i_{p+1}}, \dots, x_{i_n})]$$

and

$$[\sigma(y_{i_1}, \dots, y_{i_p}, x_{i_{p+1}}, \dots, x_{i_n}), y]$$

are commensurable segments.

In what follows, when in \mathbb{R}_{\max}^n , we will consider only max-plus segments joining points with finite coordinates. As a result, we will refer to this space as merely \mathbb{R}^n . We introduce the following definitions to make this characterization more explicit.

Definition 2. Let $(e_i)_{i=1}^n$ be the standard vector basis in \mathbb{R}^n . If $S \subseteq \{1, \dots, n\}$, define

$$v_{+S} = \sum_{i \in S} e_i,$$

$$v_{-S} = - \sum_{i \in S} e_i.$$

If $S = \emptyset$, then v_{+S} and v_{-S} are simply the zero vector.

Definition 3. A **polygonal path** $P(a_i)$ is a curve specified by a finite sequence of points $a_0, a_1, \dots, a_m \in \mathbb{R}^n$, called vertices, that consists of Euclidean line segments connecting consecutive vertices such that:

$$a_i - a_{i-1} = \lambda_i v_{\text{sgn}(S_i)S_i} \text{ for some } \lambda_i > 0, S_i \subseteq \{1, \dots, n\}, \text{ and } \text{sgn}(S_i) = \pm.$$

We call a polygonal path **empty** in the case that $S = \emptyset$, i.e. there is only one vertex in the path. Furthermore, we call a polygonal path having $\text{sgn}(S_i) = +$ (or $\text{sgn}(S_i) = -$) for all $1 \leq i \leq m$, **positive** (**negative**).

A positive or negative path is **increasingly ordered** if $i \leq j$ implies $S_i \subseteq S_j$ and **decreasingly ordered** if $i \leq j$ implies $S_i \supseteq S_j$.

Two paths $P(a_i)$ and $P(a'_i)$ are **orthogonal** if $S_i \cap S'_j = \emptyset$ for all i, j .

Definition 4. Given the sequence of weights $w = (w_1, w_2, w_3, \dots)$ with $w_i \geq 0, i \geq 1$, and $w_0 = 0$, the **length** of a polygonal path with m vertices is given by

$$d_w(P(a_i)) = \sum_{i=1}^m \lambda_i w_{|S_i|}.$$

Example 1. We define $w_i = \sqrt{i}, i \geq 0$, to be the Euclidean weight. The corresponding length is denoted by d_E .

Example 2. We similarly define $w = (0, 1, 1, \dots)$ to be the constant weight. The corresponding length is denoted by d_Q .

We note that a polygonal path is uniquely specified by the vectors $\lambda_i v_{\text{sgn}(S_i)S_i}$ and that the length of a polygonal path is invariant under a reordering of these vectors, a fact that will be taken advantage of in the proofs of Lemmas 1 and 2 in the following section.

Definition 5. Let $x, y \in \mathbb{R}^n$. Then, an equivalence relation on polygonal paths between x and y is given by:

$$P(a_i) \sim P(a'_i) \text{ if for all } S \subseteq \{1, \dots, n\}, \text{ where } \text{sgn}(S) = \pm:$$

$$\sum_{\substack{S_i=S, \\ \text{sgn}(S_i)=\text{sgn}(S)}} \lambda_i = \sum_{\substack{S'_i=S, \\ \text{sgn}(S'_i)=\text{sgn}(S)}} \lambda'_i.$$

This equivalence relation can be readily checked to be reflexive, symmetric, and transitive.

Proposition 1. *Let $x, y \in \mathbb{R}^n$. Then,*

$$[x, y] = P(a_i),$$

where the following properties are satisfied:

- (i) $x = a_1$
- (ii) $y = a_n$
- (iii) $P(a_i)$ is the concatenation of two orthogonal paths: an increasingly ordered positive polygonal path and a decreasingly ordered negative polygonal path.
 Note that either of these two paths can be empty.

Furthermore, any polygonal path satisfying these three properties is unique up to the equivalence defined in Definition 5.

