Convex Hulls and The Casas-Alvero Conjecture for the Complex Plane

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Abstract. It has been conjectured by Casas-Alvero that polynomials of degree $n$ over fields of characteristic 0, share roots with each of its $n - 1$ derivatives if and only if those polynomials have one root of degree $n$.
In this paper, using the analytic theory of polynomials, an equivalent formulation of the Casas-Alvero Conjecture is established for polynomials over the complex plane together with several special cases of it.

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1 Introduction

In 2001 Eduardo Casas-Alvero, in relation to his work on the geometry of plane curves, conjectured that certain classes of polynomials will share a root with each of its derivatives if and only if these polynomials have one unique root [1]. The conjecture is formally stated below.

**Conjecture 1.1.** (Casas-Alvero) Let $k$ be a field of characteristic 0. Any polynomial of degree $n$ in $k[x]$ shares a root with each of its $n-1$ derivatives iff $f$ has one root of multiplicity $n$, i.e. $f$ is in the form of $f(z) = \alpha(z-z_0)^n$.

Due to its elementary statement, the conjecture has immediately generated interest among mathematicians, but it has remained open. Computational methods have been used to prove the conjecture for small degrees [2], and work has been done to prove the conjecture for all polynomials of degree that is a power of a prime, $p^n$, or twice the power of a prime, $2p^n$, [3]. The smallest degree for which the Casas-Alvero Conjecture not known is for polynomials of degree 12.

In Section 2 we discuss the terminology used in the statement of the Casas-Alvero Conjecture and discuss background material about polynomials over the complex plane. In Section 3 we develop an equivalent statement to the Casas-Alvero Conjecture for polynomials over the complex plane while in Section 4 we give an equivalent condition to the conjecture for polynomials with only real roots.

2 Background

A field of characteristic 0 is a field containing a copy of the rationals $\mathbb{Q}$. Some common examples of fields of characteristic 0 include $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ and $\mathbb{Q}(X)$, the field of rational functions $\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomials with coefficients in $\mathbb{Q}$. An example of a field that is not of characteristic 0 is $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, where $p$ is a prime number.

Let $k[x]$ be the collection of all polynomials in $x$ with coefficients in $k$. For a polynomial $f \in k[x]$ let $f^{(i)}$ denote the $i^{th}$ derivative of $f$. We can recall that $f$ has a root of multiplicity $j$ at $x_0$ if and only if $f^{(i)}(x_0) = 0$ for all $0 \leq i \leq j - 1$ and $f^{(j)}(x_0) \neq 0$. We say that $x_0$ is a simple root of $f$ if $x_0$ is a root of multiplicity 1, i.e. $f(x_0) = 0$ and $f'(x_0) \neq 0$.

The reason why the conjecture is stated over a field of characteristic 0 is because counterexamples to the conjecture in characteristic $p$ are readily available. For example, the polynomial $f = x^3 - 1$ with coefficients in $\mathbb{Z}_3$ has roots at 0 and $\bar{1}$, but the polynomials $f' = 3x^2 = \bar{0}$ and $f'' = \bar{0}$ both have roots at $\bar{0}$. In other words, this particular degree 3 polynomial shares a root with each of its first two derivatives, but it does not have one root of multiplicity 3.

The rest of this paper examines polynomials over the complex plane $\mathbb{C}$ for the purpose of giving an interesting reformulation of the conjecture for this important case.
For what follows, given a set $C \subset \mathbb{C}$, we say that $z \in C$ is called a boundary point of $C$ if and only if $z$ is contained in no open subset of $C$. All other points of $C$ are called interior points. The collection of all boundary points is the boundary of $C$, denoted $\partial C$, and that the collection of all interior points is the interior of $C$, denoted $\text{int}(C)$. The exterior of $C$ is $\mathbb{C} \setminus C$ and is denoted $\text{ext}(C)$.

We can recall that a set $C$ of the complex plane is called convex if $z_1, z_2 \in C$ and $t \in (0, 1)$ implies $tz_1 + (1-t)z_2 \in C$. This definition captures the feature that the line segment connecting $z_1, z_2$ must be a part of $C$. A convex set $C$ is called strictly convex if it has the additional property that if $z_1, z_2 \in C$ then for all $t \in (0, 1)$, $tz_1 + (1-t)z_2$ must be an interior point of $C$. A circle in the plane is an example of a strictly convex set.

The Fundamental Theorem of Algebra implies that a polynomial $f$ of degree $n$ has at most $n$ complex roots and exactly $n$ roots if multiplicity is counted. Using this, one can define what a convex hull of a polynomial is.

