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Thomas Polstra[a]

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Rose-Hulman Institute of Technology
Department of Mathematics
Terre Haute, IN 47803
Email: mathjournal@rose-hulman.edu
http://www.rose-hulman.edu/mathjournal

[a]Georgia State University

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Abstract. It has been conjectured by Casas-Alvero that polynomials of degree n over fields of characteristic 0, share roots with each of its $n-1$ derivatives if and only if those polynomials have one root of degree n.
In this paper, using the analytic theory of polynomials, an equivalent formulation of the Casas-Alvero Conjecture is established for polynomials over the complex plane together with several special cases of it.

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1 Introduction

In 2001 Eduardo Casas-Alvero, in relation to his work on the geometry of plane curves, conjectured that certain classes of polynomials will share a root with each of its derivatives if and only if these polynomials have one unique root [1]. The conjecture is formally stated below.

Conjecture 1.1. *(Casas-Alvero) Let k be a field of characteristic 0. Any polynomial of degree n in $k[x]$ shares a root with each of its $n-1$ derivatives iff f has one root of multiplicity n, i.e. f is in the form of $f(z) = \alpha(z - z_0)^n$.*

Due to its elementary statement, the conjecture has immediately generated interest among mathematicians, but it has remained open. Computational methods have been used to prove the conjecture for small degrees [2], and work has been done to prove the conjecture for all polynomials of degree that is a power of a prime, p^n, or twice the power of a prime, $2p^n$, [3]. The smallest degree for which the Casas-Alvero Conjecture not known is for polynomials of degree 12.

In Section 2 we discuss the terminology used in the statement of the Casas-Alvero Conjecture and discuss background material about polynomials over the complex plane. In Section 3 we develop an equivalent statement to the Casas-Alvero Conjecture for polynomials over the complex plane while in Section 4 we give an equivalent condition to the conjecture for polynomials with only real roots.

2 Background

A field of characteristic 0 is a field containing a copy of the rationals $\mathbb{Q}$. Some common examples of fields of characteristic 0 include $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ and $\mathbb{Q}(X)$, the field of rational functions $\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomials with coefficients in $\mathbb{Q}$. An example of a field that is not of characteristic 0 is $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, where p is a prime number.

Let $k[x]$ be the collection of all polynomials in x with coefficients in k. For a polynomial $f \in k[x]$ let $f^{(i)}$ denote the i^{th} derivative of f. We can recall that f has a root of multiplicity j at x_0 if and only if $f^{(i)}(x_0) = 0$ for all $0 \leq i \leq j-1$ and $f^{(j)}(x_0) \neq 0$. We say that x_0 is a *simple root* of f if x_0 is a root of multiplicity 1, i.e. $f(x_0) = 0$ and $f'(x_0) \neq 0$.

The reason why the conjecture is stated over a field of characteristic 0 is because counterexamples to the conjecture in characteristic p are readily available. For example, the polynomial $f = x^3 - \overline{1}$ with coefficients in $\mathbb{Z}_3$ has roots at $\overline{0}$ and $\overline{1}$, but the polynomials $f' = \overline{3}x^2 = \overline{0}$ and $f'' = \overline{0}$ both have roots at $\overline{0}$. In other words, this particular degree 3 polynomial shares a root with each of its first two derivatives, but it does not have one root of multiplicity 3.

The rest of this paper examines polynomials over the complex plane $\mathbb{C}$ for the purpose of giving an interesting reformulation of the conjecture for this important case.

For what follows, given a set $C \subset \mathbb{C}$, we say that $z \in C$ is called a *boundary point* of C if and only if z is contained in no open subset of C. All other points of C are called *interior points*. The collection of all boundary points is the *boundary* of C, denoted ∂C, and that the collection of all interior points is the *interior* of C, denoted $int(C)$. The *exterior* of C is $\mathbb{C} \setminus C$ and is denoted $ext(C)$.

We can recall that a set C of the complex plane is called *convex* if $z_1, z_2 \in C$ and $t \in (0,1)$ implies $tz_1 + (1-t)z_2 \in C$. This definition captures the feature that the line segment connecting z_1, z_2 must be a part of C. A convex set C is called *strictly convex* if it has the additional property that if $z_1, z_2 \in C$ then for all $t \in (0,1)$, $tz_1 + (1-t)z_2$ must be an interior point of C. A circle in the plane is an example of a strictly convex set.