Proof. The fact that a max-plus segment satisfies properties (i)-(iii) follows from Theorem 1. The ordering and the signs of the differences $x_i - y_i$ uniquely determine whether a coordinate changes in the positive or negative direction, and for the sets S_i that are involved, $|S_i| \leq n$. The vectors v_{S_i} are linearly independent, so the sum of their coefficients λ_i is uniquely determined. Thus, the path is unique up to the equivalence given in Definition 5. \square

Remark 1. Proposition 1 implies that if we have a polygonal path between $x, y \in \mathbb{R}^n$ in the form (iii), the polygonal path is the unique max-plus segment connecting x, y . Also, the positive and negative paths correspond to the two commensurable parts of the segment.

We now proceed to introduce the notion of concavity.

Definition 6. A function $f : \mathbb{N} \rightarrow \mathbb{R}$ is **concave** if for all $M, N, T \in \mathbb{N}, M \geq N$,

$$f(M) - f(N) \geq f(M + T) - f(N + T).$$

Remark 2. This notion is easily shown to be equivalent to the property that for all $M \in \mathbb{N}, f(M + 1) - f(M) \geq f(M + 2) - f(M + 1)$.

We are ready to state our main result.

Theorem 2. *Let $x, y \in \mathbb{R}^n$. Then, if w is a sequence of weights and $i \mapsto w_i$ is concave and non-decreasing, one has*

$$d_w([x, y]) = \min d_w(P(a_i)),$$

where the minimum is taken over all polygonal paths $P(a_i)$ from x to y .

The theorem is proven in the next section. For now we point out that it suggests the following notation: $d_w(x, y) = d_w([x, y])$, that is, the distance between two points is the weighted length of the max-plus segment between them. The following corollary justifies this notation.

Corollary 1. *If w is a sequence of weights and $i \mapsto w_i$ is concave and non-decreasing, then d_w is a metric on \mathbb{R}^n .*

Proof. (i) $d_w([x, y]) \geq 0$ and $d_w([x, y]) = 0 \iff x = y$ are straightforward.

(ii) $d_w([x, y]) = d_w([y, x])$ because $[x, y] = [y, x]$ (see [5]).

(iii) The concatenation of two polygonal paths $[a, b]$ and $[b, c]$ is again a polygonal path. As $[a, c]$ has the minimum distance among all polygonal paths between a and c , the triangle inequality immediately follows. □

Corollary 2. *The lengths d_E and d_Q are metrics on \mathbb{R}^n .*

Remark 3. Since the properties described in Proposition 1 are invariant under the usual (not max-plus) scalar multiplication, any weight which may be used to define a max-plus metric may additionally be used to define a norm on the usual linear space \mathbb{R}^n .

3 Optimization: Length of Segments

We want to solve the following optimization problem:

Given two points $x, y \in \mathbb{R}^n$, what is the shortest length among all the polygonal paths connecting x and y ?

We need two key lemmas. The first one ensures that a polygonal path does not cross itself when projected down to any coordinate axis.

Lemma 1. *Suppose that $i \mapsto w_i$ is non-decreasing and concave for a sequence of weights w . Let $P(a_i)$ be a polygonal path that connects x and y with $a_i - a_{i-1} = \lambda_i v_{\pm S_i}$. If for some k and l , $\text{sgn}(S_k) \neq \text{sgn}(S_l)$ and $j \in S_k \cap S_l$ (i.e. j has overlap in $P(a_i)$), then there exists a (not necessarily equivalent) polygonal path $P(a'_i)$ (connecting x and y) for which:*

1. $d_w(P(a'_i)) \leq d_w(P(a_i))$ and
2. $j \notin S'_k \cap S'_l$
3. This process does not take another index j' , without overlap in $P(a_i)$, and cause it to have overlap in $P(a'_i)$.

Proof. Without loss of generality, assume $\lambda_k - \lambda_l \geq 0$ and denote $\text{sgn}(S_k)$ as \pm .

First, observe that

$$\lambda_k v_{\pm S_k} + \lambda_l v_{\mp S_l} = (\lambda_k - \lambda_l) v_{\pm S_k} + \lambda_l v_{\pm S_k \setminus \{j\}} + \lambda_l v_{\mp S_l \setminus \{j\}}.$$

Thus, we can take the path $P(a_i)$ and create a new path, denoted $P(a'_i)$ be replacing the vectors on the left hand side with the vectors on the right hand side, which is possible because the order of the vectors is irrelevant in determining length.

