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Kenny Goodfellow[a]

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Rose-Hulman Institute of Technology
Department of Mathematics
Terre Haute, IN 47803
Email: mathjournal@rose-hulman.edu
http://www.rose-hulman.edu/mathjournal

[a]Juniata College, Huntingdon, Pennsylvania

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Towards Bidding Connect Four

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Abstract. Allis [1] has determined that the player who goes first in Connect Four always has a winning strategy. We consider the discrete bidding variation of the game instead of alternating turns. In discrete bidding, each player holds an integer number of chips, and the players bid for the next turn. Whoever wins the bid takes a turn and gives his chips to the other player; thus, the total number of chips stays constant. Introducing bidding to the game alters a player's strategy, as multiple moves in succession are now possible. Develin and Payne [2] have completed an analysis of Tic-Tac-Toe using discrete bidding and have determined a winning strategy. We analyze bidding Connect Two on all board sizes and bidding Connect Three on a 3-by-3 board, which will give us insight into the strategy for Connect Four.

Acknowledgements: I would like to thank Dr. Catherine Stenson for her advice and support as I undertook this project.

1 Introduction

Many games involving two players, such as Connect Four, Tic-Tac-Toe, and chess, alternate turns. Adding bidding to a game changes the progression of the turn cycle. It is now possible for a player to make multiple moves in a row, changing the strategy of the game. Two players, Alice and Bob, each begin the game with an integer number of chips, and all of the chips have equal value. At the beginning of each turn, the players simultaneously reveal the number of chips that they want to bid on that turn. The player who has bid more chips wins the bid. That player gives the chips that he or she bid to the other player and makes a move. Thus, the total number of chips in a game stays constant.

David Richman studied bidding games in the 1980s, and the theory of bidding games has continued to be developed since his death. The theory can be applied to both impartial games, where any move is available to both players, as well as partisan games, where certain moves are available to one player but not the other. Real-valued bidding has been analyzed, and Lazarus et al. [3] found optimal strategies for impartial games where players know how much money the other player has and for games where this information is not known. Instead of real-valued bidding, we look at strategies for games using discrete bidding.

A problem that occurs in both real-valued and discrete bidding games is the case of equal bids. Thus, we follow Develin and Payne [2] and adopt a construct called the *tie-breaking advantage*. One of the players begins the game with the tie-breaking advantage; suppose that this player is Alice. When Alice and Bob bid the same numbers of chips, Alice has two options with the tie-breaking advantage. First, she could declare herself the winner of the bid. If she does, she gives the chips that she bid and the tie-breaking advantage to Bob, and she makes a move. Second, she could declare Bob the winner of the bid. If she does, then Bob gives the chips that he bid to Alice, and Bob makes a move. In this case, Alice keeps the tie-breaking advantage. We denote that a player has the tie-breaking advantage by the use of an asterisk, *. For example, if Alice's bidding resources are 7*, this means that she has 7 chips and the tie-breaking advantage.

Develin and Payne have proved several results regarding the tie-breaking advantage. First, it is always advantagous to a player to hold the tie-breaking advantage [2, Lemma 3.1]. That is, suppose that Alice's bidding resources are a, and Bob's are b^*. If Alice has a winning strategy for this situation, then she will also have a winning strategy when both players have the same number of chips but Alice has the tie breaking advantage instead. Second, the tie-breaking advantage is worth less than an ordinary chip [2, Lemma 3.2]. That is, suppose that Alice's bidding resources are a^*, and Bob's are $b+1$. If Alice has a winning strategy for this situation, then Alice also has a winning strategy when her bidding resources are $a+1$ and Bob's are b^*. This allows us to put an order on chip values:

$$0 < 0^* < 1 < 1^* < 2 < 2^* < 3 < 3^* < \dots$$

Third, the player who has the tie-breaking advantage should always use it unless the bids are 0 [2, Proposition 3.4].

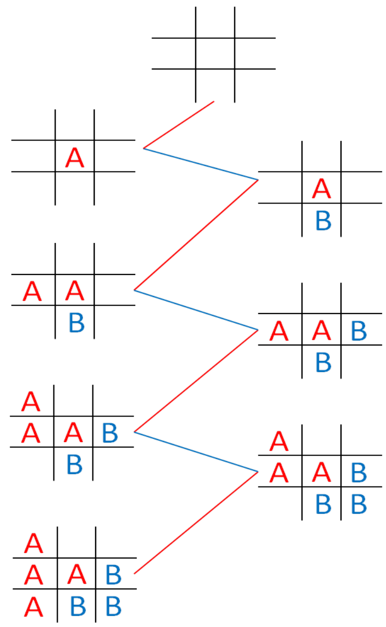

Figure 1: Possible graph for a game of Tic-Tac-Toe. Game progression is from top to bottom. Each vertex represents a position in the game, and each red or blue edge represents a move by Alice or Bob, respectively. The ending vertex shows a win for Alice.

Some games, such as Connect Four, can be represented using a graph. Each vertex of the graph represents a valid position, and each edge, colored red or blue for Alice or Bob, respectively, represents a legal move. Ending vertices represent a win for Alice or Bob or represent a draw. Figure 1 is an example of such a graph for the game of Tic-Tac-Toe. If we have a graph representing all of the positions for a game, our goal is to trim the graph into a tree of moves for Alice and Bob. At the top of the tree is the starting position, and there are two edges coming from this position: one edge leading to Alice's optimal first move, and another edge leading to Bob's optimal first move. From here, each vertex has two edges coming from it, representing the optimal moves for each player. We continue until we get to the end positions of the game: a win for Alice, a win for Bob, or a draw [2].

Up to now, we have not defined how to determine the optimal moves for each player. Following Richman's work, the *critical threshold*, R, is defined to be a number between 0 and 1 such that Alice has a winning strategy if her proportion of the bidding resources is strictly greater than R and she does not have a winning strategy if her proportion of the bidding resources is less than R [3]. Each position in the game has an associated critical threshold. To calculate the critical thresholds of each position, we start from the end positions and work backwards. If Alice has won the game, this end position has $R = 0$; that is, no matter what proportion of the chips she has, she has won the game. Otherwise, if Bob has won the game or if the game is a draw, the critical threshold is $R = 1$. This means that Alice needs more than one hundred percent of the total number of chips in order to win the game, which cannot happen [2].

For each position that is not an ending position, we calculate the critical threshold based on what the next position would be for each player. Alice's optimal move from a position is to make the critical threshold as low as possible; the closer it is to 0, the smaller proportion of the chips she needs to win. Let R_A denote the critical threshold of this optimal move by Alice. On the other hand, Bob's optimal move from a position is to make the critical threshold as high as possible; the closer it is to 1, the more difficult it will be for Alice to have a winning strategy. Let R_B denote the critical threshold of this optimal move by Bob. Following [2], the critical threshold for a position is

$$R = \frac{R_A + R_B}{2}. \tag{1}$$

For example, suppose we are in the position where the next player to move wins. Alice's optimal move is to win, which is denoted by $R_A = 0$. Bob's optimal move is also to win, which is denoted by $R_B = 1$. Thus, by equation (1), the critical threshold of this position is $R = 1/2$, which means that Alice has a winning strategy if she has more than half of the bidding resources. For any position, we can continue to work backwards in this fashion to determine the critical threshold of the starting position. We refer to this critical threshold as the critical threshold of the game.

Develin and Payne [2] have completely analyzed bidding Tic-Tac-Toe completely, and the critical threshold of the game is $R = 133/256$. One game that has not yet been analyzed from a bidding standpoint is Connect Four. A standard game of Connect Four is played on a 6×7 board that is perpendicular to the table. Thus, when a piece is played in one of the seven columns, it falls to occupy the lowest available square. A player wins when he or she claims four adjacent squares in a row or column or four squares that are connected along a diagonal. Allis [1] has determined that the player who goes first in Connect Four always has a winning strategy.

We wish to see how bidding changes the winning strategy of Connect Four. In the process, we look at Connect n for different situations of n; a player wins Connect n when he or she claims n adjacent squares in a row or a column or n squares connected along a diagonal. In Section 2, we show how chip tables are used to determine how many chips are needed for Alice to have a winning strategy given a fixed total number of chips. In Section 3, we demonstrate a notation used to analyze bidding games. In Section 4, we analyze Connect n on a $1 \times n$ board. In Section 5, we analyze Connect Two on boards of all sizes, and in Section 6, we analyze Connect 3 on a 3×3 board.

2 Chip Tables and Periodicity

In real-valued bidding games, Alice has a winning strategy as long as her proportion of the bidding resources is greater than the critical threshold. In discrete bidding, Alice may not be able to make the same bids that she can make in the real-valued bidding game. In addition, we must determine whether Alice will need the tie-breaking advantage to win the game or not. For k total chips, let $f(G, k)$ equal the minimum number of chips that Alice needs in order to have a winning strategy for game G [2].

For the rest of the analysis of games presented in this paper, a game that is not a winning game for Alice will be treated as a winning game for Bob. Thus, games that would normally be considered draws are now games in which Bob wins. This makes the analysis easier and does not change the critical threshold for Alice.

As an example, we consider Connect Two on a 1×2 board. In this game, Alice wins if she plays on both of the first two moves of the game and Bob wins otherwise. A move by Alice takes the game to a position where the next player to move wins, and a move by Bob takes the game to a winning position for Bob. Thus, using the corresponding critical thresholds and equation (1), the critical threshold of Connect Two in real-valued bidding is 3/4. Suppose that the game is played with 5 total chips. We find that to win this game, Alice needs 4^* chips.

First turn: Alice has 4^* chips, and Bob has 1. Alice bids 1^* and wins the move.

Second turn: Alice has 3 chips, and Bob has 2^*. Alice bids 3 and wins the move (and the game).

Notice that Alice could not have won with anything less. Suppose she had started with 4 chips.

First turn: Alice has 4 chips, and Bob has 1^*. Alice bids 2 and wins the move.

Second turn: Alice has 2 chips, and Bob has 3^*. Bob bids 2^* and outbids Alice to win the move.

Thus, we find that Alice needs four of the five chips plus the tie-breaking advantage in order to win. However, 4/5 is greater than 3/4, the critical threshold in real-valued bidding, yet she cannot win unless she has the tie-breaking advantage.

We return to Connect Two on a 1×2 board. Develin and Payne give the chip table for an equivalent game [2, p. 13]; here, we focus on the computational details. The goal is to fill the following table with chip values $f(\text{Connect Two}, k)$:

$k = 4n+$	+0	+1	+2	+3
$f(\text{Connect Two}, k) = 3n+$				

Develin and Payne [2, Theorems 3.13 and 3.14] have shown that these chips tables are periodic. Let p represent this periodicity, which is the number of games that we have to check. For nonnegative integer values of n, any "base case" can be checked using total number of chips $pn, pn + 1, pn + 2, \ldots, pn + (p - 1)$. In this example, the periodicity of the game is 4, which matches the denominator of the critical threshold. In addition, for this example, the number of chips needed to win the game is $3n$ plus the entry of the table corresponding to the total number of chips. The number 3 is the numerator of the critical threshold. Please see Develin and Payne's work [2] for the details on periodicity.