**Definition 2.1.** The convex hull of a polynomial $f(x)$ with coefficients in $\mathbb{C}$ is the intersection of all convex sets that contain the roots of $f$, denoted by $C_f$. That is, $C_f$ is the convex hull of the roots of $f$.

As a visual illustration of what a convex hull would look like, in the images in Figure 1 the dots represent some polynomial’s roots which lie in the complex plane, the structures containing these dots represent the convex hull of that polynomial.

Another way to visualize a convex hull of a polynomial is to imagine that the roots of a polynomial were represented by pegs lying in the complex plane, and a rubber band is stretched around those pegs. If the rubber band is allowed to contract then it will catch onto those pegs in a manner forming a convex structure. This structure will represent the convex hull of that polynomial.

![Figure 1: Examples of Convex Hulls](image_url)

We denote the convex hulls of $f, f', ..., f^{(n-1)}$ as $C_f, C_{f'}, ..., C_{f^{(n-1)}}$ respectively. A standard result on the theory of polynomials with complex coefficients is given below.
Theorem 2.2. (Gauss-Lucas,[5])
If \( f(x) \) is a polynomial with complex coefficients, then the roots of \( f'(x) \) are contained in \( C_f \).

An immediate result from the Gauss-Lucas Theorem relates the convex hull of \( f \) to those of its derivatives.

Corollary 2.3. ([5])
If \( f(x) \) is a polynomial with complex coefficients, then \( C_{f^{(n-1)}} \subseteq \ldots \subseteq C_{f'} \subseteq C_f \).

Another standard result in the study of polynomials is credited to Augustin-Louis Cauchy. It gives a description of the roots of a particular class of monic polynomials, where a monic polynomial is a polynomial whose leading coefficient is 1.

Theorem 2.4. (Cauchy’s Theorem, pg. 243, [5], pg. 2-3, [4])
Let \( f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \), where \( a_i \in \mathbb{R}_{\leq 0} \) and at least one of them is nonzero. Then \( f(x) \) has one positive root \( p \), it is simple, and all other roots have absolute value less than or equal to \( p \).

Under these conditions, \( f(x) \) will have one positive root of multiplicity 1, \( x_0 \in \mathbb{R}_{>0} \), and all other roots will lie in the disc centered at the origin with radius \( x_0 \). In Section 4 we will apply Theorem 2.4 to establish several cases of the Casas-Alvero conjecture.

By relating the roots of a polynomial to the vertices of its convex hull we give an equivalent reformulation of the Casas-Alvero Conjecture in Theorem 3.1. In order to prove Theorem 3.1 we must first look at some additional properties of the convex hull of a polynomial.

Intuitively, when considering a polygon in the plane, we sometimes think of a vertex being a boundary point where two edges of that polygon meet. But a vertex of a polygon has the feature that a line segment containing the vertex must be such that at least one of the endpoints is not a part of the polygon’s boundary nor the interior. This intuition is the motivation of the following definition.

Definition 2.5. Let \( C \subseteq \mathbb{C} \) be any set in the complex plane and \( v \in C \). Then \( v \) is a vertex for \( C \) if it has the property that, for any \( z_1, z_2 \in \mathbb{C} \) with \( tz_1 + (1-t)z_2 = v \) for some \( t \in (0,1) \), then either \( z_1 \notin C \) or \( z_2 \notin C \).

Note that the contrapositive of this definition says that if \( v \) is a vertex of \( C \), \( z_1, z_2 \in C \), then \( tz_1 + (1-t)z_2 \neq v \) for any \( t \in (0,1) \). In other words, no line segment contained in \( C \) can contain \( v \) as an interior point.

Proposition 2.6. If \( v \) is a vertex of \( C_f \), then \( v \in \partial C_f \).

Proof. Suppose by a contradiction that \( v \in \text{int}(C_f) \). This implies there exists \( D \), a disc, around \( v \) s.t. \( D \subseteq C_f \). But that implies each diameter of \( D \), a line segment containing \( v \) as an interior point is in \( C_f \), contradicting \( v \) being a vertex. \( \square \)
Proposition 2.7. A convex set, \( \gamma \subseteq C_f \), will contain a vertex, \( v \), of \( C_f \) if and only if \( v \) is also a vertex of \( \gamma \).