The Fundamental Theorem of Algebra implies that a polynomial f of degree n has at most n complex roots and exactly n roots if multiplicity is counted. Using this, one can define what a convex hull of a polynomial is.

Definition 2.1. *The* convex hull *of a polynomial $f(x)$ with coefficients in $\mathbb{C}$ is the intersection of all convex sets that contain the roots of f, denoted by C_f. That is, C_f is the convex hull of the roots of f.*

As a visual illustration of what a convex hull would look like, in the images in Figure 1 the dots represent some polynomial's roots which lie in the complex plane, the structures containing these dots represent the convex hull of that polynomial.

Another way to visualize a convex hull of a polynomial is to imagine that the roots of a polynomial were represented by pegs lying in the complex plane, and a rubber band is stretched around those pegs. If the rubber band is allowed to contract then it will catch onto those pegs in a manner forming a convex structure. This structure will represent the convex hull of that polynomial.

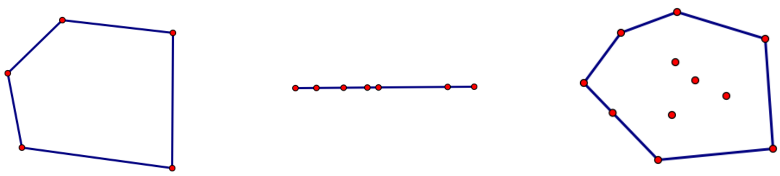

Figure 1: Examples of Convex Hulls

We denote the convex hulls of $f, f', ..., f^{(n-1)}$ as $C_f, C_{f'}, ..., C_{f^{(n-1)}}$ respectively. A standard result on the theory of polynomials with complex coefficients is given below.

Theorem 2.2. *(Gauss-Lucas,[5])*
If $f(x)$ is a polynomial with complex coefficients, then the roots of $f'(x)$ are contained in C_f.

An immediate result from the Gauss-Lucas Theorem relates the convex hull of f to those of its derivatives.

Corollary 2.3. ([5])
If $f(x)$ is a polynomial with complex coefficients, then $C_{f^{(n-1)}} \subseteq ... \subseteq C_{f'} \subseteq C_f$.

Another standard result in the study of polynomials is credited to Augustin-Louis Cauchy. It gives a description of the roots of a particular class of monic polynomials, where a *monic* polynomial is a polynomial whose leading coefficient is 1.

Theorem 2.4. *(Cauchy's Theorem, pg. 243, [5], pg. 2-3, [4])*
Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_i \in \mathbb{R}_{\leq 0}$ and at least one of them is nonzero. Then $f(x)$ has one positive root p, it is simple, and all other roots have absolute value less than or equal to p.

Under these conditions, $f(x)$ will have one positive root of multiplicity 1, $x_0 \in \mathbb{R}_{>0}$, and all other roots will lie in the disc centered at the origin with radius x_0. In Section 4 we will apply Theorem 2.4 to establish several cases of the Casas-Alvero conjecture.

By relating the roots of a polynomial to the vertices of its convex hull we give an equivalent reformulation of the Casas-Alvero Conjecture in Theorem 3.1. In order to prove Theorem 3.1 we must first look at some additional properties of the convex hull of a polynomial.

Intuitively, when considering a polygon in the plane, we sometimes think of a vertex being a boundary point where two edges of that polygon meet. But a vertex of a polygon has the feature that a line segment containing the vertex must be such that at least one of the endpoints is not a part of the polygon's boundary nor the interior. This intuition is the motivation of the following definition.

Definition 2.5. *Let $C \subseteq \mathbb{C}$ be any set in the complex plane and $v \in C$. Then v is a vertex for C if it has the property that, for any $z_1, z_2 \in \mathbb{C}$ with $tz_1 + (1-t)z_2 = v$ for some $t \in (0,1)$, then either $z_1 \notin C$ or $z_2 \notin C$.*

Note that the contrapositive of this definition says that if v is a vertex of C, $z_1, z_2 \in C$, then $tz_1 + (1-t)z_2 \neq v$ for any $t \in (0,1)$. In other words, no line segment contained in C can contain v as an interior point.