We look at Connect Two with $n = 1$. We try to determine the number of chips needed for Alice to win. With $n = 1$, the first entry in the table above represents a game with 4 total chips. We find that to win this game, Alice needs all 4 chips.

First turn: Alice has 4 chips, and Bob has 0*. Alice bids 1 and wins the move.

Second turn: Alice has 3 chips, and Bob has 1*. Alice bids 2 and wins the move (and the game).

Notice that Alice could not have won with anything less. Suppose she had started with 3* chips.

First turn: Alice has 3* chips, and Bob has 1. Alice bids 1* and wins the move.

Second turn: Alice has 2 chips, and Bob has 2*. Bob bids 2* and outbids Alice to win the move.

Thus, since the numerator of the critical threshold is 3, the first entry of the table is 1. This is because she needs 4 chips to have a winning strategy; she gets 3 of them from the numerator of the critical threshold times n, which is 1 here, and so the remaining chip will have to come from the entry in the table.

For $n = 1$, the second entry in the table represents the game with 5 total chips. We saw earlier that Alice needs 4* chips to win this game. Thus, since the numerator of the critical threshold is 3, the second entry of the table is 1*. We continue in this fashion for the final two entries of the table to find the following results:

$k = 4n+$	$+0$	$+1$	$+2$	$+3$
$f(\text{Connect Two}, k) = 3n+$	1	1*	2*	3

Thus, we have $f(\text{Connect Two}, 4n + x) = 3n + t$, where x is a number between 0 and 3, inclusive, and t is the corresponding entry in the table. For example, for $n = 1$, we have $f(\text{Connect Two}, 4(1)+1) = 3(1)+1^*$, or $f(\text{Connect Two}, 5) = 4^*$. To demonstrate the power of this periodicity, let $n = 12$. By this, we claim, for example, that $f(\text{Connect Two}, 50) = 38^*$. We test this.

First turn: Alice has 38* chips, and Bob has 12. Alice bids 12* and wins the move.

Second turn: Alice has 26 chips, and Bob has 24*. Alice bids 25 chips to win the move (and the game).

However, Alice would not be able to win if she had 38 chips.

First turn: Alice has 38 chips, and Bob has 12*. Alice bids 13 and wins the move.

Second turn: Alice has 25 chips, and Bob has 25*. Bob bids 25* and wins the move.

However, we do not have to play every game to determine the minimum number of chips needed for Alice to win the game. Similar to working backwards from end positions to find the critical thresholds in real-valued bidding games, we can work backwards from the end positions to find the number of chips needed for Alice to have a winning strategy from each position. Develin and Payne [2] have determined how this can be done. If Alice has won game G, we have $f(G, k) = 0$ because she does not need any chips to win the game. Otherwise, if Bob has won the game or if the game is a draw, we have $f(G, k) = k + 1$ because she would need more chips than are available in order to win the game.

For each position that is not an ending position, we calculate the bidding resources that Alice needs based on the optimal move for each player. Alice's optimal move makes the bidding resources that she needs to win as low as possible. Let this amount be represented by $f_A(G,k)$. Bob's optimal move makes the bidding resources that Alice needs as high as possible. Let this amount be represented by $f_B(G,k)$. In addition, following the notation of Develin and Payne [2], let $|x|$ remove the tie-breaking advantage, if necessary. In addition, $x{+}{*}{=}\, x{*}$. The following theorem [2, Theorem 3.7] is used to calculate how many chips are needed for Alice to win the game.

Theorem 1 (Develin and Payne)**.** *For any position in a game G with k total chips, the number of chips that Alice needs to win the game, $f(G,k)$, is given by*

$$f(G,k) \quad = \quad \left\lfloor \frac{|f_A(G,k)| + |f_B(G,k)|}{2} \right\rfloor + \epsilon, \tag{2}$$

where

$$\epsilon = \begin{cases} 0, & \textit{if } |f_A(G,k)| + |f_B(G,k)| \textit{ is even and } f_A(G,k) \textit{ does not have the tie-breaking advantage;} \\ 1, & \textit{if } |f_A(G,k)| + |f_B(G,k)| \textit{ is odd and } f_A(G,k) \textit{ has the tie-breaking advantage;} \\ *, & \textit{otherwise.} \end{cases}$$

3 Notation for Games

Following Develin and Payne [2], we use the following notation in order to analyze bidding games. Let **A** represent the class of games in which Alice always wins, and let **B** represent the class of games in which Bob always wins. In light of this notation and the previous discussion of critical thresholds, **A** has a critical threshold of 0, and **B** has a critical threshold of 1. In addition, let A^n represent the class of games in which Alice wins if she makes any of the next n moves and otherwise loses. Similarly, let B^n represent the class of games in which Bob wins if he makes any of the next n moves and loses otherwise. Finally, let **E** represent the class of games in which the player with more chips always wins. This is called the even game.

Next, we let G and H be two games. We say $G < H$ if $f(G,k) < f(H,k)$ for all k. This notation means that G is a better game for Alice than H and that G is a worse game for Bob than H. This allows us to order the classes of games described above. We have

$$\mathbf{A} < \cdots < A^4 < A^3 < A^2 < \mathbf{E} < B^2 < B^3 < B^4 < \cdots < \mathbf{B}.$$

If $f(G,k) = f(H,k)$ for all possible values of k, then the games G and H are *equivalent* [2].

Define $G \wedge H$ to be the game where the first player to move can choose between moving to game G or game H [2]. Then a game in which the best move Alice can make is to game

G and the best move Bob can make is to game H is equivalent to $G \wedge H$. For example, we write $\mathbf{A} \wedge (A^2 \wedge B^2)$ for a game in which Alice can move to a position in which she has won, and Bob can move to a position from which Alice can move to a position from which she can win if she makes either of the next two moves or Bob can move to a position from which he can win if he makes either of the next two moves.

We can now say that $\mathbf{E}$ is equivalent to A^1 and is also equivalent to B^1. This is because the critical threshold of $\mathbf{E}$ is 1/2, for if Alice has more than half of the chips, she will win. The position A^1 means that Alice wins only if she makes the next move, which is only guaranteed if she has more chips than Bob. A similar situation occurs for B^1. In addition, for example, we saw in Section 2 that the chip table for Connect Two on a 1×2 board, and the critical threshold is 3/4. Since Bob wins if he gets either of the next two moves and Alice wins otherwise, this position is B^2. This game can also be thought through in a different way. From the beginning, Alice can make a move to the position where the player with more chips wins, and Bob can move to a position in which he always wins. This game is $\mathbf{E} \wedge \mathbf{B}$. Thus, B^2 is equivalent to $\mathbf{E} \wedge \mathbf{B}$.

We now state two results that are useful in comparisons. The first, as stated by Develin and Payne [2, p. 22], is that A^n is equivalent to $\mathbf{A} \wedge A^{n-1}$ for all $n \geq 2$. The other result is that $\mathbf{E}$ is equivalent to $A^n \wedge B^n$ for all $n \geq 1$ and is also equivalent to $\mathbf{A} \wedge \mathbf{B}$.

For reference, we present several common positions and their corresponding critical thresholds. They are used in the calculation of critical thresholds for more complex games.

Position	Critical Threshold
$\mathbf{A}$	0
A^4	1/16
A^3	1/8
A^2	1/4
$\mathbf{E}$	1/2
B^2	3/4
B^3	7/8
B^4	15/16
$\mathbf{B}$	1

4 Bidding Connect n on a $1 \times n$ Board

We now analyze Connect n on a $1 \times n$ board. Alice wins if she gets all of the first n moves, and Bob wins otherwise. We look at this example first because the chip tables for each game up to Connect Four on a 1×4 board are easy to construct, and positions with the same critical thresholds will show up in games on larger boards, such as Connect Three on a 3×3 board.

Connect One on a 1×1 board is the game in which the player who moves first wins; thus, the player who has more chips always wins. This position was defined before as $\mathbf{E}$.

Since this game leads directly to the end positions for each player, the critical threshold of this game is 1/2. The following is the chip table for this game.

$k = 2n+$	$+0$	$+1$
$f(\mathbf{E}, k) = n+$	0^*	1

Recall, in section 2, we noted that Connect Two has a critical threshold of 3/4. In light of the analysis in Section 3, this position is B^2.

In Connect Three on a 1×3 board, Bob wins if he gets any of the first three moves, and Alice wins otherwise. Using the notation introduced in Section 3, this position is B^3. A move by Bob on the first turn allows him to win the game, but a move by Alice on the first turn means that Bob still wins if he gets either of the next two moves, and Alice wins otherwise. This is the same game as Connect Two. The critical threshold of Connect Three is 7/8. The following is the chip table for this game.

$8n+$	$+0$	$+1$	$+2$	$+3$
$0+$	1	2	3	3^*
$4+$	4^*	5^*	6^*	7

Following the notation presented in section 2, we have $f(B^3, 8n + x) = 7n + t$, where x is a number between 0 and 7, inclusive, and t is the corresponding entry in the table. For ease of reading, we use a 2×4 table instead of a 1×8 table to present the entries. In other tables, we will use a similar style with multiple rows. As an example using the above table, when $n = 1$, the second entry in the second row represents the game with 13 total chips. We have $f(B^3, 8(1) + 5) = 7(1) + 5^*$, or $f(B^3, 13) = 12^*$. Thus, Alice needs 12 chips and the tie-breaking advantage to win the game with 13 total chips.

Connect Four on a 1×4 board will give Bob the win if he gets any of the next four moves, and Alice will win otherwise. This position is B^4. Once again, a move by Bob on the first turn allows him to win the game, but a move by Alice on the first turn takes the game to B^3, where Alice will still need all of the next three moves to win. The game now is the same as Connect Three. The critical threshold of Connect Four is 15/16, and the chip table for this game is as follows.

$$f(B^4, 16n+) = 15n+$$

$16n+$	$+0$	$+1$	$+2$	$+3$
$0+$	1	2	3	4
$4+$	5	6	7	7^*
$8+$	8^*	9^*	10^*	11^*
$12+$	12^*	13^*	14^*	15

In general, Connect n on a $1 \times n$ board is B^n because Bob wins if he gets any of the first n moves and Alice wins otherwise.

In addition, we can state a result concerning the critical threshold of Connect n on a $1 \times n$ board.

Theorem 2. *Let R_n denote the critical threshold of Connect n on a $1 \times n$ board. Then $R_n = \frac{2^n-1}{2^n}$ for $n \geq 1$.*

Proof. We procede by induction. We first look at the base case of $n = 1$. From the equation above, we find that $R_1 = 1/2$. This is as we saw before in the case of the even game.