Proof. Suppose by a contradiction that \( v \) is not a vertex of \( \gamma \), hence there exists \( z_1, z_2 \in \gamma \subseteq C_f \) and \( t \in (0, 1) \) such that \( tz_1 + (1-t)z_2 = v \). This immediately contradicts \( v \) being a vertex of \( C_f \). If \( v \) is indeed a vertex of \( \gamma \) then it is trivial that \( v \in \gamma \).

Proposition 2.8. If \( f \) is a polynomial with complex coefficients, then every vertex of the convex hull of \( f, C_f \), must be a root of \( f \).

Proof. Suppose by a contradiction that a vertex, \( v \in C_f \), is such that it is not a root of \( f \), i.e. \( v \notin R \) where \( R \) is the set of all roots of \( f \). All that is needed to contradict this scenario is the existence of another convex set, \( C \), such that \( R \subseteq C \subset C_f \) where \( v \notin C \). Consider the set \( C = C_f \setminus \{v\} \), it is clear that \( v \notin C \), and that \( R \subseteq C \subset C_f \), so the only thing left to show is that \( C \) is a convex set.

Consider two points \( z_1, z_2 \in C \subset C_f \), then by convexity of \( C_f \), we have for all \( t \in (0, 1) \) that \( tz_1 + (1-t)z_2 \in C_f \). But note that \( tz_1 + (1-t)z_2 \neq v \) since \( v \) is a vertex of \( C_f \), therefore \( tz_1 + (1-t)z_2 \in C = C_f \setminus \{v\} \); implying that \( C \) is indeed a convex set.

Therefore all vertices of a polynomial’s convex hull must be among the roots of that polynomial.

3 An approach to the Casas-Alvero Conjecture

We can now state the main contribution of the paper, which is an equivalent formulation of the Casas-Alvero conjecture that could provide new insights on the conjecture for the complex plane. This formulation relies on the characterization of the vertices of a polynomial’s convex hull.

Theorem 3.1. If \( f \) is a polynomial that shares a root with each of its \( n-1 \) derivatives then each root is a vertex of \( C_f \) if and only if \( f \) has only one root of multiplicity \( n \).

Proof. If \( f \) has only one root of multiplicity \( n \), then it is clear that \( f \) shares a root with each of its \( n-1 \) derivatives and that each root, i.e. the only root, is a vertex of \( C_f \).

Let \( f \) share a root with each of its \( n-1 \) derivatives. We can assume that each root of \( f \) is a vertex of \( C_f \). By the Gauss-Lucas Theorem and Corollary 2.3 we have, \( C_{f^{(n-1)}} \subseteq ... \subseteq C_{f'} \subseteq C_f \). Thus, if \( f^{(n-1)}(z_0) = 0 \) then \( z_0 \) lies in the convex hull of all derivatives of \( f(z) \), i.e. \( z_0 \in C_{f^{(i)}} \) for all \( i \). Also, we know that \( f(z_0) = f^{(n-1)}(z_0) = 0 \), since \( f \) shares a root with each of its \( n-1 \) derivatives and the \((n-1)\)st derivative \( f^{(n-1)} \) of a degree \( n \) polynomial is a degree 1 polynomial, and hence has only one root. In particular, we know \( z_0 \in C_{f'} \), but
since $z_0$ is a vertex of $C_f$, by Proposition 2.7 we know $z_0$ is a vertex of $C_{f'}$. By Proposition 2.8, $z_0$ must be a root of $f'$.

Since $z_0 \in C_{f'}$ and $z_0$ is now known to be a vertex of $C_{f'}$, this implies by the same reasons that $f'(z_0) = 0$ that $f''(z_0) = 0$.

Continuing this process shows that $z_0$ must be a root of all $n-1$ derivatives of $f(z)$, implying that $f(z)$ has only one root, $z_0$, of multiplicity $n$.

Based upon Theorem 3.1, we know that in order to prove the Casas-Alvero Conjecture for polynomials over the complex plane, it is enough to show that a complex polynomial of degree $n$ which shares a root with each of its $n-1$ derivatives must be such that each root is a vertex of its convex hull.

Before we can give a proof of the Casas-Alvero Conjecture under some additional assumptions given in Corollary 3.5, we need to develop a few Lemmas.

**Lemma 3.2.** If $w \in \partial C$, where $C$ is a convex set, and $w$ is not a vertex of $C$, then any segment, $l$, contained in $C$ and containing $w$ as an interior point must be such that $l \subseteq \partial C$ i.e. $l$ contains no interior point of $C$.