Proposition 2.6. *If v is a vertex of C_f, then $v \in \partial C_f$.*

Proof. Suppose by a contradiction that $v \in int(C_f)$. This implies there exists D, a disc, around v s.t. $D \subseteq C_f$. But that implies each diameter of D, a line segment containing v as an interior point is in C_f, contradicting v being a vertex.

□

Proposition 2.7. *A convex set, $\gamma \subseteq C_f$, will contain a vertex, v, of C_f if and only if v is also a vertex of γ.*

Proof. Suppose by a contradiction that v is not a vertex of γ, hence there exists $z_1, z_2 \in \gamma \subseteq C_f$ and $t \in (0,1)$ such that $tz_1 + (1-t)z_2 = v$. This immediately contradicts v being a vertex of C_f. If v is indeed a vertex of γ then it is trivial that $v \in \gamma$.

□

Proposition 2.8. *If f is a polynomial with complex coefficients, then every vertex of the convex hull of f, C_f, must be a root of f.*

Proof. Suppose by a contradiction that a vertex, $v \in C_f$, is such that it is not a root of f, i.e. $v \notin R$ where R is the set of all roots of f. All that is needed to contradict this scenario is the existence of another convex set, C, such that $R \subseteq C \subset C_f$ where $v \notin C$. Consider the set $C = C_f \setminus \{v\}$, it is clear that $v \notin C$, and that $R \subseteq C \subset C_f$, so the only thing left to show is that C is a convex set.

Consider two points $z_1, z_2 \in C \subset C_f$, then by convexity of C_f, we have for all $t \in (0,1)$ that $tz_1 + (1-t)z_2 \in C_f$. But note that $tz_1 + (1-t)z_2 \neq v$ since v is a vertex of C_f, therefore $tz_1 + (1-t)z_2 \in C = C_f \setminus \{v\}$; implying that C is indeed a convex set.

Therefore all vertices of a polynomial's convex hull must be among the roots of that polynomial.

□

3 An approach to the Casas-Alvero Conjecture

We can now state the main contribution of the paper, which is an equivalent formulation of the Casas-Alvero conjecture that could provide new insights on the conjecture for the complex plane. This formulation relies on the characterization of the vertices of a polynomial's convex hull.

Theorem 3.1. *If f is a polynomial that shares a root with each of its $n-1$ derivatives then each root is a vertex of C_f if and only if f has only one root of multiplicity n.*

Proof. If f has only one root of multiplicity n, then it is clear that f shares a root with each of its $n-1$ derivatives and that each root, i.e. the only root, is a vertex of C_f.

Let f share a root with each of its $n-1$ derivatives. We can assume that each root of f is a vertex of C_f. By the Gauss-Lucas Theorem and Corollary 2.3 we have, $C_{f^{(n-1)}} \subseteq ... \subseteq C_{f'} \subseteq C_f$. Thus, if $f^{(n-1)}(z_0) = 0$ then z_0 lies in the convex hull of all derivatives of $f(z)$, i.e. $z_0 \in C_{f^{(i)}}$ for all i. Also, we know that $f(z_0) = f^{(n-1)}(z_0) = 0$, since f shares a root with each of its $n-1$ derivatives and the $(n-1)$st derivative $f^{(n-1)}$ of a degree n polynomial is a degree 1 polynomial, and hence has only one root. In particular, we know $z_0 \in C_{f'}$, but since z_0 is a vertex of C_f, by Proposition 2.7 we know z_0 is a vertex of $C_{f'}$. By Proposition 2.8, z_0 must be a root of f'.

Since $z_0 \in C_{f''}$ and z_0 is now known to be a vertex of $C_{f'}$, this implies by the same reasons that $f'(z_0) = 0$ that $f''(z_0) = 0$.

Continuing this process shows that z_0 must be a root of all $n-1$ derivatives of $f(z)$, implying that $f(z)$ has only one root, z_0, of multiplicity n. □

Based upon Theorem 3.1, we know that in order to prove the Casas-Alvero Conjecture for polynomials over the complex plane, it is enough to show that a complex polynomial of degree n which shares a root with each of its $n-1$ derivatives must be such that each root is a vertex of its convex hull.

