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Proving n-Dimensional Linking in Complete n-Complexes in $(2n+1)$-Dimensional Space

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Abstract. This paper proves that there is an intrinsic link in complete n-complexes on $(2n+4)$-vertices for $n = 1, 2, 3$ using the method of Conway and Gordon from their 1983 paper. The argument uses the sum of the linking number mod 2 of each pair of disjoint n-spheres contained in the n-complex as an invariant. We show that crossing changes do not affect the value of this invariant. We assert that ambient isotopies and crossing changes suffice to change any specific embedding to any other specific embedding. To complete the proof the invariant is evaluated on a specific embedding.
Conway and Gordon use a diagram to carry out the final step for a 3-dimensional example and we use a computer to do this in higher dimensions. Our code is written in MATLAB. Taniyama has a proof for higher dimensions that does not use a computer.

1 Introduction

An n-complex C is said to be intrinsically linked if every embedding of C in $\mathbb{R}^{2n+1}$ contains a non-trivial link. Conway and Gordon [3] proved, in their 1983 paper, that the complete graph K_6 (Figure 1) is intrinsically linked in $\mathbb{R}^3$. The aim of this paper is to outline a method for determining that a complete n-complex on $(2n+4)$-vertices in $(2n+1)$-dimensional space will ensure an n-dimensional link, and to carry this out in the cases $n = 1, 2, 3$.

Following the Conway and Gordon [3] method of proof we first define the invariant we will use to show that an n-complex is intrinsically linked, the sum of the linking numbers mod 2. Second we will show that a crossing change will have an even effect on the parity of the mod 2 linking number. Third we will assert by a general position argument that crossing changes and ambient isotopies suffice to take one specific embedding and transform it to any other specific embedding.

When this is done we will look at the case of a specific embedding and evaluate the linking number. We will use a computer program in order to do in higher dimensions what Conway and Gordon did in three dimensions with diagrams. Taniyama [5] has also followed Conway and Gordon's approach, and carries out this last step for all n without using a computer, using the fact that S^{2n+1} is homeomorphic to the join of two n-dimensional spheres.

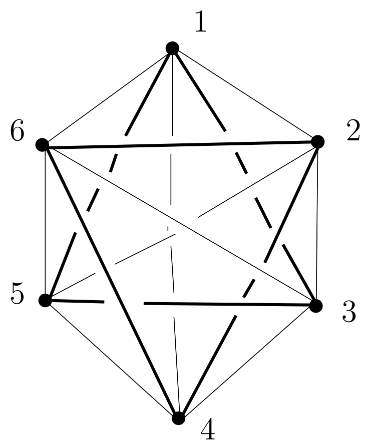

Figure 1: K_6 showing a pair of linked circles. The fact that K_6 is intrinsically linked means that, no matter how K_6 is embedded in $\mathbb{R}^3$, it is impossible to avoid thereby a pair of linked circles such as these.

2 Complete Complexes

The mathematical object that we are studying is an n-dimensional analogue of a complete graph on $(2n + 4)$-vertices. It can also be described as the n-skeleton of a $(2n + 3)$-simplex. What this means is that we take $2n+4$ vertices and join every vertex to every other vertex by a line. For the $n = 1$ case this would be the complete object, however in higher dimensions, such as $n = 2$, three points joined together by lines form the edges of a triangle, and this triangle is filled in to give a solid triangle or 2-simplex. In $n = 3$, four solid triangles that form a hollow tetrahedron are filled in to give a solid tetrahedron or 3-simplex. In general, volumes enclosed by $(k+1)$-tuples of filled areas are also filled to give solid k-simplices, and so on, until we reach dimension n.

The dimension of the complex determines the least dimension of the space in which it can be embedded. In general, to embed an n-complex requires $2n + 1$ dimensions. In an $n = 1$ example, shown in Figure 1, the link is in 3-dimensions, between 1-dimensional objects. In higher dimensions links occur between, when $n = 2$, 2-dimensional objects embedded in $\mathbb{R}^5$ and, when $n = 3$, 3-dimensional objects embedded in $\mathbb{R}^7$.