**Proof.** Suppose by a contradiction that there exists a line, $l$, contained in $C$ in which $w$ is an interior point and that $l$ contains an interior point of $C$, namely $i$. This implies there exists a line segment, $l'$, containing $i$ as one of its end points and $w$ as an interior point of $l'$. But $w$ is a boundary point of $C$ meaning that any neighborhood, $\Omega$, around $w$ must contain points in the exterior of $C$ and in the interior of $C$. So consider the disc, $\Gamma$, centered around $w$ and whose boundary contains $i$. The boundary of $C_f$ contained in $\Gamma$ separates the region of $\Gamma$ into two separate areas, one area is part of $\text{int}(C)$ and the other is in $\text{ext}(C)$. The line $l'$ contains $w \in \partial C$ implies that $l'$ must pass through the boundary of $C$, since $l'$ contains an interior point of $C$, and that it must contain points in the exterior of $C$, a contradiction.

**Lemma 3.3.** If $\gamma$ is a strictly convex set and $v \in \partial \gamma$, then $v$ is a vertex of $\gamma$.

**Proof.** Suppose by a contradiction that $v \in \partial \gamma$ but is not a vertex. This implies there exists $l$, a line segment, such that $v \in \text{int}(l)$ and that $l \subseteq \gamma$. By Lemma 3.2 $l$ can contain no interior point of $\gamma$, thus the end points of $l$ must be elements of $\partial \gamma$. But $\gamma$ is strictly convex which implies that no points that are part of $\text{int}(l)$ can be in $\partial \gamma$; since $v$ is in both, we have a contradiction.

**Theorem 3.4.** If $f$ is a polynomial with complex coefficients of degree $n$ such that its roots lie on the boundary of a strictly convex set, $\Gamma$, then each root of $f$ is a vertex of $C_f$.

**Proof.** Each root of $f$ lies on the $\partial \Gamma$ implies each root of $f$ is a vertex of $\Gamma$. Since $C_f$ is the intersection of all convex sets containing the roots of $f$, and $\Gamma$ is convex, thus $C_f \subseteq \Gamma$ and by Proposition 2.7 each root of $f$ must be a vertex of $C_f$. 

\[ \Box \]
Corollary 3.5. If \( f \) is a polynomial with complex coefficients of degree \( n \) such that its roots lie on the boundary of a strictly convex set, \( \Gamma \), then \( f \) has only one root of multiplicity \( n \).

Proof. This is immediate by Theorems 3.1 and 3.4.

A potential, but seemingly difficult, method to prove the Casas-Alvero Conjecture would be as follows. Let \( z_1, z_2, \ldots, z_n \) be roots of a degree \( n \) polynomial \( f \) that shares a root with each of its \( n - 1 \) derivatives. If we can somehow show that for every point \( z \in \mathbb{C} \) that the distance from \( z \) to \( z_i \) is the same, then we can conclude that \( z_1 = z_2 = \cdots = z_n \). Corollary 3.5 gives a simpler version of this method. If we can show that there exists at least one point in \( z \in \mathbb{C} \) such that the distance from \( z \) to \( z_i \) is the same, then all the roots of \( f \) lie on the boundary of a disc, a strictly convex set, which would prove the Casas-Alvero Conjecture for the complex plane.

4 Polynomial Functions with only Real Roots

4.1 Polynomials with real roots

Using methods different from the previous sections, we can show an equivalent condition to the Casas-Alvero Conjecture for polynomials having only real roots. It should be noted that it is not necessarily the case that a polynomial with real roots will have real coefficients; such an assumption is not made until Theorem 4.5. It will be shown in Theorem 4.3 that if \( f = \sum_{i=0}^{n} a_i x^i \) shares a root with each of its \( n - 1 \) derivatives, and \( f^{(n-2)} \) has only one root of multiplicity \( 2 \), then \( f \) has only one root of multiplicity \( n \). The proof of Theorem 4.3 relies on Viete’s Relations, formulas that relate the coefficients of a polynomial to its roots.

Theorem 4.1. (Viete’s Relations, [4])

Let \( f = \sum_{i=0}^{n} a_i x^i \) be a monic polynomial with roots \( x_1, x_2, \ldots, x_n \). Then we have the following

\[
(-1)^k a_{n-k} = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}.
\]

In particular, using the above notations,

\[
-a_{n-1} = x_1 + x_2 + \cdots + x_n
\]

\[
a_{n-2} = x_1 x_2 + x_1 x_3 + \cdots + x_1 x_n + x_2 x_3 + \cdots + x_{n-1} x_n
\]

\[
= x_1 (x_2 + x_3 + \cdots + x_n) + x_2 (x_3 + \cdots + x_n) + \cdots + x_{n-1} (x_n).
\]
Proposition 4.2. Let \( f = \sum_{i=0}^{n} a_i x^i \) be a monic polynomial such that \( f(0) = a_{n-1} = a_{n-2} = 0 \) and let \( x_1, \ldots, x_n \) be the roots of \( f \), then \( x_1^2 + \cdots + x_n^2 = 0 \).