As an induction hypothesis, suppose that the critical threshold of Connect k on a $1 \times k$ board is $R_k = \frac{2^k-1}{2^k}$. We wish to show that $R_{k+1} = \frac{2^{k+1}-1}{2^{k+1}}$. From before, we know that Connect $k+1$ on a $1 \times (k+1)$ board is B^{k+1}, which is equivalent to $B^k \wedge \mathbf{B}$. We can calculate the critical threshold of this position using equation (1). Alice's optimal move takes the game to B^k, which is Connect k on a $1 \times k$ board. By the induction hypothesis, the critical threshold of this game is $R_k = \frac{2^k-1}{2^k}$. Bob's optimal move lets him win the game, so the critical threshold of this position is 1. Thus, we have

$$R_{k+1} = \frac{\frac{2^k-1}{2^k} + 1}{2},$$

which can be simplified to

$$R_{k+1} = \frac{2^{k+1} - 1}{2^{k+1}}.$$

□

5 Bidding Connect Two

We continue by analyzing the game Connect Two on boards of all sizes. In Connect Two, Alice wins if she can make moves in two adjacent squares, whether those squares are connected vertically, horizontally, or diagonally. Bob wins if he can connect two adjacent squares in the same way. We also consider failure of either player to connect two squares a win for Bob. If there is only one row of squares, players can only connect two squares horizontally. The smallest case is the 1×2 board, which has been analyzed in the previous section. As we move to larger boards, Bob will not always be able to win with the first move that he makes. Thus, we use terminology introduced by Develin and Payne [2]. A player has a *threat* if that player can win if he or she makes the next move. Similarly, a player has a *double threat* if that player can win if he or she makes either of the next two moves. A player has made a *block* if he or she has played such that the other player, who previously had at least one threat, no longer has a threat. That is, a block removes all threats.

We first look at the 1×3 board, which is shown in Figure 2. In Figure 2, as well as in all subsequent figures, moves that lead to a win are omitted. The critical thresholds from each of the positions are also shown.

Proposition 1. *Every move in Figure 2 is optimal.*

Proof. There are three moves in the game tree of Figure 2, so we show that each of these moves is optimal. Bob's first move takes the game to $\mathbf{B}$; Alice cannot win if Bob takes the

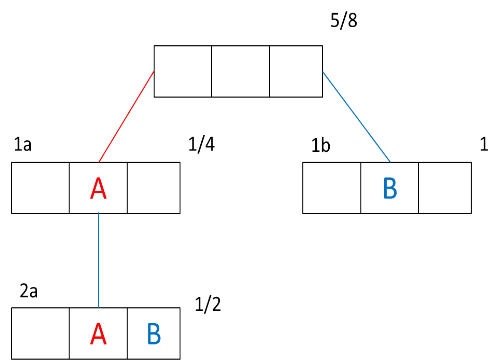

Figure 2: Optimal moves and critical thresholds for Connect Two on a 1×3 board.

center on the first move. Since Bob can never do better than this, this move is optimal for Bob. In addition, Bob's move from 1a is forced, so this move is optimal as well.

We must now show that Alice's optimal first move is in the center. By moving in the center, Alice has created the position A^2. Suppose that Alice's first move was in the leftmost square. Then the next player to move would win; Alice would win by taking the center, and Bob would prevent Alice from winning by also taking the center. The same progression of moves would take place if Alice had taken the rightmost square. Thus, Alice's move in one of the side squares on the first turn takes the game to **E**, which is worse for Alice.

□

We have already seen the chip table for position 2a; this position has a critical threshold of 1/2 and was analyzed in section 4. Position 1a has a critical threshold of 1/4; Alice wins if she gets either of the next two moves. As noted in the proof, this is position A^2. The chip table for A^2 is as follows.

$k = 4n+$	$+0$	$+1$	$+2$	$+3$
$f(A^2, k) = n+$	0	0*	0*	1

The wedge sum for the starting position of Connect Two on a 1×3 board is $A^2 \wedge \mathbf{B}$, as Alice can move to position 2a, analyzed above, and Bob can move to a winning position. The critical threshold for this position is 5/8, and the chip table for the game is as shown.

$f(A^2 \wedge \mathbf{B}, 8n+) = 5n+$

$8n+$	$+0$	$+1$	$+2$	$+3$
$0+$	1	1*	2	2*
$4+$	3	3*	4*	5

This ends the analysis of the 1×3 board.

We now move to a slightly larger board, a 1×4 board. Again, players are only able to connect two squares horizontally. Figure 3 shows the game tree for this game for Connect Two on a 1×4 board, along with the critical thresholds for each position.

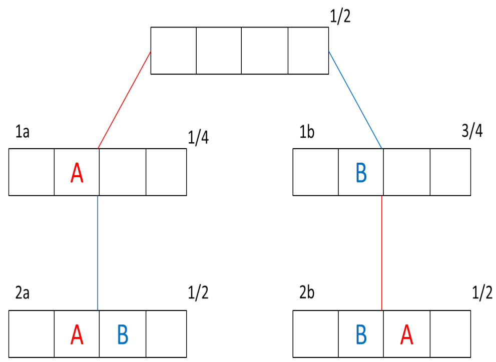

Figure 3: Optimal moves and critical thresholds for Connect Two on a 1×4 board.

Proposition 2. *Every move in Figure 3 is optimal.*

Proof. There are four moves in the game tree of Figure 3, so we analyze each of them.

Bob's move from 1a: Wherever Bob moves, Alice can win on the next turn, or Bob can win on the next turn, either by winning directly or by preventing Alice from winning. Thus, Bob can reach the game **E** wherever he moves. We choose the square directly to the right of Alice's move.

Alice's move from 1b: A move in the leftmost square does not help Alice; there are no unoccupied squares adjacent to this one. Thus, the game is B^2. If Alice moves in either of the rightmost squares, Alice or Bob can win on the next turn. Thus, this position is **E**, which is better for Alice. We choose the square directly to the right of Bob's move.

Alice's first move: Suppose that Alice moves in the leftmost square instead of in one of the two inner squares. Alice has a threat, but Bob can block by taking the inner square adjacent to Alice's move. Alice would then need the next two moves in order to win. Thus, Alice's move in the leftmost square takes the game to $\mathbf{A} \wedge B^2$. This is the same as if she had taken the rightmost square. If Alice takes one of the two inner squares, she has a double threat. Her position is A^2, which is better for her.

Bob's first move: Wherever Bob moves, he has a double threat; he can win if he gets either of the next two moves. This position is B^2. We choose that he plays in one of the inner squares.

□

For each of the positions in Figure 3, we have already seen the chip tables corresponding to the critical thresholds. The wedge sum for Connect Two on a 1×4 board is $A^2 \wedge B^2$, which is equivalent to **E**.

In playing Connect Two, we have seen that Alice needs both moves to win on a 1×2 board and needs to get the first move to win on a 1×3 board. These reasons make the critical thresholds on these boards 3/4 and 5/8; that is, they are not even games. However, we saw that no matter where Bob moves if he gets the first move on the 1×4 board, there are still two adjacent squares open in which Alice can play and win. These lead to the following result concerning larger boards when both players play optimally.

Theorem 3. *For Connect Two on a $1 \times n$ board, R=1/2 for $n \geq 4$. That is, these games are even.*

Proof. On a 1×4 board, we saw that after two turns, optimal play leads both players to a position from which either can win if neither has won yet. Thus, the game ends in at most three turns and only requires four squares for Alice to have a chance to win, even if she does not get the first move. On larger boards, Alice will take an inner square on her first turn, and Bob can take any square. If no one has won after two turns, both will still be in a position to win on the next turn. The addition of extra squares does not change the outcome of an optimally-played game.

□

We saw before that no draws occur for Connect Two on a $1 \times n$ board if both players play optimally. In particular, Bob can win by connecting two squares and does not win by creating a draw.

The next board that could be analyzed is the $n \times 1$ board; now, a player can only win vertically rather than horizontally. On each turn, each player only has one possible move, so that move has to be the optimal move. However, unlike the $1 \times n$ board, which had a critical threshold of $R = 1/2$ when $n \geq 4$, the critical threshold of the $n \times 1$ board will change depending on how large n is. This is because a draw now becomes important in determining the winner. Even with optimal play, the pieces may alternate as the players build vertically; in this case, no player will have connected two pieces. On any turn that a player cannot win, the best that that player can do is block the opponent's threat and create his or her own threat. However, where game play changes is in the top three squares of the board. If Alice gets the third square from the top and has not won, she must get the next move in order to win. Otherwise, if Bob gets the next move, then there will be one square left, making it impossible for Alice to connect two squares. Thus, the game is **E**. On the other hand, if Bob gets the third square from the top and has not won, Alice must get the next two moves in order to win; otherwise, Bob will either win or prevent Alice from winning. The position if Bob gets the third square from the top is B^2, which is worse for Alice than **E**.

By brute force, we can create trees of optimal moves to determine the critical threshold of Connect Two on an $n \times 1$ board. When $n = 2$, the critical threshold is 3/4. When $n = 3$, the critical threshold is 5/8. When $n = 4$, the critical threshold is 9/16, and when $n = 5$, the critical threshold is 17/32. We observe that the critical threshold is slightly greater than 1/2 and is approaching 1/2 as n gets larger. Thus, we have the following result.

Theorem 4. *For Connect Two on an $n \times 1$ board, the critical threshold is $R = \dfrac{2^{n-1}+1}{2^n}$ for $n \geq 2$.*

Proof. Consider three games. Let $G_1(n)$ be Connect Two on an $n \times 1$ board. Let $G_2(n)$ be Connect Two on an $n \times 1$ board on which Alice has taken the lowest square. Let $G_3(n)$

be Connect Two on an $n \times 1$ board on which Bob has taken the lowest square. We first wish to determine the critical thresholds of these games, denoted as $R(G_1(n))$, $R(G_2(n))$, and $R(G_3(n))$. Clearly, since Alice cannot win Connect Two on a 1×1 board, $R(G_1(1))$, $R(G_2(1))$, and $R(G_3(1))$ are all equal to 1.