3 General Position

The Conway and Gordon argument, and hence our generalisation, relies heavily on the idea of *general position*. A k- and an l-dimensional hyperplane in $\mathbb{R}^m$ are forced to intersect in a plane of dimension at least $k + l - m$, and are said to be in general position if their intersection is of this dimension. The term general position refers to the fact that two planes that are not in general position can be moved to general position by an arbitrarily small perturbation. This is due to the fact that the space of configurations that are not in general position is of lower dimension than the space of all configurations, and so has volume zero.

This notion of general position can be extended to non-linear maps of planes by looking at the tangent planes at points of intersection. Two such maps are said to be in general position if the tangent planes are in general position at every point of intersection. We refer the interested reader to Bredon [2, Chapter 15] for further details.

It is a central result of differential topology that maps can always be moved to be in general position by arbitrarily small perturbations. It is this feature of general position that the Conway and Gordon argument relies upon.

4 The Invariant

The invariant that we will be using to determine if our n-complex is intrinsically linked is the sum, over all pairs of disjoint n-spheres contained in the complex, of the mod 2 linking number. For K_6 any splitting of the vertices into two sets of size three gives a pair of disjoint circles, as shown in Figure 1. In an $n = 2$ example, we have the complete 2-complex on 8 vertices. These 8 vertices are split into two hollow tetrahedra, each defined by 4 of the available 8 points. In general splitting the $2n + 4$ vertices into two sets of size $n + 2$ gives a pair of disjoint n-spheres.

The mod 2 linking number of two n-spheres A and B in $\mathbb{R}^{2n+1}$ is calculated by projecting them from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^{2n}$. This projection can be put in general position, and then the intersection of the two spheres will consist of finitely many points. We refer to these as

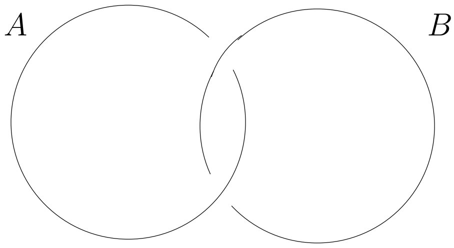

Figure 2: Projection onto $\mathbb{R}^2$ of linked circles in $\mathbb{R}^3$. The mod 2 linking number here is 1, because circle A passes over circle B once.

"crossings". The mod 2 linking number of A and B may then be defined as the number of crossings at which sphere A is above sphere B, with respect to the $(2n+1)^{th}$ co-ordinate, taken mod 2. An $n = 1$ example is shown in Figure 2. The linking number, mod 2, can be shown to be well defined for $n = 1$, using the Reidermeister moves [1], and by degree theory for $n \geq 1$ (see Rolfsen [4, pp. 132–135] for the case $n = 1$, which has all the main ideas).

For our sum of mod 2 linking numbers, an odd result ensures an odd number of links and thus at least one. So our goal is to show that this sum is always odd.

5 The Effect of Crossing Changes

The effect of a crossing change on the sum of mod 2 linking numbers is even and therefore does not affect the parity of the answer. This is easily seen in the $n = 1$ case. Crossings in K_6 occur between a pair of edges, and so involve a total of four of the six vertices. As shown in Figure 3 there are two ways these edges can be connected to the remaining two vertices to form a pair of disjoint triangles. We sum the linking number over all pairs of triangles therefore, at some point, we will cover both of these cases. It is to be noted that only crossing changes between disjoint edges can affect the linking number.

In the general case crossings occur between pairs of n-simplices, which each have $n + 1$ vertices. We use up $2(n + 1) = 2n + 2$ vertices from the entire collection of points and thus have two left, and as in the $n = 1$ case, there are two ways to connect the pair of n-simplices to the remaining two vertices to form a pair of disjoint n-spheres. This means that, if the crossing is changed, each of these two possible pairs of n-spheres is affected and so the linking number of each pair is changed by ± 1. Thus the net effect is 0, mod 2. Again it is to be noted that only crossing changes in disjoint n-simplices can affect the linking number.