Proof. Since \( a_{n-1} = a_{n-2} = 0 \) we have that by Theorem 4.1, \( x_1 + x_2 + \cdots + x_n = 0 \) and \( x_1(x_2 + x_3 + \cdots x_n) + x_2(x_3 + \cdots + x_n) + \cdots + x_{n-1}x_n = 0 \). In particular, we have that \((x_1 + x_2 + \cdots x_n)^2 = 0\). Hence

\[
0 = (x_1 + x_2 + \cdots + x_n)^2 \\
= x_1^2 + x_2^2 + \cdots + x_n^2 + x_1(x_2 + x_3 + \cdots + x_n) \\
+ x_2(x_3 + \cdots + x_n) + \cdots + x_{n-1}(x_n) \\
= x_1^2 + x_2^2 + \cdots + x_n^2 + a_{n-2} \\
= x_1^2 + x_2^2 + \cdots + x_n^2.
\]

Using Proposition 4.2 we can give an equivalent reformulation of the Casas-Alvero Conjecture for polynomials that only have real roots.

Theorem 4.3. If \( f \) a polynomial of degree \( n \) with only real roots and it shares a root with each of its \( n - 1 \) derivatives, then \( f^{(n-2)} \) has only one root of multiplicity 2 if and only if \( f \) has one root of multiplicity \( n \).

Proof. If \( f \) has only one root of multiplicity \( n \), then it is clear that \( f \) must share a root with each of its \( n - 1 \) derivatives and that \( f^{(n-2)} \) has only one root of multiplicity 2.

Without loss of generality, let \( f \) be a monic polynomial. Suppose that \( f \) shares a root with each of its \( n - 1 \) derivatives, and that \( f^{(n-2)} \) has only one root of multiplicity 2, say \( w_0 \), then \( f(w_0) = f^{(n-2)}(w_0) = 0 \).

Let \( g(x) = f(x + w_0) = \sum_{i=0}^{n} a_i x^i \). It is clear that whenever \( z_0 \) is a root of \( g^{(i)} \) then \( z_0 + w_0 \) is a root for \( f^{(i)} \). In particular, we have that \( g \) also shares a root with each of its \( n - 1 \) derivatives and that all the roots of \( g \) must be real. But we have that \( g(0) = f(w_0) = f^{(n-2)}(w_0) = f^{(n-1)}(w_0) = 0 \); i.e. \( g(0) = g^{(n-2)}(0) = g^{(n-1)}(0) = 0 \).

Note that \( g^{(n-2)}(x) = \frac{n!}{2} x^2 + a_{n-1}(n-1)! x + a_{n-2}(n-2)! \), but \( g^{(n-2)}(0) = 0 \) implies \( a_{n-2} = 0 \). Also \( g^{(n-1)}(x) = n! x + a_{n-1}(n-1)! \), so \( g^{(n-1)}(0) = 0 \) implies that \( a_{n-1} = 0 \). We now have that \( g(0) = a_{n-2} = a_{n-1} = 0 \). By Theorem 4.2 we have that for the roots \( x_1, x_2, \ldots, x_n \) of \( g \), satisfies \( x_1^2 + x_2^2 + \cdots + x_n^2 = 0 \). Since all roots of \( g \) are real, this implies \( x_1 = x_2 = \cdots = x_n = 0 \). Therefore \( g \) has only one root of multiplicity \( n \) at 0, which implies that \( f \) has only one root of multiplicity \( n \) at \( w_0 \).\]
4.2 Even and Odd Polynomial Functions

This subsection gives a proof of the Casas-Alvero conjecture for even and odd polynomial functions with real roots and under some additional assumptions. Recall that an even function is a function such that \( f(z) = f(-z) \) for all \( z \in \mathbb{C} \) and that an odd function is a function such that \( f(z) = -f(z) \) for all \( z \in \mathbb{C} \).

**Proposition 4.4.** A polynomial, \( f(x) = \sum_{i=0}^{n} a_i x^i \), is an even function if and only if \( a_i = 0 \) for all \( i \) odd.

