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A $q = -1$ Phenomenon for Pattern-Avoiding Permutations

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Abstract. We give an instance of Stembridge's $q = -1$ phenomenon for pattern-avoiding permutations. In particular, we show that setting $q = -1$ in the generating function for 132-avoiding permutations with respect to the statistic rsg, defined in [4], returns the number of 132-avoiding involutions.

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1 Introduction

In [5], Stembridge observed that for certain families of plane partitions, setting $q = -1$ in the generating function for the family returns the number of partitions in the family with a certain symmetry. For example, if $F_n(q)$ is the generating function for cyclically symmetric plane partitions in an $n \times n \times n$ cube with respect to the sum of the parts, then $F_n(-1)$ is the number of self-complementary cyclically symmetric plane partitions in an $n \times n \times n$ cube (See also [1, Ch.6]). More recently, Reiner, Stanton and White have studied this phenomenon in a much more general setting (See [2]).

Similar to the set of plane partitions in an $n \times n \times n$ cube, a set of pattern-avoiding permutations of some fixed length is also a finite set of objects. Moreover, as the Stembridge $q = -1$ phenomenon has an involution (the complement map) and a statistic on the objects (the number of boxes being partitioned), we also have involutions (such as the inverse map and the reverse-complement map) and numerous statistics on pattern-avoiding permutations.

In this paper, we investigate the $q = -1$ phenomenon and identify a permutation statistic that returns the number of involutions when we set $q = -1$ in its generating function. In Section 2, we provide the necessary background material for this paper. In Section 3, we prove that setting $q = -1$ in the generating function for the statistic rsg, introduced in [4], returns the number of 132-avoiding involutions.

2 Background

A *permutation* of a non-empty finite set is a bijection from the set to itself. A permutation π is an *involution* whenever $\pi = \pi^{-1}$. Let S_n denote the set of permutations of $\{1, \ldots, n\}$; we often identify a permutation $\pi \in S_n$ with the sequence $\pi(1)\pi(2)\cdots\pi(n)$. Let $|\pi|$ denote the *length* of the permutation, so that $|\pi| = n$ for $\pi \in S_n$. For $\pi \in S_n$ and $\sigma \in S_k$, we say a subsequence $\pi(i_1)\cdots\pi(i_k)$ has *type* σ whenever $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $\pi(i_l) < \pi(i_r)$ if and only if $\sigma(l) < \sigma(r)$, and we say σ is a *subpermutation* of π whenever π has a subsequence of type σ. For example, the subsequence 1426 of the permutation 145263 has type 1324 so 1324 is a subpermutation of 145263. We say π *avoids* σ whenever σ is not a subpermutation of π. For example, the permutation 214538769 avoids 312 and 2413, but it has 2586 as a subsequence so it does not avoid 1243. In this context σ is sometimes called a *pattern* or a *forbidden subsequence* and π is sometimes called a *restricted permutation* or a *pattern-avoiding permutation*. For all $n \geq 0$ and any set R of permutations, we write $S_n(R)$ to denote the set of permutations of length n which avoid every pattern in R. We also write $S(R)$ to denote the set of all permutations that avoid R.

To state our result, we first need the following definition.

Definition 1. *Suppose π is a permutation, and that we decompose π into a sequence of increasing runs, separated by the descents in the permutation. The statistic* $\mathrm{rsg}(\pi)$ *is the sum of the number of runs of π strictly to the right of each entry i of π which contain elements both larger and smaller than i. Equivalently,* $\mathrm{rsg}(\pi)$ *is the number of* $2{-}13$ *patterns* *in π, where the dash means other numbers may exist between the two entries with pattern 21 but the two entries with pattern 23 must be consecutive.*

We write $F_n(q)$ to denote the generating function for $S_n(132)$ with respect to rsg*, so that*

$$F_n(q) = \sum_{\pi \in S_n(132)} q^{\mathrm{rsg}(\pi)}.$$

For example, if we let | denote the end of every increasing run in a permutation, then we have

$$\mathrm{rsg}(145|27|6|3) = 0+1+1+0+0+0+0 = 2.$$

Equivalently, we have two $2-13$ patterns in 1452763, namely $4-27$ and $5-27$. If we perform a similar computation for each $\pi \in S_3(132)$ and each $\pi \in S_4(132)$, we find that the generating functions for $S_3(132)$ and $S_4(132)$ with respect to rsg are $F_3(q) = 4+q$ and $F_4(q) = 8+4q+2q^2$, respectively. As an instance of our main result, note that setting $q=-1$ in $F_3(q) = 4+q$ returns 3, the number of involutions in $S_3(132)$. (These involutions are 123, 213 and 321.) In the next section, we will prove the $q=-1$ phenomenon algebraically with a recurrence relation on the generating function of the statistic rsg and a previous result on the number of pattern-avoiding involutions first proved by Simion and Schmidt [3].

3 The Main Result

We can now state our main result.

Theorem 2. *The number of involutions in $S_n(132)$ is $F_n(-1)$.*

To prove our main theorem, we need to first introduce the following definition and lemmas. It is well-known in the pattern-avoidance literature that the permutations in $S(132)$ can be constructed recursively.

