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Austin Williams[a]

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Rose-Hulman Institute of Technology
Mathematics Department
Terre Haute, IN 47803
Email: mathjournal@rose-hulman.edu
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[a] Portland State University, Portland, Oregon

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ON THE NUMBER OF 2-PLAYER, 3-STRATEGY, STRICTLY ORDINAL, NORMAL FORM GAMES

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Abstract. The 2-player, 2-strategy, strictly ordinal, normal form games were originally studied by Anotol Rapoport and Melvyn Guyer in a paper entitled *A Taxonomy of 2x2 Games* [4]. Their paper appeared in 1966 and included an exact count, an enumeration (that is, a complete listing), and a taxonomy of such games. Since then it has been known that there are 78 such games. If we allow each player access to one additional strategy, however, the number of games explodes to nearly two billion. In this paper we compute the exact number of 2-player, 3-strategy, strictly ordinal, normal form games.

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1 Introduction

The 2-player, 2-strategy, strictly ordinal, normal form games were originally studied by Anotol Rapoport and Melvyn Guyer in a paper entitled *A Taxonomy of 2x2 Games* [4]. Their paper appeared in 1966 and included an exact count, an enumeration (that is, a complete listing), and a taxonomy of such games. In a subsequent paper by Rapoport alone, the assertion is made [5, p. 83] that an exact count, enumeration, and taxonomy of 2-player, 3-strategey strategy, strictly ordinal, normal form games exceeded the computational abilities of the time. Today, some 45 years later, computational abilities have developed that appear to allow an exact count, enumeration, and taxonomy of these aforementioned games. The exact count is performed herein, and the enumeration and taxonomy are hoped to follow shortly.

In Rapoport's paper [5, p. 83], upper and lower bounds for the count of such games were given. In particular, if N is the number of 2-player, 3-strategy, strictly ordinal, normal form games, then $\frac{9!^2}{72} < N < \frac{9!^2}{36}$. Apparently, these bounds have not been subsequently improved in print. The count, enumeration, and taxonomy of the 2-strategy case in [4] rely on a particular representation of the given games. These representations are not unique, and the appropriate equivalence relation on these representations is developed in [4] in order to produce a one-to-one correspondence between the given games and the equivalence classes of these representations. The game count is then achieved by counting equivalence classes. The count of the equivalence classes is accomplished by analyzing the action of a certain group on the set of representations of the given games referred to above. In particular, the count is achieved by a careful analysis of the various orbits of the representations under the action of this group of what could be termed "game symmetries".

We employ essentially the same approach herein to achieve our count. For convenience, we refer to 2-player, N-strategy, strictly ordinal, normal form games as *N-strategy Rapoportian games*. The paper is organized as follows. Section 2 contains basic definitions and background information. In particular, Rapoportian games are defined, along with their so-called bimatrix representations. It is shown that every 3-strategy Rapoportian game has at least one bimatrix representation, and the set, X, of all such representations is formed. In Section 3 a demonstration that a given 3-strategy Rapoportian game can have several distinct bimatrix representations is given, and the appropriate equivalence relation on the representations is generated. Further, symmetries of these representations (hence, of the games) are defined, and the group of all such symmetries is determined. The counting problem is then reduced to the problem of counting orbits under the action of this group. In Section 4 symmetric, asymmetric, and standard symmetric bimatrix representations are defined and studied. In Section 5 symmetric and asymmetric orbits are defined, and it is shown that for a given bimatrix representation, $x \in X$, the orbit of x under the group action must be precisely one of these two types. In Section 6 the order of the symmetric orbits is given. In Section 6, the order of the asymmetric orbits is determined. In Section 8 the number of symmetric orbits is counted, and from this the number of 3-strategy Rapoportian games is deduced. Finally, Section 9 contains some thoughts on future work.

2 Basic Definitions and Background Material

Suppose there are two players, each of whom has a set of strategies, S and T respectively. Suppose that for each strategy pair $(s,t) \in S \times T$ there is an outcome for each player, $\omega_1(s,t)$ and $\omega_2(s,t)$ respectively. Let $\boldsymbol{G}: S \times T \longrightarrow \Omega_1 \times \Omega_2$ be the function that maps each strategy pair to the pair of outcomes. Suppose the players have preferences over the product of outcomes, $\Omega_1 \times \Omega_2$. A *utility function* for player i, $\boldsymbol{v}_i: \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R}$, is a function that assigns a real number to each of the outcome pairs in $\Omega_1 \times \Omega_2$, such that a more preferred outcome pair receives a greater number than a less preferred outcome pair, and that assigns the *same* number to two outcomes between which the player is indifferent [3, p. 29]. As an outcome pair is assigned to each play, and each outcome pair is assigned a utility for each player, we may define for each player a *payoff function* $b_i : S \times T \longrightarrow \mathbb{R}$ given by: $b_i(s,t) = \boldsymbol{v}_i(\boldsymbol{G}(s,t))$. The payoff function maps each strategy pair to the utility assigned to the corresponding outcome pair [3 pg. 48]. As in [4], assume that $|S \times T| = |\Omega_1 \times \Omega_2|$. Note that in the case where the image of $\boldsymbol{G}$, $Im(\boldsymbol{G})$, is not onto $\Omega_1 \times \Omega_2$, $Im(\boldsymbol{G})$ is to be treated as a multiset, in order that $|S \times T| = |Im(\boldsymbol{G})| = |\Omega_1 \times \Omega_2|$.

As defined in Luce and Raiffa's *Games and Decisions* [3, p. 55], a *2-player, normal form game* consists of:

i. The set of 2 players.
ii. Two sets of strategies, S and T, one for each player.
iii. Two payoff functions, b_1 and b_2, one for each player.

If a player's preferences are *strict* (that is, given any two outcome pairs in $\Omega_1 \times \Omega_2$, the player prefers one to the other), then the player's preferences over the outcome pairs is totally ordered. If, in addition, the utility functions assign a constant difference to successively preferred outcome pairs, then such a utility function is called a *strictly ordinal utility function* [6]. As the magnitudes of the differences are unimportant, the range of the strictly ordinal utility functions may, without loss of generality, be the integers. As the strictly ordinal utility functions never assign the same number to any two outcome pairs in $\Omega_1 \times \Omega_2$, the range of the functions is, without loss of generality, $\{1, \dots, m\}$, where $m = |\Omega_1 \times \Omega_2|$ [4, p. 1]. It follows that the range of payoff functions is, without loss of generality, $\{1, \dots, m\}$. If strictly ordinal utility functions are used to reflect players' preferences over the outcome pairs of a normal form game then the game is referred to as a *strictly ordinal, normal form game.* A *2-player, N-strategy, strictly ordinal, normal form game* is a 2-player, strictly ordinal, normal form game wherein each player has precisely *N* strategies to choose among. For brevity, we refer to 2-player, *N*-strategy, strictly ordinal, normal form games as *N-strategy Rapoportian games*.

For a 3-strategy Rapoportian game, G, let $S = \{s_1, s_2, s_3\}$ denote player 1's strategy set and let $T = \{t_1, t_2, t_3\}$ denote player 2's strategy set. Then $|\Omega_1 \times \Omega_2| = |S \times T| = 9$. Let $\boldsymbol{v}_i: \Omega_1 \times \Omega_2 \leftrightarrow \{1, \dots, 9\}$ denote player i's strictly ordinal utility function and let $b_i : S \times T \leftrightarrow \{1, \dots, 9\}$ be defined by $b_i(s,t) = \boldsymbol{v}_i(\boldsymbol{G}(s,t))$. Note that b_i is player i's payoff function. As in Luce and Raiffa's *Games and Decisions* [3, p. 58] the game is represented by the following bimatrix:

	t_1	t_2	t_3
s_1	$b_1(s_1,t_1), b_2(s_1,t_1)$	$b_1(s_1,t_2), b_2(s_1,t_2)$	$b_1(s_1,t_3), b_2(s_1,t_3)$
s_2	$b_1(s_2,t_1), b_2(s_2,t_1)$	$b_1(s_2,t_2), b_2(s_2,t_2)$	$b_1(s_2,t_3), b_2(s_2,t_3)$
s_3	$b_1(s_3,t_1), b_2(s_3,t_1)$	$b_1(s_3,t_2), b_2(s_3,t_2)$	$b_1(s_3,t_3), b_2(s_3,t_3)$

The above matrix is referred to as a *bimatrix representation* of the game *G*. Abbreviate the above matrix by the notation (N, A, u), where N is the set of players, $A = S \times T$, and $u = (b_1, b_2)$. Next, the set of all bimatrix representations of 3-strategy Rapoportian games is defined. Let $N = \{p_1, p_2\}$ be a set of two players, indexed by i. Let $A = S \times T$, where $S = \{s_1, s_2, s_3\}$ is a set of three strategies available to p_1 and $T = \{t_1, t_2, t_3\}$ is a set of three strategies available to player p_2. Let $B = \{b | b : A \leftrightarrow \{1, \dots, 9\}\}$. That is, B is the set of all possible payoff functions for 3-strategy Rapoportian games. Let $U = B \times B$. That is, U is the set of all possible 2-tuples of payoff functions. Then $X = \{(N, A, U)\}$ is the set of all bimatrix representations of 3-strategy Rapoportian games. To demonstrate that every element of X is a bimatrix representation of a 3-strategy Rapoportian game, pick any $x \in X$. Then $x = (N, A, u)$, where $N = \{p_1, p_2\}$, $A = \{s_1, s_2, s_3\} \times \{t_1, t_2, t_3\}$, and $u = (b_1, b_2)$, where $b_i : A \leftrightarrow \{1, \dots, 9\}$ is p_i's payoff function. Hence every element in X represents a 3-strategy Rapoportian game. Let $x_1, x_2 \in X$. Then $x_1 = x_2$ if and only if for all $i \in \{1,2\}$, for all $s \in S$, and for all $t \in T$, $b_i(s,t) = b'_i(s,t)$. That is, two bimatrix representations are equal precisely when their corresponding entries are equal.

