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ON THE DEGREE-CHROMATIC  
POLYNOMIAL OF A TREE

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# ON THE DEGREE-CHROMATIC POLYNOMIAL OF A TREE

Diego Cifuentes

**Abstract.** The degree chromatic polynomial  $P_m(G, k)$  of a graph  $G$  counts the number of  $k$ -colorings in which no vertex has  $m$  adjacent vertices of its same color. We prove Humpert and Martin's conjecture on the leading terms of the degree chromatic polynomial of a tree.

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## 1 Introduction

George David Birkhoff defined the chromatic polynomial of a graph to attack the renowned four color problem. The chromatic polynomial  $P(G, k)$  counts the  $k$ -colorings of a graph  $G$  in which no two adjacent vertices have the same color [3].

Given a graph  $G$ , Humpert and Martin defined its  $m$ -chromatic polynomial  $P_m(G, k)$  to be the number of  $k$ -colorings of  $G$  such that no vertex has  $m$  adjacent vertices of its same color. They proved this is indeed a polynomial. When  $m = 1$ , we recover the usual chromatic polynomial of the graph  $P(G, k)$ .

The chromatic polynomial is of the form

$$P(G, k) = k^n - ek^{n-1} + o(k^{n-1})$$

where  $n$  is the number of vertices and  $e$  the number of edges of  $G$ . For  $m > 1$  the formula is no longer true, but Humpert and Martin conjectured the following formula when the graph is a tree  $T$ :

$$P_m(T, k) = k^n - \sum_{v \in V(T)} \binom{d(v)}{m} k^{n-m} + o(k^{n-m}) \quad (1)$$

where  $d(v)$  is the degree of  $v$ . Note that (1) is not true for  $m = 1$  —we will see why in the course of proving Theorem 1.

The goal of this paper is to prove this conjecture in Theorem 1. In section 2 we discuss the basic concepts required to understand the theorem, while in section 3 we provide the proof.

## 2 Background

A *finite graph*  $G$  is an ordered pair  $(V, E)$ , where  $V$  is a finite set of *vertices* and  $E$  is a set of *edges*, which are 2-element subsets of  $V$ .

Figure 1 shows the graphic representation of graph.

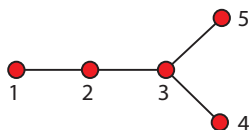


Figure 1: Graphic representation of a graph with  $V = \{1, 2, 3, 4, 5\}$  and  $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{3, 5\}\}$ .

We now present some basic definitions of graph theory.

**Definition 1.** The *degree* of a vertex  $v$  is the number of edges which contain  $v$ , and is denoted as  $d(v)$ . Two vertices  $p, q \in V$  are said to be *adjacent* if the pair  $\{p, q\} \in E$ . A *path* is a sequence of vertices  $v_0, v_1, \dots, v_k$  where  $v_i$  is adjacent to  $v_{i+1}$  for  $0 \leq i \leq k - 1$ . A *cycle* is a path  $v_0, \dots, v_k$  with  $v_0 = v_k$ . A graph is *connected* if for any pair of vertices there exists a path containing both of them. A *tree* is a connected graph with no cycles.

It is easy to see that the graph in Figure 1 is actually a tree.

A *coloring* of a graph is an assignment of colors to each of its vertices. If  $\sigma$  is a coloring, we denote by  $\sigma(v)$  the color assigned to the vertex  $v$ . A  $k$ -coloring is one in which  $\sigma(v) \in \{1, 2, \dots, k\}$  for all  $v$ , i.e. we may use at most  $k$  different colors. A graph with  $n$  vertices clearly has  $k^n$  different  $k$ -colorings, as each of its  $n$  vertices has  $k$  possible choices for its color.

A coloring is called *proper* if there is no edge connecting any two identically colored vertices. Figure 2 shows all of these colorings with  $k = 3$  for a 3-vertex tree.

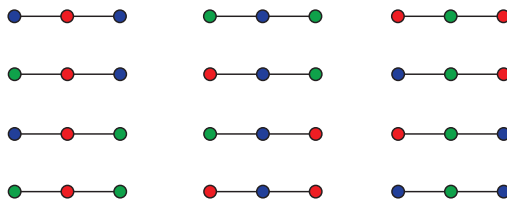


Figure 2: Proper 3-colorings of a tree with 3 vertices.

The *chromatic polynomial* of a graph  $P(G, k)$  counts the proper  $k$ -colorings of  $G$ . It is well-known to be a monic polynomial in  $k$  of degree  $n$ , the number of vertices.

**Example 1.** The chromatic polynomial of a tree  $T$  with  $n$  vertices is  $P(T, k) = k(k - 1)^{n-1}$ . To prove this, fix an initial vertex  $v_0$ . There are  $k$  possible choices for its color  $\sigma(v_0)$ . Then, consider a vertex  $v_1$  adjacent to  $v_0$ . There are  $k - 1$  ways to choose  $\sigma(v_1)$ , as it has to be different from  $\sigma(v_0)$ . Now, consider a vertex  $v_2$  adjacent to  $v_0$  or to  $v_1$ . Notice it cannot be adjacent to both of them, or there would be cycle. Thus, there are also  $k - 1$  possible choices for  $\sigma(v_2)$ . If we repeat this algorithm, we will always have a vertex adjacent to exactly one of the previously colored vertices, so it can be colored in  $k - 1$  ways. The result follows after repeating this procedure  $n - 1$  times.

### 3 Results

Now, we prove the conjecture stated by Humpert and Martin.