Definition 3. *For any permutations α and β, $\alpha \circledast \beta$ is the permutation of length $|\alpha|+|\beta|+1$ whose ith entry is given by*

$$(\alpha \circledast \beta)(i) = \begin{cases} |\beta| + \alpha(i) & \text{if } 1 \le i \le |\alpha|; \\ |\beta| + |\alpha| + 1 & \text{if } i = |\alpha|+1; \\ \beta(i - |\alpha| - 1) & \text{if } |\alpha|+2 \le i \le |\alpha|+|\beta|+1. \end{cases}.$$

The conventional notation in the permutation patterns literature for $\alpha \circledast \beta$ is $(\alpha \oplus 1) \ominus \beta$.

To visualize $\alpha \circledast \beta$, note if we graph the function $\alpha \circledast \beta$, then we obtain the graph in Figure 1, which contains shifted copies of the graphs of α and β.

Lemma 4. *If $\alpha, \beta \in S(132)$, then $\alpha \circledast \beta \in S(132)$.*

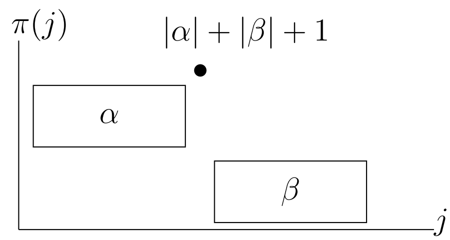

Figure 1: The permutation $\alpha \circledast \beta \in S(132)$.

Proof. It follows from the definition of $\alpha \circledast \beta$ that any entry in α is strictly larger than any entry in β. Thus, if $\alpha \circledast \beta$ contains a subsequence of type 132, then that subsequence is entirely contained in α or it is entirely contained in β. Since $\alpha, \beta \in S(132)$, the result follows. □

Lemma 5. *For any non-empty permutation $\pi \in S(132)$, there exist unique $\alpha, \beta \in S(132)$ such that $\pi = \alpha \circledast \beta$.*

Proof. For any non-empty permutation $\pi \in S(132)$ of length $|\alpha| + |\beta| + 1$, we can always pick its largest entry, namely $|\alpha| + |\beta| + 1$, and then call the permutation to its left α and the one to its right β. Observe that any entry in α is strictly larger than any entry in β. (If not, then a given entry in α, the entry $|\alpha| + |\beta| + 1$, and a given entry in β form a 132 subsequence, which is a contradiction.) It follows from $\pi \in S(132)$ that $\alpha \in S(132)$ and $\beta \in S(132)$. Thus we can always decompose π uniquely into $\alpha \circledast \beta$. □

Lemma 6. *For any non-empty permutation π, let $\mathrm{last}(\pi)$ denote the last entry of π. If π is the empty permutation, then we set $\mathrm{last}(\pi) = 0$. The statistics rsg and last satisfy the following relations:*

$$\mathrm{rsg}(\alpha \circledast \beta) = \mathrm{rsg}(\alpha) + \mathrm{rsg}(\beta) + |\alpha| - \mathrm{last}(\alpha); \tag{1}$$

if β is not empty,

$$\mathrm{last}(\alpha \circledast \beta) = \mathrm{last}(\beta); \tag{2}$$

if β is the empty permutation,

$$\mathrm{last}(\alpha \circledast \beta) = |\alpha| + 1. \tag{3}$$

Proof. Observe that the statistic $\mathrm{rsg}(\alpha \circledast \beta)$ can be decomposed into three parts: the runs in $\alpha(\mathrm{rsg}(\alpha))$, the runs in $\beta(\mathrm{rsg}(\beta))$, and the additional runs after inserting $1 + |\alpha| + |\beta|$ between α and β ($|\alpha| - \mathrm{last}(\alpha)$). The largest entry in $\alpha \circledast \beta$ extends the last run in α and it contributes $|\alpha| - \mathrm{last}(\alpha)$ runs towards the statistic rsg, because for every entry between $\mathrm{last}(\alpha)$ and $1 + |\alpha| + |\beta|$, it is a legitimate run for $\mathrm{rsg}(\alpha \circledast \beta)$. □

Before proving Theorem 2, we need the following fact about the nth central binomial coefficient $\binom{n}{[n/2]}$, which is the number of length n 132-avoiding involutions (see [3, Thm. 5.6]).

Lemma 7. *The generating function for the central binomial coefficients is*

$$\sum_{n\geq 0}\binom{n}{[n/2]}x^n = \frac{1-2x-\sqrt{1-4x^2}}{4x^2-2x}.$$

Proof. Let $n=2m$ if n is even, and $n=2m+1$ if n is odd. Then we have

$$\sum_{n\geq 0}\binom{n}{[n/2]}x^n = \sum_{m=0}^{\infty}\binom{2m}{m}(x^2)^m + x\sum_{m=0}^{\infty}\binom{2m+1}{m}(x^2)^m.$$

As a special case of the generalized binomial theorem,

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty}\binom{2n}{n}x^n.$$

Since $\binom{2n+1}{n}=\binom{2n+1}{n+1}$ and the sum of these two binomial coefficients is $\binom{2n+2}{n+1}$, we have

$$\sum_{n=0}^{\infty}\binom{2n+1}{n}x^n = \frac{\frac{1}{\sqrt{1-4x}}-1}{2x}.$$

Thus, our result follows by combining the two generating functions as shown above and replacing x with x^2. □

Now we are ready to prove Theorem 2.