Remark on the literature: in *Essentials of Game Theory* by Leyton-Brown and Shoham [2] the authors' definition of a normal form game is referred to in this paper as a *bimatrix representation of a normal form game*. The difference between the definition of a normal form game given in Luce and Raiffa's *Games and Decisions* [3, p. 55] and that given in Leyton-Brown and Shoham's *Essentials of Game Theory* [2, p. 3] is subtle. Luce and Raiffa's definition is used in this paper as it predates that found in *Essentials of Game Theory* by 43 years, and has apparently been cited more often.

3 Equivalent Representations

The representation relation between 3-strategy Rapoportian games and their bimatrix representations is not functional. That is, a given game may have several distinct bimatrix representations. Consider the following two bimatrix representations, x_1 and x_2:

$\boldsymbol{x_1}$	t_1	t_2	t_3
s_1	(9,1)	(8,2)	(7,3)
s_2	(6,4)	(5,5)	(4,6)
s_3	(3,7)	(2,8)	(1,9)

$\boldsymbol{x_2}$	t_1	t_2	t_3
s_1	(6,4)	(5,5)	(4,6)
s_2	(9,1)	(8,2)	(7,3)
s_3	(3,7)	(2,8)	(1,9)

A quick check shows that $b_1(s_1, t_1) = 9 \neq 6 = b_1'(s_1, t_1)$. Thus $x_1 \neq x_2$. Yet upon inspection x_1 and x_2 are similar in that they have simply had their first and second rows interchanged and their strategies relabeled. Thus x_1 and x_2 are two bimatrix representations of the "same game" (up to a relabeling of the strategies). More generally, let $x_1, x_2 \in X$, with $x_1 = (N, A, u), u = (b_1, b_2)$, $x_2 = (N, A, u')$, and $u' = (b'_1, b'_2)$. Let $\pi_s: \{1,2,3\} \leftrightarrow \{1,2,3\}$ and $\pi_t: \{1,2,3\} \leftrightarrow$

{1,2,3} be any permutations. Then x_1 and x_2 are to be considered representations of the same game if for all $i \in \{1, 2\}$, for all $s_i \in S$, and for all $t_k \in T$, $b_i\big(s_j, t_k\big) = b'_i(s_{\pi_s(j)}, t_{\pi_t(k)})$. Colloquially, any permutation of the rows and columns of a bimatrix representation results in a bimatrix representation of the same game up to a relabeling of the strategies. Now consider a third game, $x_3 \in X$, shown alongside x_1:

$\boldsymbol{x_1}$	t_1	t_2	t_3
s_1	(9,1)	(8,2)	(7,3)
s_2	(6,4)	(5,5)	(4,6)
s_3	(3,7)	(2,8)	(1,9)

$\boldsymbol{x_3}$	t_1	t_2	t_3
s_1	(1,9)	(4,6)	(7,3)
s_2	(2,8)	(5,5)	(8,2)
s_3	(3,7)	(6,4)	(9,1)

Again, observation shows that $b_1(s_1, t_1) = 9 \neq 1 = \; b_2{}^*(s_1, t_1)$. Thus $x_1 \neq x_3$. But x_1 and x_3 are similar in that they have had their players' indexes switched (the row player in one is the column player in the other). That is, the bimatrix has been transposed and the order of the payouts permuted. Again, x_1 and x_3 are distinct bimatrix representations of a common game up to a relabeling of the players.

As in Rapoport and Guyer's *A taxonomy of 2x2 games* [4], two bimatrix representations are considered as representing the same game if one can be obtained from the other by a finite sequence of interchanging rows, interchanging columns, and/or interchanging players . This idea is formalized by considering all the possible finite sequences of row interchanges, column interchanges, and player interchanges of bimatrix representations of 3-strategy Rapoportian games. Next, create the group of all the symmetries of bimatrix representations of 3-strategy Rapoportian games. Its elements are the finite sequences of row, column, and player interchanges. For a 3-strategy Rapoportian game the group can be thus generated via the following elements:

The interchange of rows 1 and 2, denoted $R12$.
The interchange of rows 1 and 3, denoted $R13$.
The interchange of rows 2 and 3, denoted $R23$.
The interchange of columns 1 and 2, denoted $C12$.
The interchange of columns 1 and 3, denoted $C13$.
The interchange of columns 2 and 3, denoted $C23$.
The interchange of the row and column players, denoted T.

In fact, this generating set is redundant. In particular, $R12\; R23\; R12 = R13$ and $C12\; C23\; C12 = C13$, so this group is generated by the elements: $R12$, $R23$, $C12$, $C23$,and T.

The generators relate in the following way. Each generator is its own inverse: $R12^2 = R23^2 = C12^2 = C23^2 = T^2 = e$. The row and column interchanges commute: $12\; C12 = C12\; R12$, $R12\; C23 = C23\; R12$, $R23\; C12 = C12\; R23$, and $R23\; C23 = C23\; R23$. A player switch followed by a row switch is identified with the corresponding column switch followed by a player switch: $T\; R12 = C12\; T$, and $T\; R23 = C23\; T$. Finally: $R12R23R12 = R23R12R23$, and $C12C23C12 = C23C12C23$.

This group has 72 elements. It is isomorphic to $(C_3 \times C_3) \rtimes D_4$ [F.R. Beyl 7, private communication]. This group is not constructed in this paper. All work with this group is done relative to the generators and relations. Throughout this paper we refer to this group as G_{72}. The group G_{72} has a normal subgroup of 36 elements generated by the generators of G_{72} excluding T. We refer to this subgroup as G_{36}.

Lemma 1. The quotient group ${}^{G_{72}}/_{G_{36}}$ is isomorphic to ${}^{\mathbb{Z}}/_{2\mathbb{Z}}$.

Proof: Note that $\left|{}^{G_{72}}/_{G_{36}}\right| = \frac{|G_{72}|}{|G_{36}|} = \frac{72}{36} = 2$ ■

For each $g \in G_{72}$ define a map $\rho_g: U \leftrightarrow U$ in the following way. Let $\rho_T(u) = \rho_T\big((b_1, b_2)\big) = (b'_1, b'_2) = u'$ where b'_i is define as follows. For all $j, k \in \{1, 2, 3\}$, $b'_i\big(s_j, t_k\big) = b_{3-i}(s_k, t_j)$. For the remaining generators of G_{72}, note that each only permutes the rows or columns of a bimatrix representation, not the players. Let $\pi_s: \{1,2,3\} \leftrightarrow \{1,2,3\}$ be the permutation that describes the row interchanges of such a generator g, and let $\pi_t: \{1,2,3\} \leftrightarrow \{1,2,3\}$ be the permutation that describes the column interchanges. Then $\rho_g(u) = \rho_g\big((b_1, b_2)\big) = (b'_1, b'_2) = u'$ where b'_i is defined by: for all $j, k \in \{1, 2, 3\}$, $b'_i\big(s_j, t_k\big) = b_i\big(s_{\pi_s{}^{-1}(j)}, t_{\pi_t{}^{-1}(k)}\big)$. As an illustration, consider the bimatrix representations x_1 and x_2 from before:

$\boldsymbol{x_1}$	t_1	t_2	t_3
s_1	(9,1)	(8,2)	(7,3)
s_2	(6,4)	(5,5)	(4,6)
s_3	(3,7)	(2,8)	(1,9)

$\boldsymbol{x_2}$	t_1	t_2	t_3
s_1	(6,4)	(5,5)	(4,6)
s_2	(9,1)	(8,2)	(7,3)
s_3	(3,7)	(2,8)	(1,9)

The group element that sends x_1 to x_2 is $\boldsymbol{R12}$. Let π_s be the permutation given by: $\pi_s(1) = 2$, $\pi_s(2) = 1$, and $\pi_s(3) = 3$. Let π_t be the permutation given by: $\pi_t(i) = i$ for all $i \in \{1, 2, 3\}$. Let $u = (b_1, b_2)$. Then $\rho_{R12}(u) = \rho_{R12}\big((b_1, b_2)\big) = (b'_1, b'_2) = u'$ where b'_i is define by: for all $j, k \in \{1, 2, 3\}$, $b'_i\big(s_j, t_k\big) = b_i\big(s_{\pi_s{}^{-1}(j)}, t_{\pi_t{}^{-1}(k)}\big)$.

Now that the map ρ_g has been defined for all of the generators of G_{72}, the map ρ_g for the rest of the elements of G_{72} is defined in the following way. Let g_1, g_2, $g_3 \in G_{72}$. Suppose $g_1 = g_2 g_3$. Then define ρ_{g_1} by $\rho_{g_1}(u) = \rho_{g_2} \circ \rho_{g_3}(u)$. For example, $\rho_{\mathrm{TC23C12R12}} = \rho_T \circ \rho_{C23} \circ \rho_{C12} \circ \rho_{R12}$. Finally, let $P = \{\rho_g | g \in G_{72}\}$. Note that P forms a group under composition and is isomorphic to G_{72} via the map $g \mapsto \rho_g$. Next, define a group action of G_{72} on the set, X, of all bimatrix representations of 3-strategey Rapoportian games using ρ_g. Explicitly, define $*: G \times X \to X$ via $g * x = g * (N, A, u) = (N, A, \rho_g(u))$. For convenience, write "gx" instead of "$g * x$".

Theorem 1. The set X is a G_{72}-set.