Now,

$$\begin{aligned} d_w(P(a'_i)) - d_w(P(a_i)) &= ((\lambda_k - \lambda_l)w_{|S_k|} + \lambda_l w_{|S_k|-1} + \lambda_l w_{|S_l|-1}) - (\lambda_k w_{|S_k|} + \lambda_l w_{|S_l|}) \\ &= \lambda_l (w_{|S_k|-1} - w_{|S_k|} - w_{|S_l|} + w_{|S_l|-1}) \\ &\leq 0 \end{aligned}$$

since

$$w_{|S_l|} - w_{|S_l|-1} \geq w_{|S_k|-1} - w_{|S_k|}$$

by the concavity of $i \mapsto w_i$. □

Lemma 2. *Suppose $i \mapsto w_i$ is concave. Given a positive (or negative) polygonal path $P(a_i)$, if there exist S_k, S_l with $S_k \not\subseteq S_l$ and $S_l \not\subseteq S_k$, then there exists a polygonal path $P(a'_i)$ with $d_w(P(a'_i)) \leq d_w(P(a_i))$, and S_k and S_l replaced with ordered sets.*

Proof. This proof is similar to that of the previous lemma.

Without loss of generality, assume $\lambda_l \geq \lambda_k > 0$, $\text{sgn}(S_k) = \text{sgn}(S_l) = \pm$. We see

$$\lambda_k v_{\pm S_k} + \lambda_l v_{\pm S_l} = (\lambda_l - \lambda_k) v_{\pm S_l} + \lambda_k v_{\pm S_k \cap S_l} + \lambda_k v_{\pm S_k \cup S_l}.$$

Again, we can replace the vectors on the left hand side with those on the right to obtain a new path $P(a'_i)$.

$$\begin{aligned} d_w(P(a_i)) - d_w(P(a'_i)) &= (\lambda_k w_{|S_k|} + \lambda_l w_{|S_l|}) - ((\lambda_l - \lambda_k)w_{|S_l|} + \lambda_k w_{|S_l \cap S_k|} + \lambda_k w_{|S_l \cup S_k|}) \\ &= \lambda_k (w_{|S_k|} + w_{|S_l|} - w_{|S_l \cap S_k|} - w_{|S_l \cup S_k|}) \end{aligned}$$

Now, letting $M = |S_k|$, $N = |S_l \cap S_k|$, and $T = |S_l| - |S_l \cap S_k|$, the concavity of $i \mapsto w_i$ implies

$$\begin{aligned} w_M - w_N &\geq w_{M+T} - w_{N+T} \\ w_{|S_k|} - w_{|S_l \cap S_k|} &\geq w_{|S_l|+|S_k|-|S_l \cap S_k|} - w_{|S_l|}, \end{aligned}$$

and finally

$$w_{|S_k|} + w_{|S_l|} \geq w_{|S_l \cap S_k|} + w_{|S_l \cup S_k|}.$$

$$d_w(P(a'_i)) \leq d_w(P(a_i))$$

□

Using Lemma 1 and Lemma 2, and reordering vectors as necessary, we see that a path with minimal length can be represented as a polygonal path which is the concatenation of a totally ordered positive and a totally ordered negative path. We apply Proposition 1 to obtain our desired result (Theorem 2).

4 Necessary Conditions for a Metric

In the previous section, we proved a theorem giving sufficient conditions for which a set of weights establishes a metric in max-plus geometry. In this section, we show that, as long as the set of weights is generalizable to arbitrarily high dimensions, these conditions are also necessary. This is stated in the following theorem.

Theorem 3. *If $\{w_i\}$ is an infinite set of weights that establishes a metric in all dimensions n , then the function $i \mapsto w_i$ is concave and non-decreasing.*

Proof. Suppose the function is not concave and choose $a \leq b \leq c \leq d \in \mathbb{N}$ such that $a + d = b + c$, yet $w_d - w_c > w_b - w_a$, or equivalently, $w_a + w_d > w_b + w_c$. We demonstrate a violation of the triangle inequality with points x, y , and z as follows: Let $S = \{1, \dots, b\}$, $T = \{b - a + 1, \dots, d\}$, so that $|S| = b$, $|T| = c$, $|S \cap T| = a$, and $|S \cup T| = d$.

$$x = 0$$

$$y = v_S$$

$$z = v_S + v_T$$

Thus,

$$d_w(x, y) = w_b$$

$$d_w(y, z) = w_{d-b+a} = w_c$$

$$d_w(x, z) = w_a + w_d,$$

violating the triangle inequality. Therefore, the function is concave.