**Theorem 1** ([1, 2], Conjecture). *Let  $T$  be a tree with  $n$  vertices and let  $m$  be an integer with  $1 < m < n$ . Then the equation (1) holds, where  $P_m(G, k)$  counts the number of  $k$ -colorings of  $T$  in which no vertex has  $m$  adjacent vertices of its same color.*

*Proof.* For a given coloring of  $T$ , say vertices  $v_1$  and  $v_2$  are “friends” if they are adjacent and have the same color. For each  $v$ , let  $A_v$  be the set of colorings such that  $v$  has at least  $m$  friends. We want to find the number of colorings which are not in any  $A_v$ , and we will use the inclusion-exclusion principle. As the total number of  $k$ -colorings is  $k^n$ , we have

$$P_m(T, k) = k^n - \sum_{v \in V} |A_v| + \sum_{v_1, v_2 \in V} |A_{v_1} \cap A_{v_2}| - \dots$$

We first show that  $|A_v| = \binom{d(v)}{m} k^{n-m} + o(k^{n-m})$ . Let  $A_v^{(l)}$  be the set of  $k$ -colorings such that  $v$  has exactly  $l$  friends. In order to obtain a coloring in  $A_v^{(l)}$ , we may choose the  $l$  friends in  $\binom{d(v)}{l}$  ways, the color of  $v$  and its friends in  $k$  ways, the color of the remaining adjacent vertices to  $v$  in  $(k-1)^{d(v)-l}$  ways, and the color of the rest of the vertices in  $k^{n-1-d(v)}$  ways. Then

$$\begin{aligned} |A_v| &= \sum_{l=m}^{n-1} |A_v^{(l)}| = \sum_{l=m}^{n-1} \binom{d(v)}{l} k^{n-d(v)} (k-1)^{d(v)-l} \\ &= \binom{d(v)}{m} k^{n-m} + o(k^{n-m}). \end{aligned}$$

To complete the proof, it is sufficient to see that for any set  $S$  of at least 2 vertices  $|\bigcap_{v \in S} A_v| = o(k^{n-m})$ ; clearly we may assume  $S = \{v_1, v_2\}$ . Consider the following cases:

*Case 1* ( $v_1$  and  $v_2$  are not adjacent). Split  $A_{v_1}$  into equivalence classes with the equivalence relation

$$\sigma_1 \sim \sigma_2 \Leftrightarrow \sigma_1(w) = \sigma_2(w) \text{ for all } w \neq v_2.$$

Note that each equivalence class  $C$  consists of  $k$  colorings, which only differ in the color of  $v_2$ . In addition, for each  $C$  at most  $\frac{d(v_2)}{m}$  of its colorings are in  $A_{v_2}$ , as if  $\sigma \in A_{v_2}$  there must be  $m$  vertices adjacent to  $v_2$  with the color  $\sigma(v_2)$ . Therefore

$$|A_{v_1} \cap A_{v_2}| = \sum_C |C \cap A_{v_2}| \leq \sum_C \frac{d(v_2)}{m} = \frac{|A_{v_1}|}{k} \cdot \frac{d(v_2)}{m}.$$

It follows that  $\frac{|A_{v_1} \cap A_{v_2}|}{|A_{v_1}|}$  goes to 0 as  $k$  goes to infinity, so  $|A_{v_1} \cap A_{v_2}| = o(k^{n-m})$ .

*Case 2* ( $v_1$  and  $v_2$  are adjacent). Let  $W$  be the set of adjacent vertices to  $v_2$  other than  $v_1$ . They are not adjacent to  $v_1$  as  $T$  has no cycles. Split  $A_{v_1}$  into equivalence classes with the equivalence relation

$$\sigma_1 \sim \sigma_2 \Leftrightarrow \sigma_1(w) = \sigma_2(w) \text{ for all } w \notin W.$$

Each equivalence class  $C$  consists of  $k^{|W|}$  colorings, which may only differ in the colors of the vertices in  $W$ . If  $v_1$  and  $v_2$  are friends in the colorings of  $C$ , then a coloring in  $|C \cap A_{v_2}|$  must contain at least  $m-1$  vertices in  $W$  of the same color as  $v_2$ . Therefore

$$|C \cap A_{v_2}| = \sum_{l=m-1}^{|W|} \binom{|W|}{l} (k-1)^{|W|-l} < \sum_{l=0}^{|W|} \binom{|W|}{l} k^{|W|-1} = 2^{|W|} k^{|W|-1}.$$

Notice that here we are using  $m \geq 2$  so that  $l \geq 1$ . Otherwise, if  $v_1$  and  $v_2$  are not friends in the colorings of  $C$ , then

$$|C \cap A_{v_2}| = \sum_{l=m}^{|W|} \binom{|W|}{l} (k-1)^{|W|-l} < \sum_{l=0}^{|W|} \binom{|W|}{l} k^{|W|-1} = 2^{|W|} k^{|W|-1}.$$

Therefore

$$\begin{aligned} |A_{v_1} \cap A_{v_2}| &= \sum_C |C \cap A_{v_2}| < \sum_C 2^{|W|} k^{|W|-1} \\ &= \frac{|A_{v_1}|}{k^{|W|}} \cdot 2^{|W|} k^{|W|-1} = \frac{|A_{v_1}| \cdot 2^{|W|}}{k} \end{aligned}$$

and  $|A_{v_1} \cap A_{v_2}| = o(k^{n-m})$  follows as in the first case.

This completes the proof of the theorem. □

## 4 Conclusions

In conclusion, the degree-chromatic polynomial is a natural generalization of the usual chromatic polynomial, and it has a very particular structure when the graph is a tree. The leading terms of the chromatic polynomial are determined by the number of edges. Likewise, when  $m \geq 2$ , the leading coefficients of the degree chromatic polynomial  $P_m(G)$  can be described easily in terms of  $G$ , but now they depend on the degree of the vertices of  $G$ .

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