Proof: We observe that $ex = e(N, A, u) = \big(N, A, \rho_e(u)\big) = (N, A, u) = x$, and that $(g_1g_2)x = (g_1g_2)(N, A, u) = \left(N, A, \rho_{g_1g_2}(u)\right) = \left(N, A, \rho_{g_1} \circ \rho_{g_2}(u)\right) = g_1\left(N, A, \rho_{g_2}(u)\right) = g_1\big(g_2(N, A, u)\big) = g_1(g_2x)$. Hence $*$ is an action of G_{72} on X and X is a G_{72}-set. ■

Define a relation, $\sim$, on X by $x_1 \sim x_2$ if there exists $g \in G_{72}$ such that $gx_1 = x_2$. Colloquially, $x_1 \sim x_2$ precisely when x_1 and x_2 represent the same game.

Theorem 2. The relation $\sim$ is an equivalence relation on X.

Proof: (Reflexive) $ex = x$. Thus $x \sim x$.

(Symmetric) Suppose $x_1 \sim x_2$. Then there exists $g \in G_{72} \in gx_1 = x_2$. It follows that $g^{-1}x_2 = x_1$, and $x_2 \sim x_1$.

(Transitive) Suppose $x_1 \sim x_2$ and $x_2 \sim x_3$. Then there exists $g_1 \in G_{72} \in g_1x_1 = x_2$, and there exists $g_2 \in G_{72} \in g_2x_2 = x_3$. Then $g_2g_1x_1 = g_2x_2 = x_3$. Hence $x_1 \sim x_3$. ■

Note that $x_1 \sim x_2$ precisely when x_1 and x_2 are in a common orbit of X under the action of G_{72}. Since $x_1 \sim x_2$ precisely when x_1 and x_2 represent the same game, the problem of counting games is thus reduced to the problem of counting orbits of X under the action of G_{72}.

4 Symmetric, Asymmetric, and Standard Symmetric Representations

Let $x \in X$. We will refer to x as a *symmetric bimatrix representation* if there exists a $\phi: S \leftrightarrow T$ such that for all $s \in S$, and for all $t \in T$, $b_1(s,t) = b_2(\phi^{-1}(t), \phi(s))$. Informally, a bimatrix representation, x, is a symmetric bimatrix representation if $x = gx$ for some $g \in G_{72} \backslash G_{36}$. We refer to every bimatrix representation that is not a symmetric bimatrix representation as an *asymmetric bimatrix representation*. Also, we'll refer to the outcome of a strategy pair $(s,t) \in A$ as a *central outcome* if $b_1(s,t) = b_2(s,t)$. Let c be such a central outcome. Then the *value of* c, denoted $v(c)$, will be defined by $b_1(s,t) = b_2(s,t) = v(c)$. For example, consider the following bimatrix representation, x_1:

$\boldsymbol{x_1}$	t_1	t_2	t_3
s_1	$(\mathbf{9}, \mathbf{9})$	(8,5)	(7,3)
s_2	(6,4)	(5,2)	(4,1)
s_3	(3,7)	(2,8)	(1,6)

The outcome for (s_1, t_1) is a central outcome because $b_1(s_1, t_1) = 9 = b_2(s_1, t_1)$. The value of the central outcome (s_1, t_1) is 9. The outcome for (s_2, t_1) is *not* a central outcome because $b_1(s_2, t_1) = 6 \neq 4 = b_2(s_2, t_1)$.

Theorem 3: Every symmetric bimatrix representation has exactly one central outcome in each row and column.

Proof: Let $x \in X$ be a symmetric bimatrix representation. Then there exists a $\phi: S \leftrightarrow T$ such that for all $s \in S$, and all $t \in T, b_1(s,t) = b_2(\phi^{-1}(t), \phi(s))$. Pick any $s_j \in S$.

(1) Let $\phi(s_j) = t_k$. Then $\phi^{-1}(t_k) = s_j$.

(2) Since x is a symmetric bimatrix representation, $b_1(s_j, t_k) = b_2\left(\phi^{-1}(t_k),\ \phi(s_j)\right)$.

(3) By (1), $b_2\left(\phi^{-1}(t_k), \phi(s_j)\right) = b_2(s_j, t_k)$.

(4) By (2), (3), and transitivity, $b_1(s_j, t_k) = b_2(s_j, t_k)$.

By definition (s_j, t_k) is a central outcome. Thus there is *at least* one central outcome in each row, s_j. A similar argument shows that there is at least one central outcome in each column, t_k (simply pick any $t_k \in T$, note that there exists $s_j \in S$ such that $\phi(s_j) = t_k$, and proceed precisely as before). To show that there is *at most* one central out come in each row and column, let $(s_j, t_{k'})$ be another central outcome of x in row s_j. It must be shown that $t_{k'} = t_k$.

(4) As $(s_j, t_{k'})$ is a central outcome, $b_1(s_j, t_{k'}) = b_2(s_j, t_{k'})$.

(5) Since x is a symmetric bimatrix representation, $b_1(s_j, t_{k'}) = b_2\left(\phi^{-1}(t_{k'}),\ \phi(s_j)\right)$.

By (4), (5), and transitivity, $b_2\left(\phi^{-1}(t_{k'}),\ \phi(s_j)\right) = b_2(s_j, t_{k'})$.

Since b_2 is a bijection, $\left(\phi^{-1}(t_{k'}),\ \phi(s_j)\right) = (s_j, t_{k'})$.

(6) Hence $\phi(s_j) = t_{k'}$.

By (1),(6), and transitivity, $t_{k'} = t_k$. It follows that $(s_j, t_{k'}) = (s_j, t_k)$, so there is *at most* one central outcome in each row. A similar argument shows that there is at most one central outcome in each column. Hence every symmetric bimatrix representation has exactly one central outcome in each row and column. ■

Now let $x \in X$ be a symmetric bimatrix representation with central outcomes c_1, c_2, c_3 such that $v(c_1) > v(c_2) > v(c_3)$. We refer to x as a *standard symmetric bimatrix representation* if for all $i \in \{1,2,3\}, (s_i, t_i) = c_i$. For clarity, observe three representations that are standard symmetric and three that aren't. These first three representations *are* standard symmetric bimatrix representations with central outcomes in bold:

$\boldsymbol{x_1}$	t_1	t_2	t_3
s_1	$(\mathbf{9},\mathbf{9})$	(5,6)	(4,3)
s_2	(6,5)	$(\mathbf{8},\mathbf{8})$	(2,1)
s_3	(3,4)	(1,2)	$(\mathbf{7},\mathbf{7})$

$\boldsymbol{x_2}$	t_1	t_2	t_3
s_1	$(\mathbf{9},\mathbf{9})$	(6,4)	(7,3)
s_2	(4,6)	$(\mathbf{5},\mathbf{5})$	(2,8)
s_3	(3,7)	(8,2)	$(\mathbf{1},\mathbf{1})$

$\boldsymbol{x_3}$	t_1	t_2	t_3
s_1	$(\mathbf{3},\mathbf{3})$	(7,4)	(6,8)
s_2	(4,7)	$(\mathbf{2},\mathbf{2})$	(9,5)
s_3	(8,6)	(5,9)	$(\mathbf{1},\mathbf{1})$

All three are symmetric bimatrix representations with the greatest, middle, and least valued central outcomes in the upper left, middle, and lower right positions respectively. The next three representations are *not* standard symmetric bimatrix representations. The central outcomes are in bold:

$\boldsymbol{x_4}$	t_1	t_2	t_3	$\boldsymbol{x_5}$	t_1	t_2	t_3	$\boldsymbol{x_6}$	t_1	t_2	t_3
s_1	**(8,8)**	(5,6)	(4,3)	s_1	(6,4)	**(9,9)**	(7,3)	s_1	**(3,3)**	(7,7)	(6,6)
s_2	(6,5)	**(9,9)**	(2,1)	s_2	**(5,5)**	(4,6)	(2,8)	s_2	(4,4)	**(2,2)**	(9,5)
s_3	(3,4)	(1,2)	**(7,7)**	s_3	(8,2)	(3,7)	**(1,1)**	s_3	(8,8)	(5,9)	**(1,1)**

The representation x_4 is not standard symmetric because $c_1 = (s_2, t_2) \neq (s_1, t_1)$. That is, the highest valued central outcome is not in the upper left corner of the bimatrix. The representation x_5 is not standard symmetric since $(s_1, t_1) \neq c_1 = (s_1, t_2)$. That is, the central outcomes of x_5 do not lie along the main diagonal. The representation x_6 is not standard symmetric either. It *is* the case that the central outcomes of x_6 lie along the main diagonal in the correct order, but x_6 is not a symmetric bimatrix representation, hence it cannot be standard symmetric.

Theorem 4: If $x \in X$ is a standard symmetric bimatrix representation then $ex = Tx = x$.
Proof: Since X is a G_{72}-set, $ex = x$. It must be shown that $Tx = x$. Let $x \in X$ be a standard symmetric bimatrix representation. Since x is a symmetric bimatrix representation, there exists a bijection $\phi: S \leftrightarrow T$ such that for all $s \in S$, and all $t \in T, b_1(s,t) = b_2(\phi^{-1}(t), \phi(s))$. Since x is a standard symmetric bimatrix representation, for all $i \in \{1,2,3\}, (s_i, t_i) = c_i$, were the c_i's are the central outcomes in the definition of a standard symmetric bimatrix representation. Since c_i is a central outcome, $b_1(c_i) = b_2(c_i)$.