6 Crossing Changes Suffice

By a general position argument, we claim that it is possible to use ambient isotopies and crossing changes to change any one specific embedding into any other specific embedding. Since ambient isotopy has no effect on linking number, and crossing changes do not affect

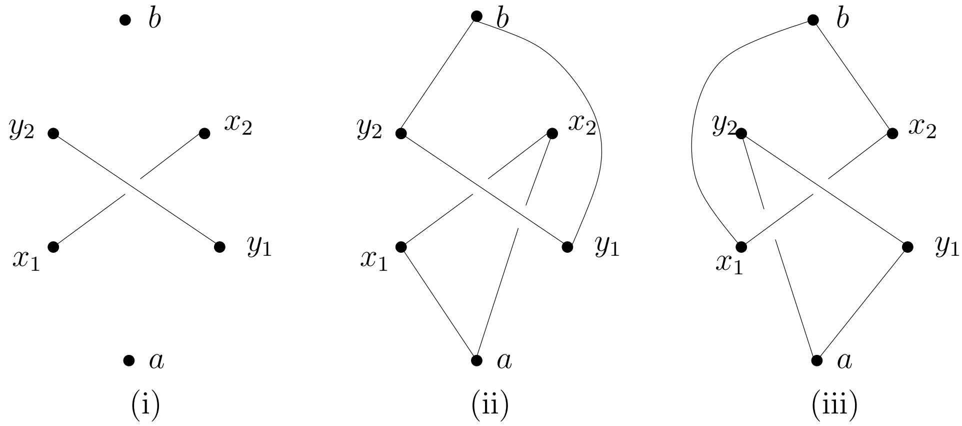

Figure 3: Points x_1, x_2, y_1 and y_2 define a pair of crossing lines, (ii) shows one possibility for joining these points to form 1-spheres and (iii) the other possibility. The crossing change affects the linking number of both pairs equally, mod 2.

the mod 2 sum, this implies that our invariant takes the same value on all embeddings. The argument is given in Taniyama [5] and we outline it below.

Given two embeddings of our complex C we take an arbitrary homotopy deforming one into the other.This gives a map $C \times [0,1] \to \mathbb{R}^{2n+1} \times [0,1]$, where the second factor corresponds to time, and we perturb this map to be in general position.

The movement of a k-simplex from our complex under this homotopy creates a $(k+1)$-dimensional trace in $\mathbb{R}^{2n+1} \times [0,1]$. Consider an intersection of two such traces, created by two simplices K and L, of dimensions k and l respectively. Since the homotopy is in general position the intersection of their traces is of dimension $(k+1)+(l+1)-(2n+2) = k+l-2n$.

When either k or l is less than n, the dimension $k+l-2n$ is negative meaning that intersections between the traces can be avoided completely. On the other hand, when $k = l = n$ the dimension of the intersection is $n+n-2n = 0$, so the intersection of the traces is a collection of points — these points represent places during the homotopy where one n-simplex passes through another, causing a crossing change.

7 Evaluation on a Specific Embedding

Using MATLAB program we will evaluate our invariant on a specific embedding for $n = 1, 2, 3$. The code is provided for the $n = 1$ and $n = 2$ cases, in Appendix A and Appendix C respectively. The code for $n = 3$ has been omitted for clarity as the code is very similar to $n = 2$ but more lengthy.

We will consider embeddings defined by simply choosing $2n+4$ points in $\mathbb{R}^{2n+1}$, and embedding each n-simplex linearly: that is, if $v_0, \ldots, v_n$ are $n+1$ of our chosen points, then we embed the simplex they define as the set

$$\left\{ \sum_{i=0}^{n} \lambda_i v_i \middle| \sum_{i=0}^{n} \lambda_i = 1, 0 \leq \lambda_i \right\}.$$

For simplicity, we choose the $2n + 4$ points randomly, since this will almost certainly (i.e. with probability 1) give us an embedding in general position. This could also have been achieved by selecting the points carefully but this is more difficult and does not give any better result.

To evaluate the invariant on our embedding we proceed as follows. First we split the $2n+4$ points into two sets of equal size, this is done using for-loops which ensure that every possible division into two equal sized groups of points is accounted for. The two sets are then used through the remainder of the code and will contribute to the final value of the summed linking numbers.

Each half of the points determines an n-dimensional sphere. One of these n-spheres must contain "point 1", we will consistently choose the n-sphere containing point 1 to be n-sphere 1 and its faces (n-simplices) will also be labeled 1. Again through for-loops, all permutations of pairs of faces are accounted for.