Before we can give a proof of the Casas-Alvero Conjecture under some additional assumptions given in Corollary 3.5, we need to develop a few Lemmas.

Lemma 3.2. *If $w \in \partial C$, where C is a convex set, and w is not a vertex of C, then any segment, l, contained in C and containing w as an interior point must be such that $l \subseteq \partial C$ i.e. l contains no interior point of C.*

Proof. Suppose by a contradiction that there exists a line, l, contained in C in which w is an interior point and that l contains an interior point of C, namely i. This implies there exists a line segment, l', containing i as one of its end points and w as an interior point of l'. But w is a boundary point of C meaning that any neighborhood, Ω, around w must contain points in the exterior of C and in the interior of C. So consider the disc, Γ, centered around w and whose boundary contains i. The boundary of C_f contained in Γ separates the region of Γ into two separate areas, one area is part of $int(C)$ and the other is in $ext(C)$. The line l' contains $w \in \partial C$ implies that l' must pass through the boundary of C, since l' contains an interior point of C, and that it must contain points in the exterior of C, a contradiction.

□

Lemma 3.3. *If γ is a strictly convex set and $v \in \partial\gamma$, then v is a vertex of γ.*

Proof. Suppose by a contradiction that $v \in \partial\gamma$ but is not a vertex. This implies there exists l, a line segment, such that $v \in int(l)$ and that $l \subseteq \gamma$. By Lemma 3.2 l can contain no interior point of γ, thus the end points of l must be elements of $\partial\gamma$. But γ is strictly convex which implies that no points that are part of $int(l)$ can be in $\partial\gamma$; since v is in both, we have a contradiction.

□

Theorem 3.4. *If f is a polynomial with complex coefficients of degree n such that its roots lie on the boundary of a strictly convex set, Γ, then each root of f is a vertex of C_f.*

Proof. Each root of f lies on the $\partial\Gamma$ implies each root of f is a vertex of Γ. Since C_f is the intersection of all convex sets containing the roots of f, and Γ is convex, thus $C_f \subseteq \Gamma$ and by Proposition 2.7 each root of f must be a vertex of C_f.

□

Corollary 3.5. *If f is a polynomial with complex coefficients of degree n such that its roots lie on the boundary of a strictly convex set, Γ, then f has only one root of multiplicity n.*

Proof. This is immediate by Theorems 3.1 and 3.4. □

A potential, but seemingly difficult, method to prove the Casas-Alvero Conjecture would be as follows. Let $z_1, z_2, ..., z_n$ be roots of a degree n polynomial f that shares a root with each of its $n-1$ derivatives. If we can somehow show that for every point $z \in \mathbb{C}$ that the distance from z to z_i is the same, then we can conclude that $z_1 = z_2 = \cdots = z_n$. Corollary 3.5 gives a simpler version of this method. If we can show there exists at least one point in $z \in \mathbb{C}$ such that the distance from z to z_i is the same, then all the roots of f lie on the boundary of a disc, a strictly convex set, which would prove the Casas-Alvero Conjecture for the complex plane.

4 Polynomial Functions with only Real Roots

4.1 Polynomials with real roots

Using methods different from the previous sections, we can show an equivalent condition to the Casas-Alvero Conjecture for polynomials having only real roots. It should be noted that it is not necessarily the case that a polynomial with real roots will have real coefficients; such an assumption is not made until Theorem 4.5. It will be shown in Theorem 4.3 that if $f = \sum_{i=0}^{n} a_i x^i$ shares a root with each of its $n-1$ derivatives, and $f^{(n-2)}$ has only one root of multiplicity 2, then f has only one root of multiplicity n. The proof of Theorem 4.3 relies on Viete's Relations, formulas that relate the coefficients of a polynomial to its roots.