Recall that because x is standard symmetric, for all $i \in \{1,2,3\}$, $c_i = (s_i, t_i)$. Also recall that since x is a symmetric bimatrix representation, for all $i \in \{1,2,3\}$, $b_1(s_i, t_i) = b_2(\phi^{-1}(t_i), \phi(s_i))$. Hence for all $i \in \{1,2,3\}$, $= b_2\big(\phi^{-1}(t_i), \phi(s_i)\big) = b_1(s_i, t_i) = b_1(c_i) = b_2(c_i) = b_2(s_i, t_i)$. Specifically, for all $i \in \{1,2,3\}$, $b_2(\phi^{-1}(t_i), \phi(s_i)) = b_2(s_i, t_i)$. Since b_2 is a bijection, it follows that for all $i \in \{1,2,3\}$, $(\phi^{-1}(t_i), \phi(s_i)) = (s_i, t_i)$.
(1) Hence for all $i \in \{1,2,3\}$, $\phi(s_i) = t_i$.
Now let $j, k \in \{1,2,3\}$.
(2) Then since x is a symmetric bimatrix representation, $b_1(s_j, t_k) = b_2\big(\phi^{-1}(t_k), \phi(s_j)\big)$.
(3) By (1), $b_2\big(\phi^{-1}(t_k), \phi(s_j)\big) = b_2(s_k, t_j)$.
(4) By (2), (3), and transitivity, $b_2(s_k, t_j) = b_1(s_j, t_k)$.
Note that (4) asserts that b_2 is expressible in terms of b_1. Consider the standard symmetric bimatrix representation, x:

$\boldsymbol{x}$	t_1	t_2	t_3
s_1	$b_1(s_1,t_1), b_2(s_1,t_1)$	$b_1(s_1,t_2), b_2(s_1,t_2)$	$b_1(s_1,t_3), b_2(s_1,t_3)$
s_2	$b_1(s_2,t_1), b_2(s_2,t_1)$	$b_1(s_2,t_2), b_2(s_2,t_2)$	$b_1(s_2,t_3), b_2(s_2,t_3)$
s_3	$b_1(s_3,t_1), b_2(s_3,t_1)$	$b_1(s_3,t_2), b_2(s_3,t_2)$	$b_1(s_3,t_3), b_2(s_3,t_3)$

Using (4), write all the entries in x in terms of b_1:

$\boldsymbol{x}$	t_1	t_2	t_3
s_1	$b_1(s_1,t_1), b_1(s_1,t_1)$	$b_1(s_1,t_2), b_1(s_2,t_1)$	$b_1(s_1,t_3), b_1(s_3,t_1)$
s_2	$b_1(s_2,t_1), b_1(s_1,t_2)$	$b_1(s_2,t_2), b_1(s_2,t_2)$	$b_1(s_2,t_3), b_1(s_3,t_2)$
s_3	$b_1(s_3,t_1), b_1(s_1,t_3)$	$b_1(s_3,t_2), b_1(s_2,t_3)$	$b_1(s_3,t_3), b_1(s_3,t_3)$

Next, we write the bimatrix Tx using (4) to express each entry in terms of b_1:

$$\begin{array}{cccc}
\boldsymbol{x} & t_1 & t_2 & t_3 \\
s_1 & b_1(s_1,t_1), b_1(s_1,t_1) & b_1(s_1,t_2), b_1(s_2,t_1) & b_1(s_1,t_3), b_1(s_3,t_1) \\
s_2 & b_1(s_2,t_1), b_1(s_1,t_2) & b_1(s_2,t_2), b_2(s_2,t_2) & b_1(s_2,t_3), b_1(s_3,t_2) \\
s_3 & b_1(s_3,t_1), b_1(s_1,t_3) & b_1(s_3,t_2), b_1(s_2,t_3) & b_1(s_3,t_3), b_1(s_3,t_3)
\end{array}$$

Comparing entry by entry, we observe that $Tx = x$. Hence $ex = Tx = x$. ■

Theorem 5: If $x \in X$ is a standard symmetric bimatrix representation and $g \in G_{72}$ such that $gx = x$ then $g = e$ or $g = T$.
Proof: Let $x \in X$ be a standard symmetric bimatrix representation. Let the central outcomes be labeled c_1, c_2, c_3, such that $v(c_1) > v(c_2) > v(c_3)$. Since x is a standard symmetric bimatrix representation, the central outcomes of x are arranged in the bimatrix of x as follows:

$$\begin{array}{c|c|c|c|}
\boldsymbol{x} & t_1 & t_2 & t_3 \\ \hline
s_1 & c_1 & & \\ \hline
s_2 & & c_2 & \\ \hline
s_3 & & & c_3 \\ \hline
\end{array}.$$

By Theorem 4, $ex = Tx = x$, but note that any $g \in G_{72}$ (other than e or T) would permute the rows and columns of x in such a way that for some $i \in \{1,2,3\}, (s_i, t_i) \neq c_i$. This is because every $g \in G_{72}$ (other than e or T) permutes – in a non-trivial way – the rows and/or columns of x. Since there is exactly one central outcome in each row and column, g would move at least one of these central outcomes from the position it must occupy for gx to be a standard symmetric bimatrix representation. Thus e and T are the only elements in G_{72} such that gx is a standard symmetric bimatrix representation. Hence if $x \in X$ is a standard symmetric bimatrix representation and $g \in G_{72}$ such that $gx = x$ then $g = e$ or $g = T$. ■

Theorem 6 (Corollary): If $x \in X$ is a standard symmetric bimatrix representation and $g \in G_{32}$ such that $gx = x$ then $g = e$.
Proof: The proof follows directly from Theorem 5 and the fact that $T \notin G_{36}$. ■

Theorem 7: If $x \in X$ is a symmetric bimatrix representation then there exists a unique $g \in G_{36}$ such that gx is a standard symmetric bimatrix representation.
Proof: Let $x \in X$ be a symmetric bimatrix representation. By Theorem 3, x has exactly one central outcome in each row and column. This can happen in one of six ways:

x	t_1	t_2	t_3
s_1	■		
s_2		■	
s_3			■

,

x	t_1	t_2	t_3
s_1	■		
s_2			■
s_3		■	

,

x	t_1	t_2	t_3
s_1		■	
s_2	■		
s_3			■

,

x	t_1	t_2	t_3
s_1		■	
s_2			■
s_3	■		

,

x	t_1	t_2	t_3
s_1			■
s_2	■		
s_3		■	

, or

x	t_1	t_2	t_3
s_1			■
s_2		■	
s_3	■		

,

where the shaded boxes represent the positions of the central outcomes of x. Let the central outcomes be labeled c_1, c_2, c_3, such that $v(c_1) > v(c_2) > v(c_3)$.

<u>Case 1</u>: The central outcomes of x occupy the the following positions:

x	t_1	t_2	t_3
s_1	■		
s_2		■	
s_3			■

Then the outcomes $c_1, c_2,$ and c_3 must be in one of the following six arrangements:

x	t_1	t_2	t_3
s_1	c_1		
s_2		c_2	
s_3			c_3

,

x	t_1	t_2	t_3
s_1	c_1		
s_2		c_3	
s_3			c_2

,

x	t_1	t_2	t_3
s_1	c_2		
s_2		c_1	
s_3			c_3

,

x	t_1	t_2	t_3
s_1	c_3		
s_2		c_1	
s_3			c_2

,

x	t_1	t_2	t_3
s_1	c_2		
s_2		c_3	
s_3			c_1

, or

x	t_1	t_2	t_3
s_1	c_3		
s_2		c_2	
s_3			c_1

<u>Case 1-1</u>: The outcomes $c_1, c_2,$ and c_3 are in the following arrangement:

x	t_1	t_2	t_3
s_1	c_1		
s_2		c_2	
s_3			c_3

.

In this case x is already a standard symmetric bimatrix representation. Since $e \in G_{36}$, and $ex = x$, there exists $g \in G_{36}$ such that gx is a standard symmetric bimatrix representation. Furthermore, suppose that for some $g' \in G_{36}$ with $g' \neq e$, $g'x$ was a standard symmetric bimatrix representation. Then $g'x = g'(ex)$. Since ex is a standard symmetric bimatrix representation, Theorem 6 states that $g' = e$. $\Rightarrow\Leftarrow$. So e is the unique element, $g \in G_{36}$, such that gx is a standard symmetric bimatrix representation. Hence there exists a unique $g \in G_{36}$ such that gx is a standard symmetric bimatrix representation.

<u>Case 1-2</u>: The outcomes $c_1, c_2,$ and c_3 are in the following arrangement:

x	t_1	t_2	t_3
s_1	c_1		
s_2		c_3	
s_3			c_2

.

Then $C_{23}R_{23}x$ is a standard symmetric bimatrix representation. Furthermore, for any $g' \in G_{36}$ with $g' \neq C_{23}R_{23}$, $g'x$ would not be a standard symmetric bimatrix representation (again using Theorem 6). Hence $C_{23}R_{23}$ is the only $g \in G_{36}$ such that gx is a standard symmetric bimatrix representation. Thus there exists a unique $g \in G_{36}$ such that gx is a standard symmetric bimatrix representation.

<u>Case 1-3</u>: The outcomes $c_1, c_2,$ and c_3 are in the following arrangement:

x	t_1	t_2	t_3
s_1	c_2		
s_2		c_1	
s_3			c_3

.

Then $C_{12}R_{12}x$ is a standard symmetric bimatrix representation and $C_{12}R_{12}$ is the only $g \in G_{36}$ such that gx is a standard symmetric bimatrix representation. Hence there exists a unique $g \in G_{36}$ such that gx is a standard symmetric bimatrix representation.

<u>Case 1-4</u>: The outcomes $c_1, c_2,$ and c_3 are in the following arrangement:

x	t_1	t_2	t_3
s_1	c_3		
s_2		c_1	
s_3			c_2

.

Then $C_{23}C_{12}R_{23}R_{12}x$ is a standard symmetric bimatrix representation and $C_{23}C_{12}R_{23}R_{12}$ is the only $g \in G_{36}$ such that gx is a standard symmetric bimatrix representation. Hence there exists a unique $g \in G_{36}$ such that gx is a standard symmetric bimatrix representation.

<u>Case 1-5</u>: The outcomes $c_1, c_2,$ and c_3 are in the following arrangement:

x	t_1	t_2	t_3
s_1	c_2		
s_2		c_3	
s_3			c_1

.