We skip $G_1(n)$ for now. In $G_2(n)$, Alice's move lets her win, which has a critical threshold of 0, and Bob's move takes the game to the $(n-1) \times 1$ board on which he has taken the lowest square. This has a critical threshold of $R(G_3(n-1))$. Thus, by equation (1), we have

$$R(G_2(n)) = \frac{0 + R(G_3(n-1))}{2}.$$

In $G_3(n)$, Alice's move takes the game to the $(n-1) \times 1$ board on which she has taken the lowest square. This has a critical threshold of $R(G_2(n-1))$. Bob's move lets him win, which has a critical threshold of 1. Thus, by equation (1), we have

$$R(G_3(n)) = \frac{R(G_2(n-1)) + 1}{2}.$$

Finally, in $G_1(n)$, Alice's move takes the game to the $n \times 1$ board on which she has taken the lowest square, which has a critical threshold of $R(G_2(n))$. Bob's move takes the game to the $n \times 1$ board on which he has taken the lowest square, which has a critical threshold of $R(G_3(n))$. Thus, by equation (1), we have $R(G_1(n)) = \dfrac{R(G_2(n)) + R(G_3(n))}{2}$. We can also substitute the previous results and simplify to get

$$R(G_1(n)) = \frac{R(G_2(n-1)) + R(G_3(n-1)) + 1}{4}.$$

We now procede to prove the theorem by induction. We first look at the base case of $n = 2$. By the equation above, we have $R(G_1(2)) = \dfrac{R(G_2(1)) + R(G_3(1)) + 1}{4}$, or $R(G_1(2)) = 3/4$, as desired.

As an induction hypothesis, suppose that the critical threshold of Connect Two on a $k \times 1$ board is $R(G_1(k)) = \dfrac{2^{k-1}+1}{2^k}$, or $\dfrac{2^{k-1}+1}{2^k} = \dfrac{R(G_2(k)) + R(G_3(k))}{2}$. We wish to show that

$$R(G_1(k+1)) = \frac{2^k + 1}{2^{k+1}}.$$

From before, we know that

$$R(G_1(k+1)) = \frac{R(G_2(k+1)) + R(G_3(k+1))}{2}.$$

Using the substitutions from earlier, we have

$$R(G_1(k+1)) = \frac{\dfrac{R(G_2(k)) + R(G_3(k))}{2} + \dfrac{1}{2}}{2}.$$

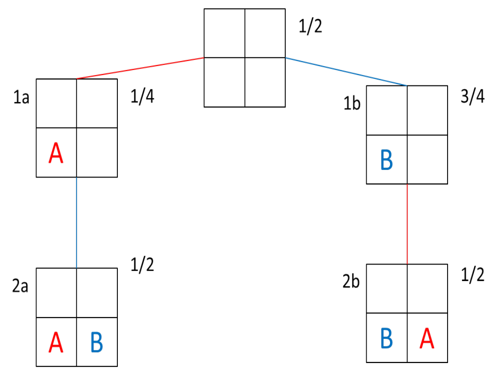

Figure 4: Optimal moves and critical thresholds for Connect Two on a 2×2 board.

By the induction hypothesis, we have

$$R = \frac{\frac{2^{k-1}+1}{2^k} + \frac{1}{2}}{2},$$

which can be simplified to

$$R = \frac{2^k + 1}{2^{k+1}}.$$

□

The next board size to analyze is the smallest board with multiple rows and columns: a 2×2 board. This is the first case where a player can win vertically, horizontally, or diagonally. Figure 4 shows the game tree of optimal moves and critical thresholds for the 2×2 board.

It is simple to see that every move in this tree is optimal. There are four moves in this tree. When Bob moves from 1a, he has two choices of where to move. Regardless of where he moves, the game becomes **E**, for the next player to make a move wins. Similarly, regardless of where Alice moves from 1b, either player can win on the next turn. Regardless of where Alice moves if she gets the first move, the game is A^2, and regardless of where Bob moves if he gets the first move, the game becomes B^2. Thus, the position at the beginning of Connect Two on a 2×2 board is $A^2 \wedge B^2$, which is equivalent to **E**. We see that the critical thresholds for all of the optimal moves in this game are the same as in Connect Two on a 1×4 board. However, we notice that every move for both Alice and Bob is an optimal move on the 2×2 board, whereas on the 1×4 board, Alice's optimal first move is to move into one of the inner squares. Due to the geometry of the 2×2 board, it is impossible for a game of Connect Two to end in a draw.

Now, we can generalize Connect Two to boards larger than 2×2. It turns out that games played on larger boards are the same as those played on the 2×2 board.

Theorem 5. *For Connect Two on an $m \times n$ board, R=1/2 for $m, n \geq 2$. That is, these games are even.*

Figure 5: a. Alice has a threat because she can win on the next turn if she moves in the center column. b. Alice has a distant threat because she can win if she takes the center square. However, this square is not available on the next turn.

Proof. We saw that Connect Two on a 2×2 board will never end in a draw. That is, after at most three turns, either Alice or Bob will have connected two pieces vertically, horizontally, or diagonally. This requires only three squares in an L-configuration. Adding additional rows or columns to the board will not change the critical values of optimal play. The game can still be finished in three turns in a 2×2 section of the larger board. After one move, a player has a threat. If a player makes a second move and does not win, then that player did not play in an adjacent square. Since winning is optimal, this is not an optimal move. □

This ends the analysis of Connect Two.

6 Bidding Connect Three

We are now in a position to analyze the game Connect Three on a 3×3 board. Let Alice win Connect Three if she gets three pieces vertically, horizontally, or diagonally, and let Bob win otherwise. As in Connect Two, the critical threshold for each position is the same for Alice as if we allowed the game to end in a draw. We introduce one more term that will be helpful in our analysis. We say that a player has a *distant threat* if he or she needs to claim one more square in order to win, but that square is not available because a piece played in that column would drop to a row lower than the square needed. Figure 5 demonstrates the difference between a threat and a distant threat.

We wish to analyze the optimal moves of Connect Three.

Proposition 3. *Every move in Figures 6 and 7 is optimal for real-valued bidding.*

Proof. There are thirty moves in the game trees of Figures 6 and 7, so we must analyze each of these moves. As before, moves that lead to a win in these figures are omitted. The following moves are easy to handle. The position 3d, created by Alice's move from 2c, is the same position as 3b. Both Alice's move and Bob's move from 3f are obvious; moving in the left column will create the mirror image of the position created by moving in the right column, and vice versa. We do not look at the optimal moves for positions in the 5 row or 4c and 4e. The positions 4e, 5a, 5b, 5e, and 5g are equivalent to **E**. The position 5c is

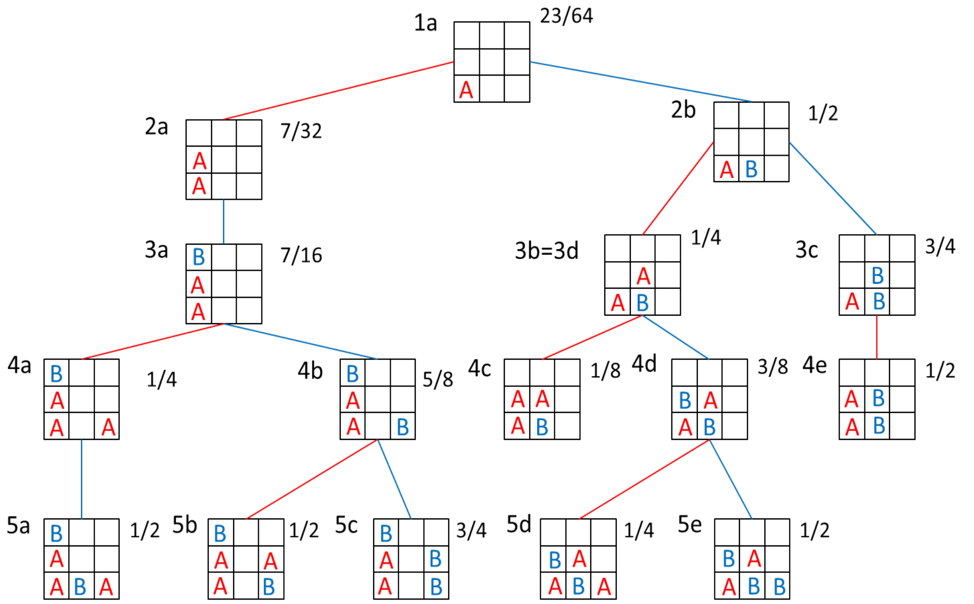

Figure 6: Optimal moves and critical thresholds for Connect Three on a 3×3 board, assuming Alice makes the first move.

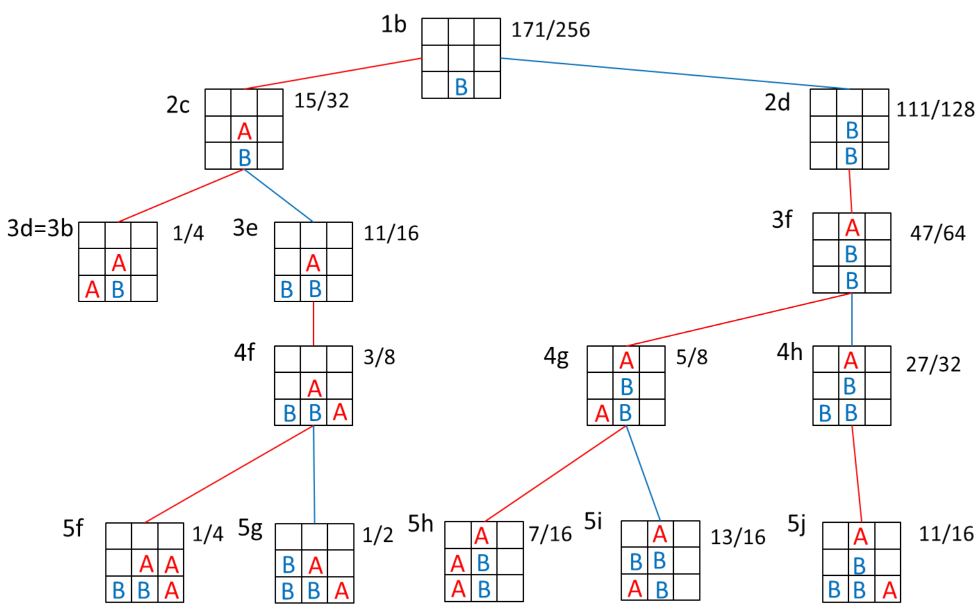

Figure 7: Optimal moves and critical thresholds for Connect Three on a 3×3 board, assuming Bob makes the first move.