**Proof.** If \( a_i = 0 \) for all \( i \) odd then it is clear that \( f(x) \) is even function. Consider a polynomial \( f(x) \) that is an even function. Let \( g(x) = \sum_{i=0}^{n} b_i x^i \) where \( b_i = a_i \) for \( i \) even and \( b_i = 0 \) for \( i \) odd. It is clear \( g(x) \) is an even function; it is also clear that \( h(x) = f(x) - g(x) \) is an even function. But the only nonzero coefficients \( h(x) \) can have is at the \( x \)'s with odd powers. So if there are nonzero coefficients, then \( h(x) \) will have nonzero coefficients at odd powers, clearly making \( h(x) \) not an even function, a contradiction.

An identical proof can be used to show that for \( f(x) = \sum_{i=0}^{n} a_i x^i \) is an odd polynomial function if and only if \( a_i = 0 \) for \( i \) even. In addition, by Proposition 4.4 we can readily see that the derivative of an even polynomial function is odd and the derivative of an odd polynomial function is even.

**Theorem 4.5.** If \( f(x) = \sum_{i=0}^{n} a_i x^i \) is an even or odd monic polynomial function that shares a root with each of its \( n-1 \) derivatives, has only real roots, and \( a_{n-1}, a_{n-2}, \ldots, a_0 \leq 0 \), then \( f(x) \) has only one root at 0 of multiplicity \( n \).

**Proof.** Assume that \( f \) is an even polynomial function. To show that \( f \) has a root at 0 consider \( f^{(n-1)}(0) = n!a_n \), which has a root at 0. Recall that by Proposition 4.4 that \( a_{n-1} = 0 \). Hence \( f^{(n-1)}(x) \) has its only root at 0, therefore \( f(x) \) must have a root at 0 as well.

Suppose by contradiction that \( f(x) \) shares a root with each of its \( n-1 \) derivatives and that 0 is not the only root of \( f(x) \); i.e, for some \( a_i \), a coefficient of \( f(x) \), is nonzero.

All coefficients of \( f(x) \) are negative or equal to zero, with at least one being nonzero. Thus by Theorem 2.4, \( f(x) \) has only one positive root \( p \) and it is simple. Since \( f(x) \) is an even function, \( f(-p) = 0 \). We also have that \(-p\) is a simple root of \( f(x) \), if not, then \( f'(x) \) is an odd function such that \( f(-p) = 0 \), which would imply \( f'(p) = 0 \), contradicting that \( p \) is a simple root of \( f(x) \).

We now have that \( f(x) = (x - p)(x + p)x^{n-2} \). Since 0 is a root of multiplicity \( n - 2 \) for \( f \) we have that \( f(0) = f'(0) = \cdots = f^{(n-3)}(0) = f^{(n-1)}(0) = 0 \), so \( f^{(n-2)}(x) \) is the only derivative in question to having a root at 0. But note that \( C_f \), the convex hull of the roots of \( f(x) \), is the line segment \([-p,p] \). Hence \( p \) and \(-p\) are vertices of \( C_f \), so if \( f^{(n-2)}(x) \) has a root at \( p \) or \(-p\) then \( f(p) = f'(-p) = \cdots \).
see Theorem 3.1. Therefore if the root shared by $f(x)$ and $f^{(n-2)}(x)$ is at $p$ or $-p$ then $f(x)$ will have a root of multiplicity $n - 1$ at $p$ or $-p$, a contradiction to $p$ and $-p$ being simple.

The proof of this Theorem for $f$ an odd polynomial function is identical to the case when $f$ is an even polynomial function.

\[ \square \]

5 Conclusions

This paper examined the properties a polynomial $f$ over $\mathbb{C}$ for the purpose to restate the Casas-Alvero Conjecture in terms of the location of the roots of $f$ with respect to its convex hull. In particular, a degree $n$ polynomial $f$ that shares a root with each of its $n - 1$ derivatives will have only one root if and only if every root of $f$ is a vertex of the convex hull of $f$. Using this reformulation of the conjecture, it might be possible use computational methods to prove the conjecture for degree 12 polynomials.

A reformulation of the Casas-Alvero Conjecture for polynomials with only real roots is given in Section 4. Theorem 4.3 shows that to prove the Casas-Alvero Conjecture for a polynomial $f$ of degree $n$ with real roots, it is enough to show that the $(n - 2)$nd derivative of $f$ has only one root of multiplicity 2.

References