Then $C_{13}C_{12}R_{13}R_{12}x$ is a standard symmetric bimatrix representation and $C_{13}C_{12}R_{13}R_{12}$ is the only $g \in G_{36}$ such that gx is a standard symmetric bimatrix representation. Hence there exists a unique $g \in G_{36}$ such that gx is a standard symmetric bimatrix representation.

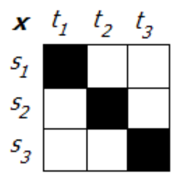
Case 1-6: The outcomes $c_1, c_2,$ and c_3 are in the following arrangement:

x	t_1	t_2	t_3
s_1	c_3		
s_2		c_2	
s_3			c_1

.

Then $C_{13}R_{13}x$ is a standard symmetric bimatrix representation and $C_{13}R_{13}$ is the only $g \in G_{36}$ such that gx is a standard symmetric bimatrix representation. Hence there exists a unique $g \in G_{36}$ such that gx is a standard symmetric bimatrix representation.

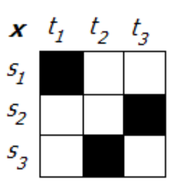
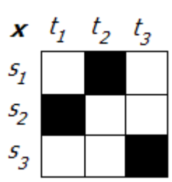
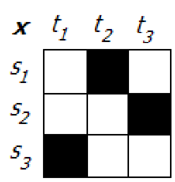
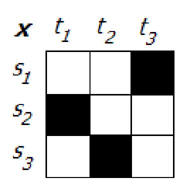
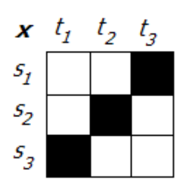
Therefore if the central outcomes of x occupy the the following shaded outcomes:



,

then there exists a unique $g \in G_{36}$ such that gx is a standard symmetric bimatrix representation.

Cases 2 through 6: For each of the five remaining cases:


, 
, 
, 
, and 
,

the proof that there exists a unique $g \in G_{36}$ such that gx is a standard symmetric bimatrix representation follows the same procedure as for Case 1, and is omitted for brevity. Therefore if $x \in X$ is a symmetric bimatrix representation then there exists a unique $g \in G_{36}$ such that gx is a standard symmetric bimatrix representation. ■

5 Symmetric and Asymmetric Orbits

In this section it is shown that if two bimatrix representations are equivalent, then either both of them are symmetric, or both of them are asymmetric. This shows that for each orbit of X under the action of G_{72} either all of the elements in the orbit are symmetric bimatrix representations, or all of the elements in the orbit are asymmetric bimatrix representations.

Lemma 2: For all $g \in G_{72} \backslash G_{36}$, there exists $g' \in G_{36}$ such that $g = Tg'$.
Proof: Let $g \in G_{72} \backslash G_{36}$. (It must be shown that there exists $g' \in G_{36}$ such that $g = Tg'$). Let $g' = Tg$. Then $g = T^{-1}g' = Tg'$. It need only be shown that $g' \in G_{36}$. By Lemma 1, ${G_{72}}/{G_{36}} \cong {\mathbb{Z}}/{2\mathbb{Z}}$. Therefore ${G_{72}}/{G_{36}}$ has precisely two elements, namely G_{36} and TG_{36}. Since $g \in G_{72} \backslash G_{36}$, $g \notin G_{36}$. In other words, $gG_{36} \neq G_{36}$. Therefore:
(1) $gG_{36} = TG_{36}$
Also, since $g' = Tg$, it follows that:
(2) $g'G_{36} = TgG_{36}$.
From **(1)** and **(2)** follows:

$$g'G_{36} = TgG_{36} = TG_{36}\, gG_{36} = TG_{36}\, TG_{36} = TTG_{36} = G_{36}$$

By transitivity, $g'G_{36} = G_{36}$. Therefore $g' \in G_{36}$ as desired. ■

Theorem 8: If $x_1, x_2 \in X$, x_1 is a symmetric bimatrix representation, and $x_1 \sim x_2$ then x_2 is a symmetric bimatrix representation.
Proof: Let $x_1, x_2 \in X$ with x_1 a symmetric bimatrix representation. Suppose $x_1 \sim x_2$. Then there exists $g \in G_{72} \ni gx_1 = x_2$. Either $g \in G_{36}$, or $g \in G_{72} \backslash G_{36}$.

Case 1: Suppose $g \in G_{36}$. Let G_R be the subgroup of G_{36} that is generated by $R12$**,** $R13$**,** and $R23$. If $g_r \in G_R$ then g_r a finite sequence of row interchanges. Similarly, let G_C be the subgroup of G_{36} that is generated by $C12$, $C13$, and $C23$. If $g_c \in G_C$ then g_c is a finite sequence of column interchanges. Thus there are four sub-cases to consider:

1) $g = e$
2) $g \in G_R$ with $g \neq e$. That is, g is a non-trivial row interchange.
3) 3) $g \in G_C$ with $g \neq e$. That is, g is a non-trivial column interchange.
4) $g \in \{G_{36} \backslash G_R\} \backslash G_C$. That is, g performs a non-trivial row interchange *and* a non-trivial column interchange.

For each case, show that x_2 is a symmetric bimatrix representation.

Case 1-1: Suppose $g = e$. Then $gx_1 = ex_1 = x_1 = x_2$. Since $x_1 = x_2$ and x_1 is a symmetric bimatrix representation, it follows that x_2 is a symmetric bimatrix representation.

Case 1-2: Suppose $g \in G_R$ with $g \neq e$. Let $x_1 = (N, A, u)$, $u = (b_1, b_2)$. Let $x_2 = (N, A, u')$, $u' = ({b_1}', {b_2}')$. Let $\pi_s: \{1,2,3\} \leftrightarrow \{1,2,3\}$ be the permutation of rows performed by g such that:

(1) $$b_i'\left(s_{\pi_s(j)}, t\right) = b_i\left(s_j, t\right).$$

Rewrite (1) as follows:

(2) $$b_i'\left(s_j, t\right) = b_i\left(s_{\pi_s{}^{-1}(j)}, t\right).$$

Since x_1 is a symmetric bimatrix representation, there exists $\phi: S \leftrightarrow T$ such that for all $s \in S,$ and for all $t \in T, b_1(s,t) = b_2(\phi^{-1}(t), \phi(s))$. Therefore:

(3) $$b_1(s,t) = b_2(\phi^{-1}(t), \phi(s)).$$

Suppose $\phi(s_j) = t_k$. Then $\phi^{-1}(t_k) = s_j$. In order to show that x_2 is a symmetric bimatrix representation it must be shown that there exists $\phi': S \leftrightarrow T$ such that for all $s \in S,$ and for all $t \in T, b_1{}'(s,t) = b_2{}'(\phi'^{-1}(t), \phi'(s))$. Let $\phi': S \leftrightarrow T$ be defined as follows:

(4) $$\phi'\left(s_{\pi_s(j)}\right) = \phi\left(s_j\right) = t_k.$$

It must be shown that for all $s \in S,$ and for all $t \in T, b_1{}'(s,t) = b_2{}'(\phi'^{-1}(t), \phi'(s))$. By (4), note that $\phi'^{-1}(t_k) = \phi'^{-1}\left(\phi\left(s_j\right)\right) = \phi'^{-1}\left(\phi'\left(s_{\pi_s(j)}\right)\right) = s_{\pi_s(j)}$. Hence:

(5) $$\phi'^{-1}(t_k) = s_{\pi_s(j)}.$$

Also note that by (4):

(6) $$\phi'\left(s_j\right) = \phi\left(s_{\pi_s{}^{-1}(j)}\right).$$

Pick any $s_m \in S$ and any $t_n \in T$.

By (2), $b_1'(s_m, t_n) = b_1\left(s_{\pi_s{}^{-1}(m)}, t_n\right)$.

By (3), $b_1\left(s_{\pi_s{}^{-1}(m)}, t_n\right) = b_2\left(\phi^{-1}(t_n), \phi\left(s_{\pi_s{}^{-1}(m)}\right)\right)$.

Let $\phi^{-1}(t_n) = s_l$ for some $s_l \in S$. Then

$b_2\left(\phi^{-1}(t_n), \phi\left(s_{\pi_s{}^{-1}(m)}\right)\right) = b_2\left(s_l, \phi\left(s_{\pi_s{}^{-1}(m)}\right)\right)$.

By (1), $b_2\left(s_l, \phi\left(s_{\pi_s{}^{-1}(m)}\right)\right) = b_2'\left(s_{\pi_s(l)}, \phi\left(s_{\pi_s{}^{-1}(m)}\right)\right)$.

By (5), $b_2'\left(s_{\pi_s(l)}, \phi\left(s_{\pi_s{}^{-1}(m)}\right)\right) = b_2'\left(\phi'^{-1}(t_n), \phi\left(s_{\pi_s{}^{-1}(m)}\right)\right)$.

By (6), $b_2'\left(\phi'^{-1}(t_n), \phi\left(s_{\pi_s{}^{-1}(m)}\right)\right) = b_2'\left(\phi'^{-1}(t_n), \phi'(s_m)\right)$.

Finally, by transitivity, $b_1'(s_m, t_n) = b_2'\left(\phi'^{-1}(t_n), \phi'(s_m)\right)$. Thus, there exists $\phi': S \leftrightarrow T$ such that for all $s \in S,$ and all $t \in T, b_1{}'(s,t) = b_2{}'(\phi'^{-1}(t), \phi'(s))$. By definition, x_2 is a symmetric bimatrix representation.

<u>Case 1-3</u>: Suppose $g \in G_C$ with $g \neq e$. The argument follows the same procedure as Case 1-2, but let the permutation operate on the columns instead of the rows. Conclude that x_2 is a symmetric bimatrix representation.