These for-loops will take one face from the first n-sphere and one face from the second n-sphere. These faces will each have $n+1$ points associated with them and these are given as arguments to a function to determine if the faces cross, and if so, which face is on top. The function to determine crossings is given for $n = 1$ in Appendix B and for $n = 2$ in Appendix D, and is described below.

To determine whether two faces cross, we write the equations that define these faces with parameters μ_i and ν_i. The intersections between these faces have the general form

$$u_1 + \sum_{i=2}^{n+1} (\mu_i(u_i - u_1)) = v_1 + \sum_{i=2}^{n+1} (\nu_i(v_i - v_1)),$$

where $u_1, \ldots, u_{n+1}$ are the vertices of the first face, and $v_1, \ldots, v_{n+1}$ are the vertices of the second.

Projecting the equations to $\mathbb{R}^{2n}$ by ignoring the $(2n+1)^{th}$ coordinate gives us a system of $2n$ equations in $2n$ unknowns. Solving the system of equations in $2n$-dimensional space allows us to determine if the faces have a crossing within the bounds of their vertices. The test for determining that the crossing is indeed within the bounds is that $0 \leq \mu_i$ and $\sum \mu_i \leq 1$. Any crossing which occurs outside of these bounds does not contribute to the linking number of the embedding as the crossing does not take place within the embedding; it is simply the product of modeling the complex using lines, planes and hyperplanes, which have infinite extent.

Once any crossing point is determined, the value of the $(2n + 1)^{th}$ coordinate of the crossing point is obtained for both faces, and the face with the higher value is said to be "on top". Note that the program checks the determinant on line 5 of Appendix B to avoid

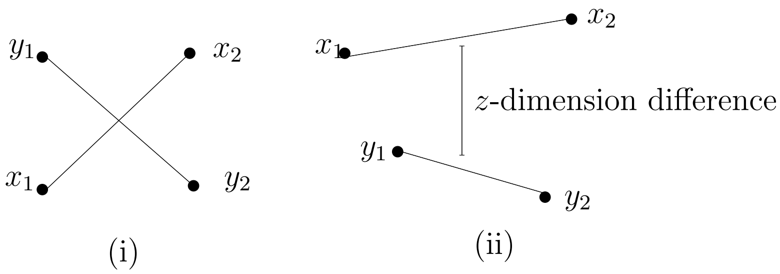

Figure 4: A crossing shown as a projection onto the x, y-plane (i), and in 3 dimensions to show the difference in their z-values (ii).

parallel lines. If the face on top is face 1 then a value of $+1$ is assigned for that pair of faces, a -1 is assigned if face 1 is "underneath" and the answer is 0 if they do not cross.

Because this was done in MATLAB, a tolerance check was introduced to determine how close the faces may get before they are considered to occupy the same value in the $(2n+1)^{th}$ dimension. This is intended to account for round-off error.

Once a value of $+1$, -1 or 0 is obtained from the function (or an error is given to indicate that parallel lines have occurred or that tolerance has been breached) then this value is returned to the original function. The returned values for every division are collected in a matrix and the positive entries are summed since these correspond to crossings in which the first sphere is on top. If this sum of sums is odd then we can determine that this embedding (and hence every possible embedding) is linked.

8 Results

When we ran the code to determine if a randomly generated embedding of our complex is linked, the output we received indicated that the embedding was, in fact, linked. The output indicated this by giving us an odd answer, an answer which is odd has an odd number of links and thus, at least one. This result was obtained for the cases of $n = 1, 2, 3$. All three cases were shown to be linked in the specific embedding, and thus, intrinsically linked.

A K6

This is the code used to calculate the sum of the linking numbers for the case of K_6. This program generates a matrix of points in 3-dimensions and iterates through divisions of the points into two groups, it then iterates through every pair of points within each group, inputting them into the crossing function given on the next page. The output is then summed and the value of the sum mod 2 is returned to the user. For more details see Section 7.

```
Points = rand(3,6)
entry = 1;
Results = zeros(1,10);
for i = 2:5
    for j = i+1 : 6
        T1 = [(Points(:,1)) Points(:,i) (Points(:,j))];
        T2 = [Points(:,(2:(i-1))) Points(:,((i+1):(j-1)))
           Points(:,((j+1):6))];
        c = 0;
        for k = 1:2
            for l = k+1:3
                pairs1 = [T1(:,k) T1(:,l)];
                for m=1:2
                    for n=m+1:3
                        pairs2 = [T2(:,m) T2(:,n)];
                        cross = crossing(pairs1(:,1),pairs1
                           (:,2),pairs2(:,1),pairs2(:,2));
                        if cross == 1
                          c = c+cross;
                        end
                    end
                end
            end
        end
Results(entry) = c;
entry = entry+1;
    end
end
Results
mod(sum(Results),2)
```