Now, suppose that $w_k - w_{k+1} = \epsilon > 0$ for some k . By concavity, we have that $w_{k+1} - w_{k+2} \geq \epsilon$ and $w_k - w_{k+2} \geq 2\epsilon$. By induction, we have $w_k - w_{n+k} \geq n\epsilon$ for all n , so by the Archimedean property of the reals, $w_N < 0$ for some N . This is a contradiction, so $i \mapsto w_i$ must be nondecreasing. \square

5 Quasi-Convexity

Definition 7. We define a metric d to be quasi-convex if

$$z \in [a, b] \Rightarrow d(c, z) \leq \max\{d(c, a), d(c, b)\}, \forall c \in \mathbb{R}^n.$$

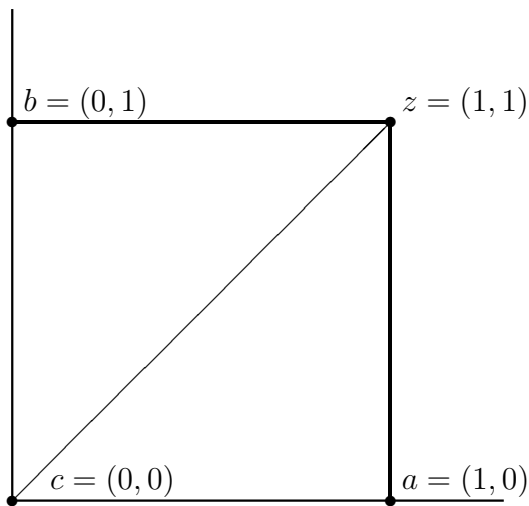
We now show that d_E is not quasi-convex. Consider the $n = 2$ case. Let $a = (1, 0)$, $b = (0, 1)$, $z = (1, 1)$, and $c = (0, 0)$. It is easy to verify that the segment from a to b is the horizontal segment from a to z , followed by the vertical segment from z to b . Thus, $z \in [a, b]$ and we have

$$d_E(c, a) = 1$$

$$d_E(c, b) = 1$$

$$d_E(c, z) = \sqrt{2}.$$

Example



This completes our counterexample. One may see from this example the motivation to consider a second metric, where the vector $(1,1)$ is given weight 1 instead of its Euclidean length $\sqrt{2}$, as in our definition of d_Q . We proceed to show that d_Q is indeed quasi-convex. We begin by finding an explicit formula for this metric. Because max-plus segments are translation invariant, we assume, without loss of generality, that the starting point of the segment is the origin.

Proposition 2. $d_Q(0, x) = \max_{i,j} \{|x_i|, |x_i - x_j|\}$

Proof. Let $x = (x_1, \dots, x_n)$, where $x_{i_1} \geq \dots \geq x_{i_k} \geq 0 \geq x_{i_{k+1}} \geq \dots \geq x_{i_n}$. By our characterization of the paths, $P(a_i) = [0, x]$ has for its values of $a_{j+1} - a_j$

$$\lambda_1 v_{\{i_1\}}, \dots, \lambda_k v_{\{i_1, \dots, i_k\}}, \lambda_{k+1} v_{-\{i_{k+1}, \dots, i_n\}}, \dots, \lambda_n v_{-\{i_n\}}.$$

By definition of the metric, $d(0, x) = \sum_{i=1}^k \lambda_i + \sum_{i=k+1}^n \lambda_i = x_{i_1} - x_{i_n}$. If $x_{i_1} > 0 > x_{i_n}$, then clearly, $x_{i_1} - x_{i_n}$ maximizes $|x_i - x_j|$ and is greater than all $|x_i|$. If $x_i \geq 0$ for all i , then there is no x_{i_n} term such that $x_{i_n} < 0$, and so $d_Q(0, x) = |x_{i_1}|$. Also, $|x_i| \geq |x_i - x_j|$ for all i, j . Similar reasoning holds when all $x_i < 0$. Thus, our claim holds in all cases. \square

The following is a lemma that will help us show that max-plus segments are preserved under coordinate projection.