<u>Case 1-4</u>: Suppose $g \in \{G_{36} \backslash G_R\} \backslash G_C$ with $g \neq e$. Note that $g = g_c g_r$ for some $g_c \in G_C$ and $g_r \in G_R$ with $g_c \neq e \neq g_r$ (otherwise either $g \in G_C$ or $g \in G_R$. $\Rightarrow\Leftarrow$). Since $g x_1 = x_2$, by substitution, $g_c g_r x_1 = x_2$. Note that $x_1 \sim g_r x_1$. Since $g_r \in G_R$ with $g_r \neq e$, and x_1 is a symmetric bimatrix representation, Case 1-2 applies. Therefore $g_r x_1$ is a symmetric bimatrix representation. Now note that $(g_r x_1) \sim g_c(g_r x_1)$. Since$g_c \in G_C$ with $g_c \neq e$, and since $(g_r x_1)$ is a symmetric bimatrix representation, Case 1-3 applies.

Therefore $g_c(g_r x_1)$ is a symmetric bimatrix representation. Finally, since $g_c g_r x_1 = x_2$, x_2 is a symmetric bimatrix representation.

Case 2: Suppose $g \in G_{72} \backslash G_{36}$. This case reduces to case to Case 1 as follows. Since x_1 is a symmetric bimatrix representation, Theorem 7 says there exists a unique $g^* \in G_{36}$ such that $g^* x_1$ is a standard symmetric bimatrix representation. Since $g x_1 = x_2$, it follows that $g^{-1} x_2 = x_1$. Operating on the right by g^*:

(1) $g^{-1} g^* x_2 = g^* x_1$

Since $g \in G_{72} \backslash G_{36}$ and ${G_{72}}/{G_{36}} \cong {\mathbb{Z}}/{2\mathbb{Z}}$, it follows that $g^{-1} \in G_{72} \backslash G_{36}$. Since $g^{-1} \in G_{72} \backslash G_{36}$, Lemma 2 guarantees there exists $g' \in G_{36}$ such that $g^{-1} = Tg'$. Substituting Tg' for g^{-1} into equation (1):

(2) $(Tg'g^*) x_2 = g^* x_1$

Since $g^* x_1$ is a standard symmetric bimatrix representation it follows that $(Tg'g^*) x_2$ is a standard symmetric bimatrix representation. Then by Theorem 4:

(3) $(Tg'g^*) x_2 = T(Tg'g^*) x_2 = e(g'g^*) x_2 = g'g^* x_2$

(4) By (2), (3), and transitivity, $g'g^* x_2 = g^* x_1$.

(5) By (4) and cancellation, $g' x_2 = x_1$.

(6) By (5), $g'^{-1} x_1 = x_2$.

Since $g' \in G_{36}$ and G_{36} is a group:

(7) $g'^{-1} \in G_{36}$.

By (6) and (7), Case 1 applies. Thus x_2 is a symmetric bimatrix representation.■

Theorem 8 establishes that for each orbit of X either all of the elements in the orbit are symmetric bimatrix representations, or all of the elements in the orbit are asymmetric bimatrix representations. If all of the elements in the orbit of x under the action of G_{72} are symmetric bimatrix representations, then we refer to the orbit of x as a *symmetric orbit*. If all of the elements in the orbit of x under the action of G_{72} are asymmetric bimatrix representations, then we refer to the orbit of x as an *asymmetric orbit*.

6 The Order of a Symmetric Orbit

Now we will show that every symmetric orbit contains precisely one standard symmetric bimatrix representation. Then we will show that the order of the orbit of any standard symmetric bimatrix representation is precisely 36. We will thus conclude that each symmetric orbit contains exactly 36 elements.

Theorem 9: If $x \in X$ is a symmetric bimatrix representation then x is equivalent to precisely one standard symmetric bimatrix representation.

Proof: Let $x \in X$ be a symmetric bimatrix representation. Then by Theorem 7 there exists a unique $g \in G_{36}$ such that gx is a standard symmetric bimatrix representation. By the definition of ~, $x \sim gx$. Since gx is a standard symmetric bimatrix representation, x is equivalent to a standard symmetric bimatrix representation. To show the uniqueness of gx, suppose that there exists $g' \in G_{72}$ such that $g'x$ is a standard symmetric bimatrix representation. It must be shown that $g'x = gx$. Either $g' \in G_{36}$ or $g' \in G_{72} \backslash G_{36}$.

Case 1: Suppose $g' \in G_{36}$. Then since Theorem 7 guarantees uniqueness of g, $g' = g$, and thus $g'x = gx$.

Case 2: Suppose $g' \in G_{72} \backslash G_{36}$. By Theorem 3, there exists $g^* \in G_{36}$ such that $g' = Tg^*$. Since, $g'x$ is a standard symmetric bimatrix representation, Tg^*x is a standard symmetric bimatrix representation. Since , Tg^*x is a standard symmetric bimatrix representation, Theorem 4 tells us that $Tg^*x = g^*x$. Hence g^*x is a standard symmetric bimatrix representation. As $g^* \in G_{36}$, and g^*x is a standard symmetric bimatrix representation, Theorem 7 asserts that $g^* = g$. Thus $g^*x = gx$. Finally, since $g'x = Tg^*x = g^*x = gx$, it follows that $g'x = gx$, as desired. Thus if $x \in X$ is a symmetric bimatrix representation, then x is equivalent to exactly one standard symmetric bimatrix representation. ∎

This theorem establishes that every symmetric bimatrix representation has a unique standard symmetric bimatrix representation in its orbit. It also follows that the number of symmetric orbits is precisely the number of standard symmetric bimatrix representations. This fact is used in section 8 to determine the number of symmetric orbits. At present, though, we simply note that each symmetric orbit is can be thought of as the orbit of some unique standard symmetric representation. The following theorem gives a lower bound on the order of the orbit of a symmetric bimatrix representation.

Theorem 10: If $x \in X$ is a symmetric bimatrix representation and $g_1, g_2 \in G_{36}$ then $g_1x = g_2x \implies g_1 = g_2$.

Proof: Let $x \in X$ be a symmetric bimatrix representation and let $g_1, g_2 \in G_{36}$. By Theorem 7, there exists a unique $g^* \in G_{36}$ such that g^*x is a standard symmetric bimatrix representation.

(1) Suppose $g_1x = g_2x$.

It must be shown that $g_1 = g_2$. Beginning with (1), operate on the right by ${g^*}^{-1}g^*$:

(2) $(g_1{g^*}^{-1}g^*)x = (g_2{g^*}^{-1}g^*)x$.

(3) Note that $x \sim (g_1{g^*}^{-1}g^*)x$.

Since x is a symmetric bimatrix representation, and by noting (2) and (3), Theorem 8 tells us:

(4) $(g_1{g^*}^{-1}g^*)x$ is a symmetric bimatrix representation.

By (2) and (4):

(5) $(g_2{g^*}^{-1}g^*)x$ is also a symmetric bimatrix representation.

(6) By Theorem 7, there exists a unique $g^\# \in G_{36}$ such that $(g^\# g_1{g^*}^{-1}g^*)x$ is a standard symmetric bimatrix representation.

(7) By (2) and (6), $(g^\# g_2{g^*}^{-1}g^*)x$ is a standard symmetric bimatrix representation.

Since $(g^\# g_1{g^*}^{-1}g^*)x$, $(g^\# g_2{g^*}^{-1}g^*)x$, and $(g^*)x$ are all standard symmetric bimatrix representations, Theorem 9 tells us that all three of these standard symmetric bimatrix representations must be the same bimatrix representation. Thus:

(8) $g^{\#} g_1 g^{*-1} g^* x = g^{\#} g_2 g^{*-1} g^* \mathrm{x} = g^* x$.
Rewrite (8) with parenthesis as follows:
(9) $g^{\#} g_1 g^{*-1}(g^* x) = g^{\#} g_2 g^{*-1}(g^* \mathrm{x}) = (g^* x)$.
Since $(g^* x)$ is a standard symmetric bimatrix representation, Theorem 5 asserts that:
(10) $g^{\#} g_1 g^{*-1} = e$ or $g^{\#} g_1 g^{*-1} = T$ and
(11) $g^{\#} g_2 g^{*-1} = e$ or $g^{\#} g_2 g^{*-1} = T$.
Well $g^{\#}$, g_1, and g^{*-1} are all elements of G_{36}. Hence $g^{\#} g_1 g^{*-1} \in G_{36}$. But $T \in G_{72} \backslash G_{36}$, so
(12) $g^{\#} g_1 g^{*-1} \neq T$.
By (10) and (12), $g^{\#} g_1 g^{*-1} = e$.
(13) A similar argument shows that $g^{\#} g_2 g^{*-1} = e$.
By (12), (13), and transitivity, $g^{\#} g_1 g^{*-1} = g^{\#} g_2 g^{*-1}$. By cancelation, $g_1 = g_2$. ■

Theorem 11: If $x \in X$ is a symmetric bimatrix representation then the orbit of x has order ≥ 36.
Proof: The proof follows immediately from Theorem 10. ■

A lower bound on the order of each symmetric orbit has been established. Theorems 12 through 14 establish an upper bound for the order of each symmetric orbit.