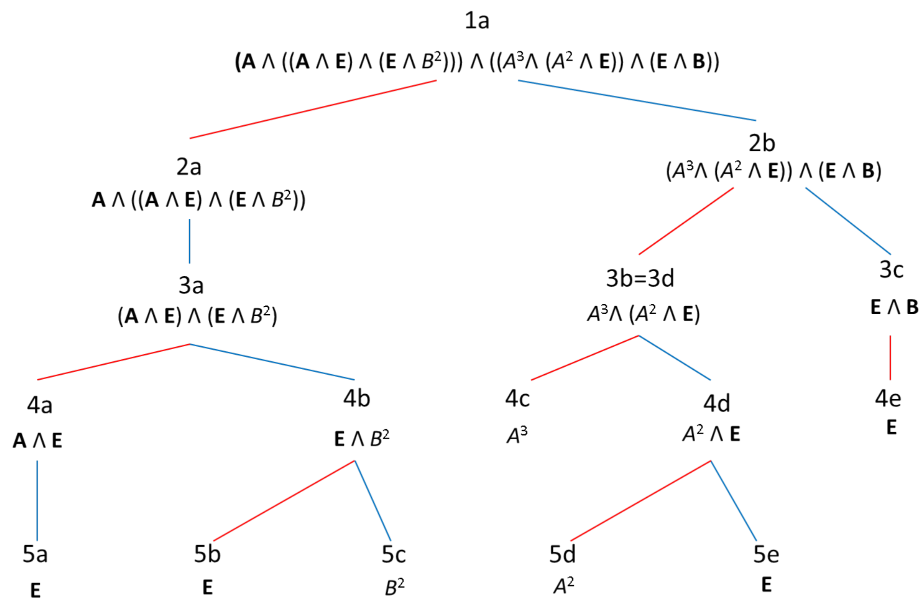

Figure 8: Wedge sums for the positions of Connect Three on a 3×3 board, assuming Alice makes the first move.

equivalent to B^2, and the positions 5d and 5f are equivalent to A^2. Position 4c is equivalent to A^3. Position 5h is $\mathbf{A} \wedge B^3$, 5i is $B^2 \wedge B^3$, and 5j is $\mathbf{E} \wedge B^3$. Figures 8 and 9 show these wedge sums and how they are used to construct the wedge sum for every other position. We analyze the remaining moves going from left to right and from bottom to top.

Bob's move from 4a: If Bob moves in the right column, Alice will still be able to win on the next turn. Thus, the best game that Bob can reach is $A^2 \wedge B^2$, which he achieves by moving in the center.

Alice's move from 4b: If Alice moves in the center column, Bob's distant threat changes to a threat, and the position $A^2 \wedge \mathbf{B}$ is created. The best game that Alice can reach is $\mathbf{E}$, which she does by playing in the right column. Both players can win by taking the center.

Bob's move from 4b: If Bob takes the center bottom square, his distant threat becomes a threat, but Alice can create both a distant threat and a threat on her next turn, making this position $A^2 \wedge \mathbf{B}$. By moving in the right column, Bob creates both a threat and a distant threat, and the best that Alice can do is to create a distant threat of her own. Thus, this position is $\mathbf{E} \wedge \mathbf{B}$, which is strictly better for Bob.

Alice's move from 4d: Taking the center square does not help Alice; since she already has a distant threat that requires the top-right corner, there is no additional benefit to starting to create another threat requiring the same empty square. Thus, Alice's move in the center creates the position $A^2 \wedge \mathbf{E}$. If Alice takes either available corner, she will have both a threat and a distant threat. If the threat is blocked, the game becomes even and depends only on

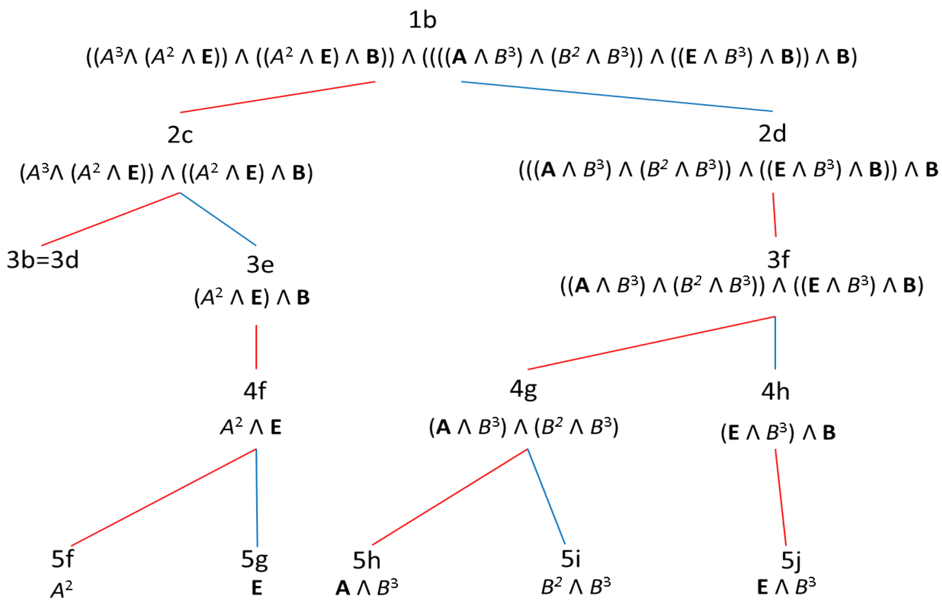

Figure 9: Wedge sums for the positions of Connect Three on a 3×3 board, assuming Bob makes the first move.

who gets the top-right corner. By moving in either corner, Alice creates the position $\mathbf{A} \wedge \mathbf{E}$, which is strictly better for Alice. We let her move in the right column.

Bob's move from 4d: If Bob moves in the center column, Alice can create a threat on the next turn. This move creates the position $\mathbf{A} \wedge \mathbf{E}$. Bob's optimal move is to reach $\mathbf{E}$, which he can do by playing in either the left column or right column. If he does, the game depends on who takes the top-right corner. We let him move in the right column.

Alice's move from 4f: A move in the center column does not help Alice; this move creates the position $A^2 \wedge \mathbf{E}$. If Alice moves in either column, she creates at least a double threat and will win if she gets either of the next two moves. The move in either column creates the position $\mathbf{A} \wedge \mathbf{E}$, which is better for Alice. We let her move in the right column.

Bob's move from 4f: If Bob moves in the center column, Alice can create a double threat on the next turn by moving in the right column. Thus, the position created is $A^2 \wedge \mathbf{E}$. If Bob moves in either the left column or the right column, the game now depends only on which player takes the upper-left corner. Bob has created $\mathbf{E}$, which is better for Bob.

Alice's move from 4g: If Alice moves in the left column, she creates a threat and can win on the next turn. If Alice does not win, Bob will block, and he can win if he gets any of the next three moves. Thus, this position is $\mathbf{A} \wedge B^3$. If Alice moves in the right column instead, her next optimal move is to create a threat by playing in the right column again. From here, Alice can win, or Bob can block, and he will win if he gets either of the next two moves. If instead Bob gets the next move after Alice moves in the right column, he will also move in the right column, and he will prevent Alice from winning if he gets either of the next two moves. Thus, the position created from Alice moving in the right column initially is $(\mathbf{A} \wedge B^2) \wedge B^2$. The right side of the wedge sum is better for Alice, since this side has a critical threshold of 3/4, but the left side is much worse, as this side has a critical threshold of 3/8. Thus, Alice's initial move in the right column is worse because it has a critical threshold of 9/16.

Bob's move from 4g: Suppose that Bob moves in the right column. The position that Bob has created is $\mathbf{E} \wedge B^3$, since Alice can create the even game by playing in the left column on the next turn. If instead Bob moves in the left column, then the best position that Alice can hope to reach is equivalent to $\mathbf{E} \wedge \mathbf{B}$, which she would do by also moving in the left column. This makes position 5i $(\mathbf{E} \wedge \mathbf{B}) \wedge B^3$, which is strictly better for Bob.

Alice's move from 4h: A move for Alice in the left column does not help her, and Bob still has a threat. Thus, the best position that Alice can reach is $\mathbf{E} \wedge B^3$, which she does by moving in the right column.

Alice's move from 3a: Suppose that Alice moves in the center column. Alice has created a threat, but Bob would block on the next turn, creating a threat of his own. Alice would then block, creating a double threat. Thus, Alice's original move in the center column creates the position $\mathbf{A} \wedge (A^2 \wedge \mathbf{B})$.

If Alice moves in the right corner, then Bob would block Alice's threat, creating the game where the player who gets two moves before the other plays. Alice's move in the right corner creates the position $\mathbf{A} \wedge (A^2 \wedge B^2)$, which is equivalent to $\mathbf{A} \wedge \mathbf{E}$ and is better for Alice.

Bobs's move from 3a: If Bob moves in the center column, then Alice can create a double distant threat by taking the center square. Thus, to at least prevent Alice from winning, Bob needs to get the center and top squares in the right column. By initially moving in the right column, Bob creates a distant threat which requires the center square to win the game. The best game that Alice can then reach is the even game, since all of Alice's possibilities for winning also require the center square.

Alice's move from 3b: Alice already has a distant threat that requires the upper right square. If Alice moves into the right column, the position created is equivalent to $A^3 \wedge (\mathbf{A} \wedge \mathbf{E})$. On the next turn, Alice's optimal move into the left column creates two additional threats, or Bob's next optimal move in the right column creates a position from which Alice can win or he can create the even game. If Alice moves into the center column, the position created is equivalent to $A^3 \wedge (\mathbf{A}^2 \wedge \mathbf{E})$. On the next turn, Alice's optimal move into the left column creates another threat and a distant threat, or Bob's move into the left column creates a position from which Alice can move to win in either the lower or upper right squares or Bob can move to create the even game. This position is worse for Alice. However, if Alice moves into the left column directly, she creates a threat and an additional distant threat. Even if Bob blocks in the left column, he still needs to get the two highest squares in the right column to prevent Alice from winning. Thus, the position created is $\mathbf{A} \wedge (\mathbf{A} \wedge \mathbf{E})$, which is equivalent to A^3 and is best for Alice.

Bob's move from 3b: Again, we note that Alice already has a distant threat in the upper right square. If Bob moves into the right column, then Alice can create a double threat by moving in the left column on the next turn, or Bob can create the even game by moving in the right column on the next turn. The position created is $A^2 \wedge \mathbf{E}$. If Bob moves in the center column, then Alice can create a threat and an additional distant threat by taking the left column on the next turn. This position is equivalent to A^3 and is worse for Bob. If Bob moves in the left column, then Alice can create a threat by moving in the left column on the next turn, or Bob can create the even game by moving in the left column again. The position created is $A^2 \wedge \mathbf{E}$, which is the same as if Bob had moved in the right column. Since these positions are equivalent, we let Bob move in the left column.

Alice's move from 3c: If Alice moves in the left column, she creates the position $\mathbf{E}$. We show that she can do no better moving elsewhere.

Suppose Alice moves in the center column to block Bob's threat. If Alice plays after her block, she can create her own threat by moving in the left column. If Alice doesn't win after moving in the left column, then Bob will block, and Alice will need to take the entire right column to win. If, instead, Bob plays after Alice's block, he would move in the left column. Then if Alice moves next, wherever she moves, she will need the two upper squares in the right column to win, and if Bob moves next, wherever he moves, Alice will need the next three moves to win. Thus, if Alice initially blocks, the position is $(\mathbf{A} \wedge B^3) \wedge (B^2 \wedge B^3)$, which is worse for Alice.