Lemma 3 (Projection Lemma). *Let $x, y \in \mathbb{R}^n$ and $[x, y]$ be the max-plus segment between them. Let π be the projection onto an l -dimensional Euclidean hyperplane spanned by standard basis vectors $\{e_j\}_{j \in J}$ with $|J| = l$. Then, $\pi([x, y]) = [\pi(x), \pi(y)]$.*

Proof. Let the path between x and y be given by the polygonal path $P(a_i)$. Then, since π is linear, we have $\pi(a_{i+1} - a_i) = \pi(a_{i+1}) - \pi(a_i)$ for all i . Additionally, $\pi(\lambda_i v_{\pm S_i}) = \lambda_i v_{\pm S_i \cap J}$. Thus, the polygonal path $P(\pi(a_i))$ satisfies the conditions of Proposition 1 since

- π preserves the positivity or negativity of polygonal paths
- If $S_i \subseteq S_k$, then $S_i \cap J \subseteq S_k \cap J$. Thus, π preserves totally ordered paths.

□

Theorem 4. d_Q is quasi-convex.

Proof. Since the metric and segments are both invariant under translation, let $c = \vec{0}$. By Proposition 2, our theorem may be rephrased as: For any $z \in [a, b]$,

$$\max_{i,j} \{|z_i|, |z_i - z_j|\} \leq \max\{\max_{i,j} \{|a_i|, |a_i - a_j|\}, \max_{i,j} \{|b_i|, |b_i - b_j|\}\}$$

Case 1: Suppose the maximum on the left hand side is obtained by $|z_i|$. Since the projection of a max-plus segment onto one coordinate cannot cross itself, this means either $a_i \leq z_i \leq b_i$ or $b_i \leq z_i \leq a_i$. Either way, $|z_i| \leq \max\{|a_i|, |b_i|\}$, completing the proof of this case.

Case 2: Suppose this maximum is obtained by $|z_i - z_j|$ and that $|z_i - z_j| > |a_i - a_j|, |b_i - b_j|$. Let π be the projection onto the plane spanned by the i -th and j -th coordinates. Then, $|z_i - z_j|$ can be seen to be the straight-line distance of $(z_i, z_j) = \pi(z)$ from the main diagonal $x_i = x_j$. Thus, $\pi(z)$ is further from the diagonal than $\pi(a)$ and $\pi(b)$. So, we can construct a diagonal line parallel to the main diagonal separating $\pi(z)$ from $\pi(a)$ and $\pi(b)$. But, $[\pi(a), \pi(b)]$ cannot cross a diagonal twice since it is composed of at most one segment not parallel to the diagonal, so $\pi(z) \notin [\pi(a), \pi(b)]$. By Lemma 3, we have $z \notin [a, b]$, which is a contradiction. Thus, $|z_i - z_j| \leq \max\{|a_i - a_j|, |b_i - b_j|\}$ for all i, j , completing the proof of the theorem. □

Corollary 3. Let $R, S \subseteq \mathbb{R}^n$ be nonempty closed sets such that $R \cup S$ is convex. Then, $R \cap S \neq \emptyset$ and

$$d_Q(z, R \cap S) = \max\{d_Q(z, R), d_Q(z, S)\}.$$

Proof. See [4]. □

The following proposition demonstrates that d_Q is the only such quasi-convex metric.

Proposition 3. Suppose d_w is a metric on \mathbb{R}^n induced by $w = (w_0, w_1, \dots, w_n)$. Then, if d_w is quasi-convex, $d_w = d_Q$.

Proof. We show that $w_k = 1$ for all $k > 0$ by strong induction on k .

Base case: We assume $w_1 = 1$ via normalization.

Induction hypothesis: Suppose $w_1 = \dots = w_k = 1$. The following two cases will put constraints on w_{k+1} which yield $w_{k+1} = 1$.

Let $S = \{1, \dots, k\}$ and $T = \{k + 1\}$. Let $a = v_S, b = v_T, z = v_{S \cup T}$, and $c = \vec{0}$. Then, we can again see that the segment from a to b is the Euclidean segment from a to z concatenated with the Euclidean segment from z to b . Also, by induction,

$$d_w(c, a) = w_k = 1$$

$$d_w(c, b) = 1$$

$$d_w(c, z) = w_{k+1}$$

This example shows that $w_{k+1} \leq 1$.