Theorem 12: If $x \in X$ is a symmetric bimatrix representation and $g_1, g_2 \in G_{72} \backslash G_{36}$ then $g_1 x = g_2 x \implies g_1 = g_2$.
Proof: Let $x \in X$ be a symmetric bimatrix representation and $g_1, g_2 \in G_{72} \backslash G_{36}$.
(1) Suppose $g_1 x = g_2 x$.
(2) By Lemma 2, there exists $g_1{}^*, g_2{}^* \in G_{36}$ such that $g_1 = T g_1{}^*$ and $g_2 = T g_2{}^*$.
Substituting these in (1) gives:
(3) $T g_1{}^* x = T g_2{}^* x$.
Operating on the left by T^{-1}:
(4) $g_1{}^* x = g_2{}^* x$.
By (4) and Theorem 10, $g_1{}^* = g_2{}^*$. Operating on the left by T:
(5) $T g_1{}^* = T g_2{}^*$.
By substituting (2) into (5), $g_1 = g_2$. ■

Theorem 13: If $x \in X$ is a symmetric bimatrix representation then for all $g_1 \in G_{36}$, there exists a unique $g_2 \in G_{72} \backslash G_{36}$ such that $g_1 x = g_2 x$.
Proof: Let $x \in X$ be a symmetric bimatrix representation and let $g_1 \in G_{36}$. Let $x = (N, A, u)$. x is a symmetric bimatrix representation, so by Theorem 7 there exists a unique $g' \in G_{36}$ such that $g'x$ is a standard symmetric bimatrix representation. Since $g'x$ is a standard symmetric bimatrix representation, Theorem 4 tells us that $Tg'x = g'x$. Operating on the right by g_1:
(1) $Tg' g_1 x = g' g_1 x$.
Then acting on the left by g'^{-1}:
(2) $(g'^{-1} T g' g_1)x = (g'^{-1} g' g_1)x = g_1 x$.
Since g'^{-1}, g', and g_1 are all elements of G_{36} while only T is an element of $G_{72} \backslash G_{36}$, it follows that:

(3) $(g'^{-1}Tg'g_1) \in G_{72} \backslash G_{36}$.
Conclude:
(4) $(g'^{-1}Tg'g_1) \in G_{72} \backslash G_{36}$ such that $g_1 x = (g'^{-1}Tg'g_1)x$.
To show uniqueness:
(5) Let $g_2 \in G_{72} \backslash G_{36}$ such that $g_2 x = g_1 x$.
It must be shown that $g_2 = g'^{-1}Tg'g_1$.
(6) By (2) and (5), $g'^{-1}Tg'g_1 x = g_1 x = g_2 x$.
By (3), (5), and Theorem 12 we have $g'^{-1}Tg'g_1 = g_2$. ■

Theorem 14 (Corollary): If $x \in X$ is a symmetric bimatrix representation then the orbit of x has order ≤ 36.
Proof: The proof follows immediately from Theorem 13. ■

Theorem 15 (Corollary): If $x \in X$ is a symmetric bimatrix representation then the orbit of x has order 36.
Proof: The proof follows immediately from Theorems 11 and 14. ■

7 The Order of an Asymmetric Orbit

In this section we show that every asymmetric orbit contains precisely 72 elements. We accomplish by showing that the only element of G_{72} that sends an asymmetric representation to itself is the identity element.

Theorem 16: If $x \in X$ and $g \in G_{72}$ then $(gx = x) \implies (g = e$ or x is symmetric$)$.
Proof: Let $x = (N, A, u) \in X$ where $u = (b_1, b_2)$. Let $g \in G_{72}$. Let $gx = (N, A, u')$, where $u' = (b_1', b_2')$. Suppose $gx = x$. Either $g \in G_{36}$ or $g \in G_{72} \backslash G_{36}$.
Case 1: Suppose $g \in G_{36}$. The bimatrix representations x and gx are shown below:

$\boldsymbol{x}$	t_1	t_2	t_3
s_1	$b_1(s_1,t_1), b_2(s_1,t_1)$	$b_1(s_1,t_2), b_2(s_1,t_2)$	$b_1(s_1,t_3), b_2(s_1,t_3)$
s_2	$b_1(s_2,t_1), b_2(s_2,t_1)$	$b_1(s_2,t_2), b_2(s_2,t_2)$	$b_1(s_2,t_3), b_2(s_2,t_3)$
s_3	$b_1(s_3,t_1), b_2(s_3,t_1)$	$b_1(s_3,t_2), b_2(s_3,t_2)$	$b_1(s_3,t_3), b_2(s_3,t_3)$

$\boldsymbol{gx}$	t_1	t_2	t_3
s_1	$b_1'(s_1,t_1), b_2'(s_1,t_1)$	$b_1'(s_1,t_2), b_2'(s_1,t_2)$	$b_1'(s_1,t_3), b_2'(s_1,t_3)$
s_2	$b_1'(s_2,t_1), b_2'(s_2,t_1)$	$b_1'(s_2,t_2), b_2'(s_2,t_2)$	$b_1'(s_2,t_3), b_2'(s_2,t_3)$
s_3	$b_1'(s_3,t_1), b_2'(s_3,t_1)$	$b_1'(s_3,t_2), b_2'(s_3,t_2)$	$b_1'(s_3,t_3), b_2'(s_3,t_3)$

Since $gx = x$, compare the matrices component wise and see that for all $j, k \in \{1, 2, 3\}, b_1(s_j, t_k) = b_1'(s_j, t_k)$. Remember, all of the $b_1(s_j, t_k)$'s are distinct elements of $\{1, \ldots, 9\}$, and g only acts on x by some combination of row and column interchanges (never player interchanges because $g \in G_{36}$). This means the row and column interchanges must have been the trivial ones. That is, when g acts on x is doesn't interchange any rows or columns. If it did, then for some $j, k \in \{1, 2, 3\}$, we'd have $b_1(s_j, t_k) \neq b_1{}'(s_j, t_k)$, which is not the case since $gx = x$. Hence $g = e$.

Case 2: Suppose $g \in G_{72} \backslash G_{36}$. Since $g \in G_{72} \backslash G_{36}$, and $e \in G_{36}$, $g \neq e$. Since $g \in G_{72} \backslash G_{36}$, Lemma 2 asserts that there exists $g^* \in G_{36}$ such that $g = Tg^*$. By supposition $x = gx$.
(1) It follows that $x = gx = Tg^*x$

In order to show that x satisfies the definition of a symmetric bimatrix representation, construct a bijection, $\Phi: S \leftrightarrow T$, such that for all $s \in S$, and for all $t \in T, b_1(s,t) = b_2(\Phi^{-1}(t), \Phi(s))$. Define $\Phi: S \leftrightarrow T$ in terms of two other functions, φ_{g^*} and φ_T, which are define as follows. Let π_s and π_t be the permutations on the set $\{1,2,3\}$ given in the definition of the map ρ_{g^*}.

Let $\varphi_{g^*}: S \cup T \leftrightarrow S \cup T$ be defined by $\varphi_{g^*}(\alpha_i) = \begin{cases} s_{\pi_s(i)} & if\ \alpha_i \in S \\ t_{\pi_t(i)} & if\ \alpha_i \in T \end{cases}$

Note that $\varphi_{g^*}[S]$ is onto S, and $\varphi_{g^*}[T]$ is onto T.

Also note that $\varphi_{g^*}{}^{-1}(\alpha_i) = \begin{cases} s_{\pi_s{}^{-1}(i)} & if\ \alpha_i \in S \\ t_{\pi_t{}^{-1}(i)} & if\ \alpha_i \in T \end{cases}$

Let $\varphi_T: S \cup T \leftrightarrow S \cup T$ be defined by $\varphi_T(\alpha_i) = \begin{cases} t_i & if\ \alpha_i \in S \\ s_i & if\ \alpha_i \in T \end{cases}$

Note that $\varphi_T[S]$ is onto T, and $\varphi_T[T]$ is onto S. Also note that $\varphi_T{}^{-1} = \varphi_T$. Finally, let $\Phi: S \leftrightarrow T$ be defined by $\Phi(s_i) = \varphi_T \circ \varphi_{g^*}(s_i)$. Note that $\Phi^{-1}(t_i) = \varphi_{g^*}{}^{-1} \circ \varphi_T(t_i)$.

Now that $\Phi: S \leftrightarrow T$ is defined, it remains to be shown that for all $s \in S$, and for all $t \in T, b_1(s,t) = b_2(\Phi^{-1}(t), \Phi(s))$. Let $s_j \in S$ and $t_k \in T$. By definition of Φ,
$b_i(\Phi^{-1}(t_k), \Phi(s_j)) = b_i\left(\varphi_{g^*}{}^{-1} \circ \varphi_T(t_k), \varphi_T \circ \varphi_{g^*}(s_j)\right) = b_i\left(\varphi_{g^*}{}^{-1}(s_k), \varphi_T \circ \left(s_{\pi_s(j)}\right)\right)$
$= b_i\left(s_{\pi_s{}^{-1}(k)}, t_{\pi_s(j)}\right)$.
(2) In particular, $b_2(\Phi^{-1}(t_k), \Phi(s_j)) = b_2\left(s_{\pi_s{}^{-1}(k)}, t_{\pi_s(j)}\right)$.
Now let's look at the bimatrix representations x, g^*x, and Tg^*x:

$\boldsymbol{x}$	t_1	t_2	t_3
s_1	$b_1(s_1,t_1), b_2(s_1,t_1)$	$b_1(s_1,t_2), b_2(s_1,t_2)$	$b_1(s_1,t_3), b_2(s_1,t_3)$
s_2	$b_1(s_2,t_1), b_2(s_2,t_1)$	$b_1(s_2,t_2), b_2(s_2,t_2)$	$b_1(s_2,t_3), b_2(s_2,t_3)$
s_3	$b_1(s_3,t_1), b_2(s_3,t_1)$	$b_1(s_3,t_2), b_2(s_3,t_2)$	$b_1(s_3,t_3), b_2(s_3,t_3)$