Theorem 4.1. *(Viete's Relations, [4])*
Let $f = \sum_{i=0}^{n} a_i x^i$ be a monic polynomial with roots $x_1, x_2, ..., x_n$. Then we have the following

$$(-1)^k a_{n-k} = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

In particular, using the above notations,

$$-a_{n-1} = x_1 + x_2 + \cdots + x_n$$

$$\begin{aligned} a_{n-2} &= x_1x_2 + x_1x_3 + \cdots + x_1x_n + x_2x_3 + \cdots + x_{n-1}x_n \\ &= x_1(x_2 + x_3 + \cdots x_n) + x_2(x_3 + \cdots + x_n) + \cdots + x_{n-1}(x_n). \end{aligned}$$

Proposition 4.2. *Let $f = \sum_{i=0}^{n} a_i x^i$ be a monic polynomial such that $f(0) = a_{n-1} = a_{n-2} = 0$ and let $x_1, \cdots, x_n$ be the roots of f, then $x_1^2 + \cdots + x_n^2 = 0$.*

Proof. Since $a_{n-1} = a_{n-2} = 0$ we have that by Theorem 4.1, $x_1 + x_2 + \cdots + x_n = 0$ and $x_1(x_2 + x_3 + \cdots x_n) + x_2(x_3 + \cdots + x_n) + \cdots + x_{n-1}x_n = 0$. In particular, we have that $(x_1 + x_2 + \cdots x_n)^2 = 0$. Hence

$$\begin{aligned} 0 &= (x_1 + x_2 + \cdots x_n)^2 \\ &= x_1^2 + x_2^2 + \cdots + x_n^2 + x_1(x_2 + x_3 + \cdots + x_n) \\ &\quad + x_2(x_3 + \cdots + x_n) + \cdots + x_{n-1}(x_n) \\ &= x_1^2 + x_2^2 + \cdots + x_n^2 + a_{n-2} \\ &= x_1^2 + x_2^2 + \cdots + x_n^2. \end{aligned}$$

□

Using Proposition 4.2 we can give an equivalent reformulation of the Casas-Alvero Conjecture for polynomials that only have real roots.

Theorem 4.3. *If f a polynomial of degree n with only real roots and it shares a root with each of its $n-1$ derivatives, then $f^{(n-2)}$ has only one root of multiplicity 2 if and only if f has one root of multiplicity n.*

Proof. If f has only one root of multiplicity n, then it is clear that f must share a root with each of its $n-1$ derivatives and that $f^{(n-2)}$ has only one root of multiplicity 2.

Without loss of generality, let f be a monic polynomial. Suppose that f shares a root with each of its $n-1$ derivatives, and that $f^{(n-2)}$ has only one root of multiplicity 2, say w_0, then $f(w_0) = f^{(n-2)}(w_0) = 0$.

Let $g(x) = f(x + w_0) = \sum_{i=0}^{n} a_i x^i$. It is clear that whenever z_0 is a root of $g^{(i)}$ then $z_0 + w_0$ is a root for $f^{(i)}$. In particular, we have that g also shares a root with each of its $n-1$ derivatives and that all the roots of g must be real. But we have that $g(0) = f(w_0) = f^{(n-2)}(w_0) = f^{(n-1)}(w_0) = 0$; i.e. $g(0) = g^{(n-2)}(0) = g^{(n-1)}(0) = 0$.

Note that $g^{(n-2)}(x) = \frac{n!}{2}x^2 + a_{n-1}(n-1)!x + a_{n-2}(n-2)!$, but $g^{(n-2)}(0) = 0$ implies $a_{n-2} = 0$. Also $g^{(n-1)}(x) = n!x + a_{n-1}(n-1)!$, so $g^{(n-1)}(0) = 0$ implies that $a_{n-1} = 0$. We now have that $g(0) = a_{n-2} = a_{n-1} = 0$. By Theorem 4.2 we have that for the roots $x_1, x_2, ..., x_n$ of g, satisfies $x_1^2 + x_2^2 + \cdots + x_n^2 = 0$. Since all roots of g are real, this implies $x_1 = x_2 = \cdots = x_n = 0$. Therefore g has only one root of multiplicity n at 0, which implies that f has only one root of multiplicity n at w_0. □

4.2 Even and Odd Polynomial Functions

This subsection gives a proof of the Casas-Alvero conjecture for even and odd polynomial functions with real roots and under some additional assumptions. Recall that an *even function* is a function such that $f(z) = f(-z)$ for all $z \in \mathbb{C}$ and that an *odd function* is a function such that $f(z) = -f(z)$ for all $z \in \mathbb{C}$.