Suppose Alice instead moves into the right column. Bob can still win on the next turn. In this situation, Alice's optimal next move is not to block but rather to create her own threat. After getting the initial move in the right column, Alice will still need to get the next two moves in order to win. Thus, the position created is B^2, which is also worse for Alice.

Alice's move from 3e: Alice's optimal move in this game is to block, creating the position $A^2 \wedge \mathbf{E}$. We show that she can do no better moving elsewhere.

Suppose Alice moves in the left column. Bob still has a threat, and her optimal next move is to block. Alice can now win by moving in either the upper left corner or the center square in the right column and will lose otherwise. Thus, by moving in the left column initially, Alice creates the position $A^2 \wedge \mathbf{B}$, which is worse for Alice.

Suppose now that Alice moves into the center column. Bob still has a threat, and her optimal next move is to block. If Alice plays again after her block, she will play in the right column again and will win if she gets either of the next two turns. If, instead, Bob plays after Alice's block, he will play in the left column, creating the even game. Thus, Alice's position by moving initially in the center column is $(A^2 \wedge \mathbf{E}) \wedge \mathbf{B}$, which is also worse for Alice.

Bob's move from 2a: If Bob moves into the left column and blocks Alice's threat, the game becomes $(\mathbf{A} \wedge \mathbf{E}) \wedge (\mathbf{E} \wedge B^2)$. We show that Bob can do no better by moving elsewhere.

Suppose that Bob moves into the center column. Alice still has a threat. Bob's optimal move from this position is now to block, and the optimal moves of both players are now in the center square. If Alice takes the center, then she wins if she takes either of the two highest squares in the right column. If Bob takes the center square, he can prevent Alice from winning by taking any square in the right column. Thus, if Bob moves in the center column instead of blocking, he creates the position $\mathbf{A} \wedge (A^2 \wedge B^3)$, which is worse for Bob.

Suppose now that Bob moves into the right column. Alice still has a threat, and Bob's optimal next move is to block. We have now reached position 4b. Thus, if Bob moves in the right column instead of blocking, he creates the position $\mathbf{A} \wedge (\mathbf{E} \wedge B^2)$. The right side of the wedge sum is just as good for Bob, but the left side is better for Alice.

Alice's move from 2b: If Alice moves in the center, the game becomes $A^3 \wedge (A^2 \wedge \mathbf{E})$, which has a critical threshold of 1/4. We show that Alice can do no better by moving elsewhere.

Suppose that Alice moves in the left column and creates a threat. Bob's optimal move is to block. The optimal moves of both players are now in the center square. If Alice takes the center, then she wins if she takes either of the two highest squares in the right column. If Bob takes the center square, he can prevent Alice from winning by taking any square in the right column. Thus, by moving in the left column, Alice creates the position $\mathbf{A} \wedge (A^2 \wedge B^3)$. Although the left side is better for Alice, the right side of the wedge sum is much better for Bob, making the entire position worse for Alice. We verify this by calculating the critical threshold of this position. Using the table in section 3 and equation (1), the critical threshold is

$$R = \frac{0 + \frac{\frac{1}{4} + \frac{7}{8}}{2}}{2} = \frac{9}{32},$$

which is larger than 1/4.

Suppose now that Alice moves in the right column. Both players now have an optimal next move of taking the center. Suppose that Alice takes the center. Her next move is to take the left column; from this position, she can win if she gets any of the next three moves. Bob's next move after Alice takes the center is also in the left column, a position from which Alice can win if she gets the upper left or upper right squares. Thus, if Alice moves first, she creates $A^3 \wedge A^2$. Suppose now that Bob takes the center square. Bob wins if he gets either of the next two moves because Alice's optimal next moves are to fill the right column and win (see Alice's move from 3c). Thus, if Bob moves first, he creates B^2. Thus, the position created when Alice moves in the right column is $(A^3 \wedge A^2) \wedge B^2$, which is worse for Alice.

Bob's move from 2b: If Bob moves in the center, he creates $\mathbf{E} \wedge \mathbf{B}$. We must show that he cannot do better than this with a move elsewhere.

Suppose Bob moves in the left column. Then both players have an optimal next move in the center square. If Alice moves in the center column, she creates position 4d, which has wedge sum $A^2 \wedge \mathbf{E}$. If Bob moves in the center column, he has a threat. If Alice gets the next move, she moves in the right column to create B^2. Thus, we have $B^2 \wedge \mathbf{B}$, which is B^3. Thus, by moving in the left column, Bob creates the position $(A^2 \wedge \mathbf{E}) \wedge B^3$, which is worse for Bob.

Suppose now that Bob plays in the right column. If Alice moves next, she will create a threat in the left column. Bob will block to create his own threat in the center, which Alice would then block to create a double threat. Thus, Alice's move creates $\mathbf{A} \wedge (A^2 \wedge \mathbf{B})$. If instead Bob moves after his initial move in the right column, he will take the center, and he can win if he gets either of the next two moves, since Alice's next move will be to create her own threat in the left column (see Alice's move from 3c). Thus, Bob's move creates B^2. Putting these results together, Bob's initial move in the right column creates the position $(\mathbf{A} \wedge (A^2 \wedge \mathbf{B})) \wedge B^2$. The left side, which has a critical threshold of 5/16, is better for Alice than $\mathbf{E}$, and the right side, which has a critical threshold of 3/4, is worse for Bob than $\mathbf{B}$. Thus, $(\mathbf{A} \wedge (A^2 \wedge \mathbf{B})) \wedge B^2$, which has a critical threshold of 17/32, is worse for Bob than $\mathbf{E} \wedge \mathbf{B}$.

Alice's move from 2c: If Alice moves in the left column, she creates the position $A^3 \wedge (A^2 \wedge \mathbf{E})$. We show that she can do no better by moving in the center column.

If Alice does move in the center column, the next player to move is forced to play in either the left or right column. Since the board is symmetric, we choose the left column. Suppose that Alice moves next and creates a distant threat in the upper right corner. If she gets another move, then she will play in the left column, allowing her to reach a position from which she wins if she gets the upper left square, the upper right square, or the right center square. If Bob moves after Alice's first move in the left column, he will also play in the left column to reach a position from which Alice can play to create another threat in addition to the distant threat or he can play to create the even game. Thus, Alice's initial move in the left column creates $A^3 \wedge (A^2 \wedge \mathbf{E})$. Suppose instead that Bob moves after Alice's move in the center and creates a threat. Alice would then block to create a position from which she can create a double threat or Bob can create the even game. Thus, Bob's move in the left column creates the position $(A^2 \wedge \mathbf{E}) \wedge \mathbf{B}$. Thus, Alice's initial move in the center creates $(A^3 \wedge (A^2 \wedge \mathbf{E})) \wedge ((A^2 \wedge \mathbf{E}) \wedge \mathbf{B})$, which is worse for Alice.

Bob's move from 2c: If Bob moves in the left corner, he reaches the position $(A^2 \wedge \mathbf{E}) \wedge \mathbf{B}$. We show that he cannot do better by moving in the center column.

If Bob does move in the center column, the next player is forced to take the left corner (or right, by symmetry). Since Bob does not create a threat by moving in the center column, it suffices to show that Alice's move in the left corner, which creates a distant threat, is better for Alice than $A^2 \wedge \mathbf{E}$. If Alice moves and then moves again in the left column, she can win if she gets the upper left square, the upper right square, or the right center square. This position is A^3. If Bob moves after Alice's move in the left corner, he will also play in the left column. Bob's move creates a position from which Alice can create another threat in addition to her distant threat or Bob can create the even game. Thus, this position is $A^2 \wedge \mathbf{E}$. As a result, the wedge sum of Alice's taking the corner is $A^3 \wedge (A^2 \wedge \mathbf{E})$, which is better for Alice than $A^2 \wedge \mathbf{E}$.

Alice's move from 2d: If Alice moves in the center column, the resulting position is $((\mathbf{A} \wedge B^3) \wedge (B^2 \wedge B^3)) \wedge ((\mathbf{E} \wedge B^3) \wedge \mathbf{B})$, which has a critical threshold of 111/128. We show that she cannot do better by moving in the left column (or right, by symmetry).

If Alice does move in the left column, she reaches position 3c, which has a wedge sum of $\mathbf{E} \wedge \mathbf{B}$. The left side of the wedge sum is a better position for Alice, but the right side is much worse for Alice. Using the table in section 3 and equation (1), the critical threshold of this position is 3/4, which is larger than 111/128. This is worse for Alice.

Alice's move from 1a: If Alice moves in the left column, the position has a wedge sum of $\mathbf{A} \wedge ((\mathbf{A} \wedge \mathbf{E}) \wedge (\mathbf{E} \wedge B^2))$. We show that Alice cannot do any better by moving elsewhere. Since Alice will be able to create a threat in both cases, it suffices to show that Bob's next move is better for him than $(\mathbf{A} \wedge \mathbf{E}) \wedge (\mathbf{E} \wedge B^2)$.

Suppose Alice moves in the right column. Bob's optimal move is to block. Both players will then want to take the center. This position has already been analyzed (see Alice's move from 2b). If Alice takes the center, she reaches $A^3 \wedge A^2$, and if Bob takes the center, he reaches B^2. Thus, Bob's move after Alice's move in the right column takes him to $(A^3 \wedge A^2) \wedge B^2$. This is better for him.

Suppose Alice moves in the center. Again, Bob's optimal move is to block. Both players will again want to take the center. If Alice takes the center, she will win if she can get the next turn. Bob's optimal move is to play in the right column and create the even game. Thus, the position if Alice takes the center is $\mathbf{A} \wedge \mathbf{E}$. If Bob takes the center square, then Alice will create the even game on the next turn, or Bob will move in the left column to create a position from which he prevents Alice from winning if he gets any of the next four moves. Thus, Bob's move in the center takes the game to position $\mathbf{E} \wedge B^4$. Thus, Bob's original block takes him to $(\mathbf{A} \wedge \mathbf{E}) \wedge (\mathbf{E} \wedge B^4)$, which is better for him as well.

Bob's move from 1a: If Bob moves in the center column, he moves to position $(A^3 \wedge (A^2 \wedge \mathbf{E})) \wedge (\mathbf{E} \wedge \mathbf{B})$. We show that Bob can do no better by moving elsewhere.

Suppose Bob moves in the left column. The optimal move for both players is to take the bottom right square. If Alice gets it, she can win on the next turn, so Bob's optimal move is to block. Now, both players want to take the center square. If Alice gets it, the game is equivalent to A^2, and if Bob gets it, the position is B^2. Thus, if Alice takes the bottom right square, we have $\mathbf{A} \wedge (A^2 \wedge B^2)$. This is just as good for Bob as the left side of the original wedge sum, since both this position and $A^3 \wedge (A^2 \wedge \mathbf{E})$ have critical thresholds of 1/4. If instead Bob gets the bottom right square, then if Alice gets the next move, she will move in the center column. If Alice then gets the center square, she will reach position A^2, and if Bob gets the center square, he will reach position B^4. If Bob moves after taking the bottom right square, he will move in the right column again. He will reach the position $\mathbf{E} \wedge \mathbf{B}$. Thus, if Bob gets the bottom right square, the position is $(A^2 \wedge B^4) \wedge (\mathbf{E} \wedge \mathbf{B})$. This is worse for Bob than the right side of the original wedge sum.