Proof of Theorem 2. We define another generating function for $S_n(132)$ with respect to the statistic rsg:

$$G(q,t,s) = \sum_{\pi\in S(132)} q^{\mathrm{rsg}(\pi)}(-t)^{\mathrm{last}(\pi)}s^{|\pi|}; \tag{4}$$

hence,

$$G(-1,-1,s) = \sum_{n\geq 0} F_n(-1)s^n.$$

Therefore, to prove the Theorem, we only need to show that

$$G(-1,-1,s) = \sum_{n\geq 0}\binom{n}{[n/2]}s^n.$$

We divide the permutations in $S(132)$ into three classes and consider the members of each class: π is the empty permutation; $\pi=\alpha\circledast\beta$ is not empty and β is not the empty permutation; π is not empty and β is the empty permutation. Thus,

$$G(q,t,s) = 1 + \sum_{\substack{\pi\in S(132)\\ \mathrm{last}(\pi)\neq|\pi|}} q^{\mathrm{rsg}(\pi)}(-t)^{\mathrm{last}(\pi)}s^{|\pi|} + \sum_{\substack{\pi\in S(132)\\ \mathrm{last}(\pi)=|\pi|}} q^{\mathrm{rsg}(\pi)}(-t)^{\mathrm{last}(\pi)}s^{|\pi|}. \tag{5}$$

It follows from the recurrence relations (1), (2), (3) in Lemma 6 that

$$G(q,t,s) = 1 + sG(q,-q^{-1},qs)[G(q,t,s)-1] - tsG(q,-q^{-1},-qts). \tag{6}$$

If we let $q=-1$ and $t=-1$ in (6), then we find

$$G(-1,-1,s) = 1 + sG(-1,1,-s)[G(-1,-1,s)-1] + sG(-1,1,-s), \tag{7}$$

from which it follows that

$$G(-1,-1,s)(1-sG(-1,1,-s)) = 1. \tag{8}$$

Similarly, if we let $q=-1$ and $t=1$ in (6), then we find

$$G(-1,1,s) = 1 + sG(-1,1,-s)[G(-1,1,s)-1] - sG(-1,1,s). \tag{9}$$

Now replace s with $-s$ in (9), to obtain

$$G(-1,1,-s) = 1 - sG(-1,1,s)[G(-1,1,-s)-1] + sG(-1,1,-s). \tag{10}$$

Notice equations (9) and (10) form a system of two equations in the unknowns $G(-1,1,-s)$ and $G(-1,1,s)$; when we solve this system we find

$$G(-1,1,-s) = \frac{2s+1 \pm \sqrt{1-4s^2}}{2s}. \tag{11}$$

Since $\lim_{s\to 0^-} \frac{2s+1+\sqrt{1-4s^2}}{2s} = -\infty$,

$$G(-1,1,-s) = \frac{2s+1-\sqrt{1-4s^2}}{2s}. \tag{12}$$

When we substitute (12) into (8), we have

$$G(-1,-1,s) = \frac{1-2s-\sqrt{1-4s^2}}{4s^2-2s}. \tag{13}$$

Now the result follows from Lemma 7. □

4 Conclusions and Future work

We have proved algebraically that setting $q=-1$ in the generating function for 132-avoiding permutations with respect to the statistic rsg returns the number of 132-avoiding involutions. We can use techniques similar to those in the proof of Theorem 2 to find an expression for $G(q^{-1},-q,q^{-1}s)$; we leave this as an exercise for the diligent reader. In addition, we are still looking for a combinatorial proof of our results, in which other Catalan objects may serve as intermediate steps (bracketing sequences, binary trees, non-crossing partitions, to name a few). One can also ask in general whether a $q=-1$ phenomenon exists in permutations that avoid multiple subsequences or a single longer subsequence.

References

[1] D. Bressoud, *"Proofs and Confirmations: The Story of the Alternating-Sign Matrix Conjecture"*, Cambridge University Press, 1999.

[2] V. Reiner, D. Stanton and D. White, *The cyclic sieving phenomenon*, J. Combin. Theory Ser. A **108** (2004), no. 1, 17–50.

[3] R. Simion and F.W. Schmidt, *Restricted permutations*, European J. Combin. **6** (1985), no. 4, 383–406.

[4] R. Simion and D. Stanton, *Octabasic Laguerre polynomials and permutation statistics*, J. Comput. Appl. Math. **68** (1996), no. 1-2, 297–329.

[5] J. Stembridge, *Some hidden relations involving the ten symmetry classes of plane partitions*, J. Combin. Theory Ser. A **68** (1994), no. 2, 372–409.