$\boldsymbol{g^*x}$	t_1	t_2	t_3
s_1	$b_1(s_{\pi_s{}^{-1}(1)}, t_{\pi_t{}^{-1}(1)}), b_2(s_{\pi_s{}^{-1}(1)}, t_{\pi_t{}^{-1}(1)})$	$b_1(s_{\pi_s{}^{-1}(1)}, t_{\pi_t{}^{-1}(2)}), b_2(s_{\pi_s{}^{-1}(1)}, t_{\pi_t{}^{-1}(2)})$	$b_1(s_{\pi_s{}^{-1}(1)}, t_{\pi_t{}^{-1}(3)}), b_2(s_{\pi_s{}^{-1}(1)}, t_{\pi_t{}^{-1}(3)})$
s_2	$b_1(s_{\pi_s{}^{-1}(2)}, t_{\pi_t{}^{-1}(1)}), b_2(s_{\pi_s{}^{-1}(2)}, t_{\pi_t{}^{-1}(1)})$	$b_1(s_{\pi_s{}^{-1}(2)}, t_{\pi_t{}^{-1}(2)}), b_2(s_{\pi_s{}^{-1}(2)}, t_{\pi_t{}^{-1}(2)})$	$b_1(s_{\pi_s{}^{-1}(2)}, t_{\pi_t{}^{-1}(3)}), b_2(s_{\pi_s{}^{-1}(2)}, t_{\pi_t{}^{-1}(3)})$
s_3	$b_1(s_{\pi_s{}^{-1}(3)}, t_{\pi_t{}^{-1}(1)}), b_2(s_{\pi_s{}^{-1}(3)}, t_{\pi_t{}^{-1}(1)})$	$b_1(s_{\pi_s{}^{-1}(3)}, t_{\pi_t{}^{-1}(2)}), b_2(s_{\pi_s{}^{-1}(3)}, t_{\pi_t{}^{-1}(2)})$	$b_1(s_{\pi_s{}^{-1}(3)}, t_{\pi_t{}^{-1}(3)}), b_2(s_{\pi_s{}^{-1}(3)}, t_{\pi_t{}^{-1}(3)})$

$\boldsymbol{Tg^*x}$	t_1	t_2	t_3
s_1	$b_2(s_{\pi_s{}^{-1}(1)}, t_{\pi_t{}^{-1}(1)}), b_1(s_{\pi_s{}^{-1}(1)}, t_{\pi_t{}^{-1}(1)})$	$b_2(s_{\pi_s{}^{-1}(2)}, t_{\pi_t{}^{-1}(1)}), b_1(s_{\pi_s{}^{-1}(2)}, t_{\pi_t{}^{-1}(1)})$	$b_2(s_{\pi_s{}^{-1}(3)}, t_{\pi_t{}^{-1}(1)}), b_1(s_{\pi_s{}^{-1}(3)}, t_{\pi_t{}^{-1}(1)})$
s_2	$b_2(s_{\pi_s{}^{-1}(1)}, t_{\pi_t{}^{-1}(2)}), b_1(s_{\pi_s{}^{-1}(1)}, t_{\pi_t{}^{-1}(2)})$	$b_2(s_{\pi_s{}^{-1}(2)}, t_{\pi_t{}^{-1}(2)}), b_1(s_{\pi_s{}^{-1}(2)}, t_{\pi_t{}^{-1}(2)})$	$b_2(s_{\pi_s{}^{-1}(3)}, t_{\pi_t{}^{-1}(2)}), b_1(s_{\pi_s{}^{-1}(3)}, t_{\pi_t{}^{-1}(2)})$
s_3	$b_2(s_{\pi_s{}^{-1}(1)}, t_{\pi_t{}^{-1}(3)}), b_1(s_{\pi_s{}^{-1}(1)}, t_{\pi_t{}^{-1}(3)})$	$b_2(s_{\pi_s{}^{-1}(2)}, t_{\pi_t{}^{-1}(3)}), b_1(s_{\pi_s{}^{-1}(2)}, t_{\pi_t{}^{-1}(3)})$	$b_2(s_{\pi_s{}^{-1}(3)}, t_{\pi_t{}^{-1}(3)}), b_1(s_{\pi_s{}^{-1}(3)}, t_{\pi_t{}^{-1}(3)})$

By (1), $x = gx = Tg^*x$. By definition this means that the corresponding components of x, gx, and Tg^*x are equal. Equating the corresponding components of x, gx, and Tg^*x we observe that:

(3) $b_1(s_j, t_k) = b_2(s_{\pi_s{}^{-1}(k)}, t_{\pi_t{}^{-1}(j)})$.

By (2), (3), and transitivity, $b_1(s_j, t_k) = b_2(\Phi^{-1}(t_k), \Phi(s_j))$ as desired. Hence $\Phi: S \leftrightarrow T$ is a bijection such that for all $s \in S$, and for all $t \in T, b_1(s,t) = b_2(\Phi^{-1}(t), \Phi(s))$. By definition x is a symmetric bimatrix representation. ■

Theorem 17: If $x \in X$ is an asymmetric bimatrix representation and $g \in G_{72}$ then $gx = x \Rightarrow g = e$.

Proof: Let $x \in X$ be an asymmetric bimatrix representation and $g \in G_{72}$. Suppose $gx = x$. Since x is not a symmetric bimatrix representation, Theorem 16 asserts that $g = e$. ■

Theorem 18 (Corollary): If $x \in X$ is an asymmetric bimatrix representation, then the orbit of x has order 72.

Proof: The proof is immediate from Theorem 17. ■

8 The Number of 3-Strategy Rapoportian Games

The goal of this section is to count number orbits of X under the action of G_{72}. We first establish a formula for the number of orbits of X under the action of G_{72} in terms of the number of symmetric and asymmetric orbits. Next, the symmetric orbits of X are counted, and finally, we count the orbits of X (and thus the number of 3-strategy Rapoportian games).

Theorem 19: If r_{72} and r_{36} are the number of asymmetric orbits and symmetric orbits respectively, then $72 \cdot r_{72} + 36 \cdot r_{36} = 9!^2$.

Proof: The proof follows immediately from Theorems 15 and 18, and noting that $|x| = 9!^2$. ■

Theorem 20: The number, r_{36}, of symmetric orbits is 60,480.

Proof: Let r_{36} be the number of mutually distinct symmetric bimatrix representations. By Theorem 9, each symmetric orbit of X contains exactly one standard symmetric bimatrix representation. Hence r_{36} equals the number of standard symmetric bimatrix representations. We set out to count the number of standard symmetric bimatrix representations.

All standard symmetric bimatrix representations, by definition, have central outcomes c_1, c_2, c_3 such that $v(c_1) > v(c_2) > v(c_3)$ where for all $i \in \{1,2,3\}, (s_i, t_i) = c_i$. Remember that $1 \le v(c_i) \le 9$. As shown in the proof of Theorem 4, if x is a standard symmetric bimatrix representation then it can be expressed using only the utility function b_1 as follows:

$\boldsymbol{x}$	t_1	t_2	t_3
s_1	$b_1(s_1,t_1), b_1(s_1,t_1)$	$b_1(s_1,t_2), b_1(s_2,t_1)$	$b_1(s_1,t_3), b_1(s_3,t_1)$
s_2	$b_1(s_2,t_1), b_1(s_1,t_2)$	$b_1(s_2,t_2), b_1(s_2,t_2)$	$b_1(s_2,t_3), b_1(s_3,t_2)$
s_3	$b_1(s_3,t_1), b_1(s_1,t_3)$	$b_1(s_3,t_2), b_1(s_2,t_3)$	$b_1(s_3,t_3), b_1(s_3,t_3)$

Note that there are only 9 values in this expression. Three of the 9 values are the central outcomes: $b_1(s_1,t_1) = v(c_1)$, $b_1(s_2,t_2) = v(c_2)$, and $b_1(s_3,t_3) = v(c_3)$. The remaining six are: $b_1(s_2,t_1)$, $b_1(s_3,t_1)$, $b_1(s_1,t_2)$, $b_1(s_3,t_2)$, $b_1(s_1,t_3)$, and $b_1(s_2,t_3)$.

First note that the number of ways values may be assigned to $v(c_1)$, $v(c_2)$, and $v(c_3)$ from the the set $\{1, \dots, 9\}$ is $\binom{9}{3} = 84$. This is because any choice of three integers from $\{1, \dots, 9\}$ can be assigned to $v(c_1)$, $v(c_2)$, and $v(c_3)$ in exactly one way due to the requirement that $v(c_1) > v(c_2) > v(c_3)$. The remaining 6 values can be assigned in any of $6! = 720$ ways. Hence the number of standard symmetric bimatrix representations is $6!\binom{9}{3} = 720 \times 84 = 60{,}480 = r_{36}$. ■

Theorem 21: The number orbits of X under the action of G_{72} is 1,828,945,440.
Proof: By Theorem 19, $72 \cdot r_{72} + 36 \cdot r_{36} = 9!^2$. By Theorem 20, $r_{36} = 60{,}480$. By substitution, $72 \cdot r_{72} + 36 \cdot 60{,}480 = 9!^2$. Solving for r_{72}, observe that $r_{72} = 1{,}828{,}884{,}960$. Hence the number orbits of X under the action of $G_{72}$is: $r_{36} + r_{72} = 60{,}480 + 1{,}828{,}884{,}960 = 1{,}828{,}945{,}440$. ■

We thus conclude, in a very precise sense, that the number of 2-player, 3-strategy, strictly ordinal, normal form games is 1,828,945,440.

9 Final Thoughts

Now that the 3-strategy Rapoportian games have been counted, the next step is to enumerate them (that is, to list them) using a computer. This can be done by selecting one bimatrix representation from each of the orbits of X under the action of G_{72}, and constructing a game that is represented by it. In the case of symmetric orbits this is very straightforward – simply choose the unique standard symmetric bimatrix representation from each symmetric orbit. Since we counted the standard symmetric bimatrix representations directly, we can generate them efficiently with a computer using a method similar to the method we used to count them. Once the games have been enumerated we can use a computer to classify them based on their strategic properties. The full taxonomy can then be published online in a searchable format.

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