Suppose now that Bob moves in the right column. If Alice goes next, she will create a threat in the left column. Bob will block, creating his own distant threat that requires the center square. If Bob goes again, he will play in the right column and create a threat, or if Alice goes, she will also play in the right column, creating a distant threat also requiring the center square. Thus, Alice's move after Bob's creates a position equivalent to $\mathbf{A} \wedge (\mathbf{E} \wedge \mathbf{B})$. If instead Bob goes after initially moving in the right column, he will create a threat in the right column. Alice will block, creating her own distant threat. If Alice goes again, she will play in the left column and create a threat, or if Bob goes, he will also play in the left column, creating a distant threat requiring the center square. Thus, Bob's move after his initial move creates position $(\mathbf{A} \wedge \mathbf{E}) \wedge \mathbf{B}$. The resulting game for Bob's move in the right column is $(\mathbf{A} \wedge (\mathbf{E} \wedge \mathbf{B})) \wedge ((\mathbf{A} \wedge \mathbf{E}) \wedge \mathbf{B})$. Using the table in section 3 and equation (1), both this position and $(A^3 \wedge (A^2 \wedge \mathbf{E})) \wedge (\mathbf{E} \wedge \mathbf{B})$ have critical thresholds of 1/2. Thus, we let Bob move in the center.

Alice's move from 1b: If Alice moves in the center column, the position created is $(A^3 \wedge (A^2 \wedge \mathbf{E})) \wedge ((A^2 \wedge \mathbf{E}) \wedge \mathbf{B})$. We show that she can do no better by moving in either of the side columns. If she moves in either of the side columns, she creates position 2b, which has a wedge sum of $(A^3 \wedge (A^2 \wedge \mathbf{E})) \wedge (\mathbf{E} \wedge \mathbf{B}))$. The left side of the wedge sum is the same for Alice, but the right side is worse. Thus, Alice should move in the center column.

Bob's move from 1b: If Bob moves in the center, the resulting game is $(((\mathbf{A} \wedge B^3) \wedge (B^2 \wedge B^3)) \wedge ((\mathbf{E} \wedge B^3) \wedge \mathbf{B})) \wedge \mathbf{B}$. We show that Bob can do no better by moving in one of the side columns. Since a move in the left column (or right column, by symmetry) creates a threat for Bob, it suffices to show that the left side of the wedge sum is worse for Bob. If Bob moves in the left column, Alice's optimal move is to block. This position has already been analyzed (see Bob's move from 2b). This position is $(\mathbf{A} \wedge (A^2 \wedge \mathbf{B})) \wedge B^2$. Compared to the left side of the wedge sum above, this is worse for Bob.

It is not easy to see that Alice and Bob's first moves are optimal by wedge sums alone. Thus, we use a combination of wedge sum analysis and calculation of cricital thresholds in order to show that the moves presented in Figures 6 and 7 are optimal.

Alice's first move: If Alice's first move is in the left column (or by symmetry, the right column), the wedge sum is $(\mathbf{A} \wedge ((\mathbf{A} \wedge \mathbf{E}) \wedge (\mathbf{E} \wedge B^2))) \wedge ((A^3 \wedge (A^2 \wedge \mathbf{E})) \wedge (\mathbf{E} \wedge \mathbf{B})))$, which has a critical threshold of $R = 23/64$. We must show that Alice can do no better by moving in the center.

If Alice initially moves in the center, then Bob's optimal move is to also move in the center. This position is equivalent to $(\mathbf{A} \wedge (\mathbf{E} \wedge B^4)) \wedge ((\mathbf{E} \wedge B^4) \wedge (B^3 \wedge B^4))$, which has a critical threshold of 75/128. Thus, we must show that the resulting position from Alice's optimal second move is greater than 17/128.

If Alice makes her second move in the left column, the resulting position has a wedge sum of $\mathbf{A} \wedge (A^2 \wedge (\mathbf{E} \wedge B^4))$, which has a critical threshold of 31/128. If Alice makes her second move in the center column, the resulting position is $\mathbf{A} \wedge ((\mathbf{A} \wedge (A^2 \wedge \mathbf{E})) \wedge ((A^2 \wedge \mathbf{E}) \wedge B^2))$, which has a critical threshold of 3/16=24/128. Thus, if Alice's first move is in the center column, then her second move should also be in the center column. However, this position has a critical threshold greater than 17/128. Hence, Alice's first move should be in the left column.

Bob's first move: If Bob's initial move is in the center, the wedge sum is $((A^3 \wedge (A^2 \wedge \mathbf{E})) \wedge ((A^2 \wedge \mathbf{E}) \wedge \mathbf{B})) \wedge ((((\mathbf{A} \wedge B^3) \wedge (B^2 \wedge B^3)) \wedge ((\mathbf{E} \wedge B^3) \wedge \mathbf{B})) \wedge \mathbf{B})$, which has a critical threshold of $R = 171/256$. We show that Bob cannot do better by moving in the left column (or by symmetry, the right column) on the first move.

If Bob's initial move is in the left column, then Alice's optimal move from this position is to move in the right column. This position has already been analyzed (see Bob's move from

1a); the wedge sum is $(\mathbf{A} \wedge (A^2 \wedge \mathbf{B})) \wedge B^2$ and its critical threshold is 1/2. Thus, we need to show that the resulting position from Bob's optimal second move is less than 107/128.

If Bob moves in the left column on his second move, the resulting position has a wedge sum of $((A^2 \wedge \mathbf{E}) \wedge ((\mathbf{E} \wedge B^3) \wedge \mathbf{B})) \wedge \mathbf{B}$, which has a critical threshold of 103/128. If Bob moves in the center column on his second move, the resulting positon has a wedge sum of $((\mathbf{A} \wedge (A^2 \wedge \mathbf{B})) \wedge B^2) \wedge \mathbf{B}$, which has a critical threshold of 49/64=98/128. If Bob moves in the right column on his second move, the resulting position has a wedge sum of $((\mathbf{A} \wedge B^2) \wedge ((B^2 \wedge B^3) \wedge B^4)) \wedge \mathbf{B}$, which has a critical threshold of 52/64=104/128. Thus, if Bob moves in the left column on his first move, then his optimal second move is to move in the right column, but the critical threshold of this position is less than 107/128. Thus, Bob's initial move should be in the center column.

□

From Figures 6 and 7, we determine the critical threshold of Connect Three on a 3×3 board to be $R = 263/512$. Alice needs a smaller proportion of the bidding resources in order to win Connect Three than she does to win Tic-Tac-Toe, as Tic-Tac-Toe had a critical threshold of $R = 133/256 = 266/512$. Connect Three is essentially Tic-Tac-Toe with restricted moves. In Tic-Tac-Toe, nine moves are available at the beginning of the game, and the number of moves decreases because players are claiming squares. In Connect Three, however, only three moves are available at the beginning of the game due to gravity. The number of moves available decreases when an entire column has been claimed.

We can make several observations from the analysis of the optimal moves of Connect Three. We notice that in almost every case in which Alice can create a threat, she will create a threat. However, Alice's move from 2b is an exception. One might think that Alice would play in the left column to create a threat. However, Alice plays in the center, creating a distant threat that requires the upper right corner. This leads us to believe that there may be something special about the center square of the 3×3 board. Indeed, in every case when the center becomes available, the optimal move for both Alice and Bob is to take the center on the next turn, unless the opponent has a threat.

Based on the tree of optimal moves, we observe that, except in case 2b, if Alice can create a threat, she should do so on her next turn. Otherwise, if Bob has a threat, she should block his threat. For example, we see that Alice's move from 3c is to create a threat of her own and create **E**, even though Bob can win in his move from 3c. However, in Alice's move from 2d, which is the same as 3c except that Alice does not have a claim to the lower left corner, Alice blocks Bob's threat.

In addition, we can also make an observation about when Bob should block. In every position in the tree where Alice has a threat, Bob blocks that threat on the next move. Thus, Bob should block unless he can win on the next turn because a draw in the game is also considered a win for Bob. Bob does not necessarily have to connect three pieces in order to win; he can also win by preventing Alice from winning.

Finally, we create a chip table for Connect Three. We work backwards from the end positions using equation (2) and the periodicity results in section 2. From the ending positions, chip tables were created for positions further up the tree.

$f(\text{Connect Three}, 512n+) = 263n+$

$512n+$	+0	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11
0+	0	1	1*	2	2*	3	4	4	4*	5*	5*	6*
12+	6*	7	7*	8*	8*	9*	10	10	11	11*	11*	12*
24+	13	14	14	14	14*	15*	16	16*	17	18	18*	18*
36+	18*	19*	20	21	21	21*	22	22*	23	24	24	25
48+	25	25*	26	27	27	28	28*	28*	29*	30	30	31
60+	31*	32*	32*	32*	33	34	34*	35	35*	36	37	37
72+	37	38	38*	39*	39*	40	40*	41	41*	42*	42*	43
84+	44	44	44*	45	45*	46*	47	47	47*	48*	48*	49*
96+	49*	50*	51	51*	51*	52*	53	53*	54	54*	55	55*
108+	55*	56*	57	58	58	58*	59	60	60	61	61	61*
120+	62*	63	63	63*	64	65	65*	65*	66	67	67*	68
132+	68	68*	69*	70	70	71	71	72	72*	73	73	74
144+	74*	75*	75*	76	76*	77	77*	78	78*	79*	79*	80
156+	80*	81*	81*	82*	82*	83*	84	84	84*	85*	86	86*
168+	87	87	88	88*	88*	89*	90	90*	91	91*	91*	92*
180+	93	94	94	94	95	95*	96	96*	97	98	98*	98*
192+	98*	99*	100	101	101	101*	102*	102*	103	104	104	105
204+	105*	106	106	107	107	108	108*	109	109*	110	110	111
216+	111*	112*	112*	112*	113	114	114*	115	115*	116*	117	117
228+	117	118	118*	119*	119*	120	121	121	121*	122*	122*	123*
240+	124	124*	124*	125*	125*	126*	127	127	128	128*	129	129*
252+	130	131	131	131*	131*	132*	133	133*	134	134*	135*	135*
264+	136	137	137	138	138	138*	139	140	140	141	141*	141*
276+	142*	143	143	144	144*	145*	145*	145*	146	147	147*	148
288+	148*	149*	150	150	150	151	151*	152*	152*	153	153*	154
300+	154*	155*	155*	156*	156*	157	157*	158*	158*	159*	160	160
312+	161	161*	161*	162*	163	164	164	164	164*	165*	166	166*
324+	167	167*	168*	168*	168*	169*	170	171	171	171*	172	172*
336+	173	174	174	174*	175*	175*	176	176*	177	178	178*	178*
348+	179	180	180	181	181	182	182*	183	183	184	184*	185
360+	185*	186	186*	187	187	188	188*	189*	189*	190	190*	191*
372+	191*	192*	192*	193	194	194*	194*	195	195*	196*	197	197
384+	197*	198*	199	199*	199*	200	201	201*	201*	202*	202*	203*
396+	204	204*	204*	205*	206	207	207	207*	208	208*	209	209*
408+	210	211	211	211*	212	213	213	214	214	215	215*	215*
420+	216	217	217*	218	218*	218*	219*	220	220	221	221*	222
432+	222*	223	223	224	224*	225*	225*	225*	226*	227	227*	228
444+	228*	229*	230	230	230	231	231*	232*	232*	233	234	234
456+	234*	235*	235*	236*	237	237*	237*	238*	238*	239*	240	240*