Now, let $c = \vec{0}, a = 2v_{S \cup T}, b = v_S$, and $z = v_S + v_{S \cup T}$. Again, it follows that $z \in [a, b]$ and

$$d_w(c, a) = 2w_{k+1}$$

$$d_w(c, b) = 1$$

$$d_w(c, z) = 1 + w_{k+1}$$

Thus, quasi-convexity implies that $w_{k+1} \geq 1$. Combining the inequalities gives us $w_{k+1} = 1$, and proceeding by induction gives us the result for the entire vector w . \square

6 Generalizing Quasi-Convexity for the Euclidean weighting

We have shown that quasi-convexity fails under the Euclidean weighting. In this section we consider the lowest bound which generalizes quasi-convexity for the Euclidean weighting. Because we exclusively consider the Euclidean weighting, d will refer to d_E throughout this section.

It is natural to ask what the smallest coefficient is for the following equation:

$$d(c, z) \leq K_n \max \{d(a, c), d(b, c)\}, \forall a, b, c \in \mathbb{R}^n, z \in [a, b]$$

Once we find any such coefficient K_n , we would prove the following theorem:

Theorem 5. *Let $R, S \subseteq \mathbb{R}^n$ be nonempty closed sets such that $R \cup S$ is convex. Then, $R \cap S \neq \emptyset$ and*

$$d(z, R \cap S) \leq K_n \max \{d(z, R), d(z, S)\}$$

Proof. See [4]. His proof may be exactly replicated up to multiplication of the right hand side by the constant K_n . \square

6.1 Bounds on K_n

We first establish an upper bound.

Proposition 4. *Let $a, b, c \in \mathbb{R}^n$ and let $z \in [a, b]$, then*

$$d(c, z) \leq 2 \max\{d(a, c), d(b, c)\}.$$

Thus, $K_n \leq 2$ in any dimension.

Proof. First observe that by the triangle inequality,

$$d(c, z) \leq d(c, a) + d(a, z)$$

$$d(c, z) \leq d(c, b) + d(b, z)$$

Summing the two gives us:

$$2d(c, z) \leq d(c, a) + d(c, b) + d(a, z) + d(b, z)$$

Since segments minimize distance, $[a, b] = [a, z] + [z, b]$. Thus, $d(a, b) = d(a, z) + d(z, b)$ and the above reduces to

$$2d(c, z) \leq d(c, a) + d(c, b) + d(a, b)$$

Applying the triangle inequality once more gives us

$$2d(c, z) \leq d(c, a) + d(c, b) + d(a, c) + d(b, c) = 2(d(c, a) + d(c, b)) \leq 4 \max\{d(c, a), d(c, b)\}$$

Dividing by 2 gives us the final result. \square

Note that this proof holds for any weighting which is concave and non-decreasing.

We now proceed by applying some reduction arguments and finding a lower bound on each K_n before outlining our proposed proof. Because our segments are invariant under translation, we define $\|\cdot\| := d(0, \cdot)$. Now, to find our lowest K_n , our problem reduces to the following:

$$\|z\| \leq K_n \max\{\|a\|, \|b\|\}, \forall a, b \in \mathbb{R}^n, z \in [a, b]$$

Observation 1. In \mathbb{R}^n , let

$$a = (1 \ 0 \ 0 \ \cdots \ 0)$$

$$b = (0 \ (n-1)^{-1/2} \ \cdots \ (n-1)^{-1/2})$$

$$z = (1 \ (n-1)^{-1/2} \ \cdots \ (n-1)^{-1/2})$$

It is easy to check that $\|a\| = \|b\| = 1$, $\|z\| = \frac{\sqrt{n} + \sqrt{n-1} - 1}{\sqrt{n-1}}$, $z \in [a, b]$

$$\implies K_n \geq \frac{\sqrt{n} + \sqrt{n-1} - 1}{\sqrt{n-1}}.$$

This lower bound for each K_n leads us to our conjecture:

Conjecture 1. $K_n = \frac{\sqrt{n} + \sqrt{n-1} - 1}{\sqrt{n-1}}$ in \mathbb{R}^n .

6.2 Proposed Outline of Proving the Conjecture

Using our lower bound we get $K_n > 1$ for $n > 1$. In the $n = 1$ case, we trivially have $K_1 = 1$.

For further reduction, we note if $a = \vec{0}$, then $[0, z] \subseteq [0, b]$ for $z \in [a, b]$.