Proposition 4.4. *A polynomial, $f(x) = \sum_{i=0}^{n} a_i x^i$, is an even function if and only if $a_i = 0$ for all i odd.*

Proof. If $a_i = 0$ for all i odd then it is clear that $f(x)$ is even function. Consider a polynomial $f(x)$ that is an even function. Let $g(x) = \sum_{i=0}^{n} b_i x^i$ where $b_i = a_i$ for i even and $b_i = 0$ for i odd. It is clear $g(x)$ is an even function; it is also clear that $h(x) = f(x) - g(x)$ is an even function. But the only nonzero coefficients $h(x)$ can have is at the x's with odd powers. So if there are nonzero coefficients, then $h(x)$ will have nonzero coefficients at odd powers, clearly making $h(x)$ not an even function, a contradiction.

□

An identical proof can be used to show that for $f(x) = \sum_{i=0}^{n} a_i x^i$ is an odd polynomial function if and only if $a_i = 0$ for i even. In addition, by Proposition 4.4 we can readily see that the derivative of an even polynomial function is odd and the derivative of an odd polynomial function is even.

Theorem 4.5. *If $f(x) = \sum_{i=0}^{n} a_i x^i$ is an even or odd monic polynomial function that shares a root with each of its $n-1$ derivatives, has only real roots, and $a_{n-1}, a_{n-2}, ..., a_0 \leq 0$, then $f(x)$ has only one root at 0 of multiplicity n.*

Proof. Assume that f is an even polynomial function. To show that f has a root at 0 consider $f^{(n-1)}(x) = n!x$, which has a root at 0. Recall that by Proposition 4.4 that $a_{n-1} = 0$. Hence $f^{(n-1)}(x)$ has its only root at 0, therefore $f(x)$ must have a root at 0 as well.

Suppose by a contradiction that $f(x)$ shares a root with each of its $n-1$ derivatives and that 0 is not the only root of $f(x)$; i.e, for some a_i, a coefficient of $f(x)$, is nonzero.

All coefficients of $f(x)$ are negative or equal to zero, with at least one being nonzero. Thus by Theorem 2.4, $f(x)$ has only one positive root p and it is simple. Since $f(x)$ is an even function, $f(-p) = 0$. We also have that $-p$ is a simple root of $f(x)$, if not, then $f'(x)$ is an odd function such that $f(-p) = 0$, which would imply $f'(p) = 0$, contradicting that p is a simple root of $f(x)$.

We now have that $f(x) = (x-p)(x+p)x^{n-2}$. Since 0 is a root of multiplicity $n-2$ for f we have that $f(0) = f'(0) = \cdots = f^{(n-3)}(0) = f^{(n-1)}(0) = 0$, so $f^{(n-2)}(x)$ is the only derivative in question to having a root at 0. But note that C_f, the convex hull of the roots of $f(x)$, is the line segment $[-p, p]$. Hence p and $-p$ are vertices of C_f, so if $f^{(n-2)}(x)$ has a root at p or $-p$ then $f(p) = f'(p) = \cdots f^{(n-2)}(p) = 0$ or $f(-p) = f'(-p) = \cdots f^{(n-2)}(-p) = 0$, see Theorem 3.1. Therefore if the root shared by $f(x)$ and $f^{(n-2)}(x)$ is at p or $-p$ then $f(x)$ will have a root of multiplicity $n-1$ at p or $-p$, a contradiction to p and $-p$ being simple.

The proof of this Theorem for f an odd polynomial function is identical to the case when f is an even polynomial function.

□

5 Conclusions

This paper examined the properties a polynomial f over $\mathbb{C}$ for the purpose to restate the Casas-Alvero Conjecture in terms of the location of the roots of f with respect to its convex hull. In particular, a degree n polynomial f that shares a root with each of its $n-1$ derivatives will have only one root if and only if every root of f is a vertex of the convex hull of f. Using this reformulation of the conjecture, it might be possible use computational methods to prove the conjecture for degree 12 polynomials.

A reformulation of the Casas-Alvero Conjecture for polynomials with only real roots is given in Section 4. Theorem 4.3 shows that to prove the Casas-Alvero Conjecture for a polynomial f of degree n with real roots, it is enough to show that the $(n-2)$nd derivative of f has only one root of multiplicity 2.

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