$512n+$	+0	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11
468+	241	241*	241*	242*	243	244	244	244	244*	245*	246	246*
480+	247	248	248*	248*	248*	249*	250	251	251	251*	252*	252*
492+	253	254	254	255	255*	256	256	257	257	258	258*	258*
504+	259*	260	260*	261	261*	262*	262*	263				

As an example, we calculate the entry in the table for position 3a for 4 total chips. This entry is 2. Using Figure 6 as a reference, we see that we need a value from position 4a and from position 4b to calculate the entry in 3a. Position 4a is equivalent to A^2; using the table in section 5 and the periodicity result, this value is 1. To calculate an entry from postition 4b, we need a value from position 5b and from position 5c. Position 5b is equivalent to **E**. Using the table in section 4 and the periodicity result, this value is 2^*. Position 5c is equivalent to B^2; using the table in section 2 and the periodicity result, this value is 4. Using equation (2) for the values from positions 5b and 5c, the value for position 4b is 3^*. Using equation (2) again for the values from positions 4a and 4b, the entry for position 3a for 4 total chips is 2, as desired.

Below are the chip tables used to calculate the chip table for Connect Three.

From position 5h, we have $f(\text{Connect Three}_{5h}, 16n+) = 7n + .$

$16n+$	+0	+1	+2	+3	+4	+5	+6	+7
0+	0*	1	1*	1*	2	2*	3	3*
8+	4	4*	5	5	5*	6*	6*	7

From position 5i, we have $f(\text{Connect Three}_{5i}, 16n+) = 13n + .$

$16n+$	+0	+1	+2	+3	+4	+5	+6	+7
0+	1	2	3	3	4	5	6	6*
8+	7*	8*	9*	9*	10*	11*	12*	13

From position 5j, we have $f(\text{Connect Three}_{5j}, 16n+) = 11n + .$

$16n+$	+0	+1	+2	+3	+4	+5	+6	+7
0+	1	1*	2*	2*	3*	4	5	5*
8+	6*	7	8	8	9	9*	10*	11

From position 4b, we have $f(\text{Connect Three}_{4b}, 8n+) = 5n + .$

$8n+$	+0	+1	+2	+3
0+	1	1	2	2*
4+	3*	3*	4*	5

From position 4c, we have $f(\text{Connect Three}_{4c}, 8n+) = n + .$

$8n+$	+0	+1	+2	+3
0+	0	0	0	0*
4+	0*	0*	0*	1

From position 4d (or 3e), we have $f(\text{Connect Three}_{4d}, 8n+) = 3n + .$

$8n+$	+0	+1	+2	+3
0+	0	1	1	1*
4+	1*	2*	2*	3

From position 4g, we have $f(\text{Connect Three}_{4g}, 16n+) = 10n + .$

$16n+$	+0	+1	+2	+3	+4	+5	+6	+7
0+	1	1*	2*	2*	3	4	4*	5
8+	5*	6*	7	7	8	9	9*	10

From position 4h, we have $f(\text{Connect Three}_{4h}, 32n+) = 27n+$

$32n+$	+0	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11
0+	1	2	3	3*	4*	5	6	7	8	8*	9*	10
12+	11	12	13	13*	14*	15*	16*	17	18	18*	19*	20*
24+	21*	22	23	23*	24*	25*	26*	27				

From position 3a, we have $f(\text{Connect Three}_{3a}, 16n+) = 7n + .$

$16n+$	+0	+1	+2	+3	+4	+5	+6	+7
0+	0*	1	1*	1*	2	2*	3	3*
8+	4	4*	5	5	5*	6	6*	7

From position 3e, we have $f(\text{Connect Three}_{3e}, 16n+) = 11n + .$

$16n+$	+0	+1	+2	+3	+4	+5	+6	+7
0+	0*	1*	2	3	3*	4*	5	5*
8+	6	7	7*	8*	9	10	10*	11

From position 3f, we have $f(\text{Connect Three}_{3f}, 64n+) = 47n + .$

$64n+$	+0	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11
0+	1	2	3	3	3*	4*	5*	6	7	7*	8	9*
12+	9*	10*	11*	11*	12*	13*	14*	15	15*	16	17	17*
24+	18*	19*	20	20	21	22	23	23*	24*	25*	26*	26*
36+	27	28	29	29*	30*	31	31*	32	33	34	35	35
48+	36	37	38	38*	39	39*	40*	41	42	43	43*	43*
60+	44*	45*	46*	47								

From position 2a, we have $f(\text{Connect Three}_{2a}, 32n+) = 7n+$.

$32n+$	+0	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11
0+	0	0*	0*	0*	1	1	1*	1*	2	2	2*	2*
12+	2*	3	3	3*	3*	4	4	4	4*	4*	5	5
24+	5*	5*	6	6	6	6*	6*	7				

From position 2c, we have $f(\text{Connect Three}_{2c}, 32n+) = 15n+$.

$32n+$	+0	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11
0+	0	1	1*	2	2	3	3*	3*	4	5	5	5*
12+	6	7	7	7*	7*	8*	9	9*	9*	10*	11	11
24+	11*	12*	12*	13	13*	14*	14*	15				

From position 2d, we have $f(\text{Connect Three}_{2d}, 128n+) = 111n+$.

$128n+$	+0	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11
0+	1	2	3	3*	4*	5*	6*	7	8	9	9*	10*
12+	11*	12*	13*	14	15	16	17	17*	18*	19	20	21
24+	22	23	23*	24	25	26	27	28	29	30	31	31*
36+	32	33	34	35	36	36*	37*	38	39	40	41	41*
48+	42*	43*	44*	45*	46	47	48	48*	49*	50*	51*	52
60+	53	54	55	55*	56*	57*	58*	59	60	61	62	62*
72+	63*	64*	65	66	67	68	69	69*	70*	71*	72*	73
84+	74	74*	75*	76*	77*	78*	79	79*	80*	81*	82*	83*
96+	84*	85*	86*	87	87*	88*	89*	90*	91*	92	93	93*
108+	94*	95*	96*	97	98	99	100	101	101*	102*	103*	104
120+	105	106	107	107*	108*	109*	110*	111				

From position 1a, we have $f(\text{Connect Three}_{1a}, 64n+) = 23n+$.

$64n+$	+0	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11
0+	0	1	1	1*	1*	2	2*	3	3	3*	4	4*
12+	4*	5	5	6	6	6*	6*	7	7*	8	8	8*
24+	9	9*	9*	10	10	11	11	11*	11*	12*	12*	13
36+	13	13*	14	14*	14*	15	15*	16	16	16*	16*	17*
48+	17*	18	18	18*	19	19*	19*	20	20*	21	21	21*
60+	21*	22*	22*	23								

From position 1b, we have $f(\text{Connect Three}_{1b}, 256n+) = 171n+$.

$256n+$	+0	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11
0+	0*	1*	2*	2*	3	4	5	5*	6	7	7	8
12+	8*	9*	10	11	11*	12*	13	13*	14	15	15*	16
24+	17	18	18	18*	19*	20*	21	21*	22	23	24	24
36+	24*	25*	26*	27	27*	28	28*	29*	30	31	31*	32
48+	32*	33*	34	35	35*	36*	37	37	38	39	39*	40
60+	41	42	42*	42*	43	44	45	45*	46	47	48	48
72+	48*	49*	50	51	51*	52*	53	53*	54	55	55*	56*
84+	57	57*	58	58*	59*	60*	61	61	62	63	63*	64
96+	64*	65*	66*	67	67	68	69	69*	70	71	71*	72
108+	72*	73*	74	75	75*	76*	77	78	78	79	79*	80
120+	81	82	82*	82*	83*	84*	85	85*	86	87	88	88
132+	88*	89*	90*	91	91*	92*	92*	93*	94	95	95*	96*
144+	97	98	98*	99	99*	100*	101	101*	102*	103*	103*	104
156+	105	106	106*	107	107*	108*	109*	109*	110	111	112	112*
168+	113	113*	114	115	115*	116*	117	117*	118	119	119*	120*
180+	121	122	122*	122*	123*	124*	125	125*	126*	127*	128	128
192+	128*	129*	130*	131	131*	132*	133*	133*	134	135	135*	136*
204+	137	138	138*	139	139*	140*	141	142	142*	143	143*	144
216+	145	146	146*	146*	147*	148*	149	149*	150	151	152	152*
228+	152*	153*	154*	155	155*	156*	157	157*	158	159	159*	160*
240+	161	162	162*	163*	163*	164*	165	165*	166*	167*	168	168
252+	169	170	170*	171								

7 Conclusions

In this paper, we have discussed and analyzed bidding games in the style of Connect Four. We started with Connect n on a $1 \times n$ board. We determined the critical thresholds and the trees of optimal moves for Connect Two on boards of various sizes and for Connect Three on a 3×3 board. We also constructed the chip table for Connect Three on a 3×3 board. We found that the corner is the optimal first move for Alice in real-valued bidding. However, in discrete bidding, whether the corner or the center is an optimal first move for Alice may depend on the total number of chips. Thus, we would need to construct the chip table for Connect Three assuming that Alice's first move is in the center and do a cell-by-cell comparison between the two tables.

There are other obvious future possible extensions for this project. We would like to analyze Connect Four on a 4×4 board and beyond to the standard-size board. We hope that some of the conclusions drawn from Connect Three will carry over to bidding Connect Four. For example, we want to know if Alice will still have the same strategy for blocking;

that is, we want to know if she should create a threat if she can and otherwise block. In addition, we want to determine if Bob should always block unless he can win on the next turn. Finally, we wish to be able to determine if the inner squares have more value than the outer squares. In Connect Three, we found that the center square should be taken as soon as it is available. However, larger boards may not have one center square, but rather several central squares. We wish to know how much more desirable these squares are than the outer squares and whether any of these central squares are the most valuable.

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