$\implies \|z\| \leq \|b\|$. Since $K_n > 1$, we can disregard the case when a or b is 0 and state our problem as follows:

$$K_n = \sup_{a, b \in \mathbb{R}^n_{\max} \setminus \{0\}, z \in [a, b]} \frac{\|z\|}{\max\{\|a\|, \|b\|\}}$$

Idea 1. Given $a, b \in \mathbb{R}^n \setminus \{0\}$, $a \leq b$, then

$$\max_{z \in [a, b]} \|z\| = \max\{\|a\|, \|b\|\}$$

Remark 4. Given any segment $[a, b]$, if it is a commensurable segment then the above idea implies that $\max_{z \in [a, b]} \frac{\|z\|}{\max\{\|a\|, \|b\|\}} = 1$. If the segment is incommensurable then we have shown there is a unique point $z \in [a, b]$ with both $[a, z]$ and $[z, b]$ commensurable. It is easy to see from our characterization that this point can be described by

$$z_i = \max\{a_i, b_i\}$$

We will denote this point as $z_{a,b}$.

By Observation 1, we have $K_n > 1$, so by the above remark we are only interested in the incommensurable case. Using Remark 5, we have

$$\sup_{a, b \in \mathbb{R}^n \setminus \{0\}, z \in [a, b]} \frac{\|z\|}{\max\{\|a\|, \|b\|\}} = \sup_{a, b \in \mathbb{R}^n_{\max}} \frac{\|z_{a,b}\|}{\max\{\|a\|, \|b\|\}}$$

Idea 2. Given $a, b \in \mathbb{R}^n \setminus \{0\}$, $a \leq b$, Define a', b' by

$$a'_i = \begin{cases} a_i & : a_i = \max\{a_i, b_i, 0\} \\ 0 & : \text{otherwise} \end{cases}$$

$$b'_i = \begin{cases} b_i & : b_i = \max\{a_i, b_i, 0\} \wedge a_i \neq \max\{a_i, b_i, 0\} \\ 0 & : \text{otherwise} \end{cases}$$

Then

$$\frac{\|z_{a,b}\|}{\max\{\|a\|, \|b\|\}} \leq \frac{\|z_{a',b'}\|}{\max\{\|a'\|, \|b'\|\}}$$

Definition 8. We will call a, b a **k-l-partition** ($k \leq \frac{n-l}{2}$) if there exists a partition Z, P, Q of $\{1 \dots n\}$ with $|Z| = l, |P| = k, |Q| = n - l - k$

$$a_i, b_i = 0 \quad \forall i \in Z$$

$$a_i > 0, b_i = 0 \quad \forall i \in P$$

$$b_i > 0, a_i = 0 \quad \forall i \in Q$$

Idea 3. For $1 < k$ and $a, b \in \mathbb{R}^n$ if a, b is a k - l -partition for some l , then there exists a $(k-1)$ - $(l+1)$ partition a', b' with

$$\frac{\|z_{a,b}\|}{\max\{\|a\|, \|b\|\}} \leq \frac{\|z_{a',b'}\|}{\max\{\|a'\|, \|b'\|\}}$$

Idea 4. Given $a, b \in \mathbb{R}^n$, a 1 - l partition, we have

$$\frac{\|z_{a,b}\|}{\max\{\|a\|, \|b\|\}} \leq \frac{\sqrt{n-l} + \sqrt{n-l-1} - 1}{\sqrt{n-l-1}}$$

6.3 Comments on Conjecture

We believe that all the above are true after extensive testing in $\mathbb{R}^3, \mathbb{R}^4, \mathbb{R}^5$, and \mathbb{R}^6 . The validity of the above ideas would be enough to prove the conjecture.

We believe the ideas have simple proofs, with the exception of Idea 3.

We can prove our conjecture in the \mathbb{R}^2 case without using the above by showing a more basic observation:

$$d_E(a, b) \leq \sqrt{2}d_Q(a, b),$$

where Q is our quasi-convex weighting.

$$\implies K_2 \leq \sqrt{2}.$$

Our observation would show that $K_2 \geq \sqrt{2}$. Thus, $K_2 = \sqrt{2}$ in this case.

Note that there are no 2 - 1 partitions in \mathbb{R}^3 , so we can prove our bound for \mathbb{R}^3 without Idea 3.

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