B K_6 Crossing Function

This is the function `crossing` called on line 15 of Appendix A. It takes as input two pairs of points in $\mathbb{R}^3$, and determines whether the line segments joining each pair cross when projected to $\mathbb{R}^2$. The output is $+1$ if there is a crossing in which the first line segment is on top; -1 if they have an intersection in which the second line segment is on top; and 0 if the line segments do not cross.

The function returns an error if the line segments are close to parallel, or if the segments are considered too close to say which is on top.

```
function c = crossing(u,v,x,y)
    tolerance = 10^(-6);
  A = [ u-v y-x ];
    A = A(1:2,:);
      if abs(det(A)) < 1/10000
      error('Matrix determinant is out of limit')
      end
    b = [(y(1) - v(1));(y(2) - v(2))];
r = A\b;
t = r(1);
s = r(2);
zdiff = ((t*u + (1-t)*v) - (s*x + (1-s)*y));
zdiff = zdiff(3);
 if ((0<s) && (s<1)) && ((0<t) && (t<1))
   if (abs(zdiff)< tolerance)
       error('Lines are not within tolerance: may intersect')
   else
       c = sign(zdiff);
   end
 else
     c = 0;
 end
```

C Complete 2-Complex on 8 Points in 5-Dimensions

This code's function and methods are equivalent to those in Appendix A.

```
Points = rand(5,8)
entry = 1;
Results = zeros(1,35);
for i = 2:6
    for j = i+1:7
        for k = j+1:8
            Tetrahedra1 = [(Points(:,1)) Points(:,i) (Points(:,j)) (Points(:,k))];
            Tetrahedra2 = [Points(:,(2:(i-1))) Points(:,((i+1):(j-1))) Points(:,((j+1):(k-1))) Points(:,((k+1):8))];
            c = 0;
        for l = 1:2
            for m = l+1:3
                for n = m+1:4
                    triplets1 = [Tetrahedra1(:,l) Tetrahedra1(:,m) Tetrahedra1(:,n)];
                    for o = 1:2
                        for g = o+1:3
                            for q = g+1:4
                                triplets2 = [Tetrahedra2(:,o) Tetrahedra2(:,g) Tetrahedra2(:,q)];
                                cross = crossing5(triplets1(:,1),triplets1(:,2),triplets1(:,3),triplets2(:,1),triplets2(:,2),triplets2(:,3));
                                if cross == 1
                                    c = c+cross;
                                end
                            end
                        end
                    end
                end
            end
        end
Results(entry) = c;
entry = entry+1;
```

```
        end
    end
end
Results
mod(sum(Results),2)
```

D Crossing Function for the Complete 2-Complex on 8 Points in 5-Dimensions

The function `crossing5`, used to determine whether the planes joining each triple of points cross when projected from $\mathbb{R}^5$ to $\mathbb{R}^4$. It is analogous to the code for 3 dimensions, given in Appendix B.

```
function c = crossing5(u1,u2,u3,v1,v2,v3)
    tolerance = 10^(-6);
  A = [ (u2-u1) (u3-u1) (v1-v2) (v1-v3) ];
    A = A(1:4,:);
      if abs(det(A)) < 1/1000000
      error('Matrix determinant is out of limit')
      end
    b = [v1-u1];
      b = b(1:4,:);
r = A\b;
t = r(1);
s = r(2);
p = r(3);
q = r(4);
zdiff = (u1+t*(u2-u1)+s*(u3-u1)-(v1+p*(v2-v1)+q*(v3-v1)));
zdiff = zdiff(5);
 if ((0<s && 0<t && s+t<1) && (0<p && 0<q && p+q<1))
   if (abs(zdiff)< tolerance)
       error('Lines are not within tolerance: may intersect')
   else
       c = sign(zdiff);
   end
 else
     c = 0;
end
```

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