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Computing Fixed Point Floer Homology via the Hochschild Homology of a sequence of curves

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Abstract. In the summer of 2009, our group developed a computer program that computes Hochschild Homology, a topological invariant. While we must assume that the reader has at least encountered algebraic topology, in this paper we provide the mathematical background and motivation for our algorithm. After presenting a number of definitions, we will explain how the algorithm works. Specifically, we first define the Floer complex of two curves on surface; the resulting homology is invariant under isotopies. Then, we introduce the Fukaya category associated to a sequence of curves. Next, we define the Hochschild complex of the Fukaya category. And finally, we describe an algorithm for computing Hochschild Homology and provide some examples.

Acknowledgements: This project was completed with funding from NSF's RTG grant DMS-0739392, and with guidance from our professors and their teaching assistants.

1 Introduction

The notion of a chain complex and its homology groups emerged in algebraic topology, but is inspiring advances throughout mathematics. Different definitions of the same homology groups have also led to substantial advances. In particular, in differential topology sometimes we employ Morse homology to analyze a space. Rather than studying simplices on a space itself, Morse homology investigates the functions on the space. Using Morse homology, which is defined in terms of the critical points of a Morse function, it is possible to exploit ordinary homology to obtain information about differentiable functions. We will briefly review Morse homology in Section 2.

Morse homology is an invariant of finite dimensional manifolds; and the condition of finiteness has led to the natural question: how can we study infinite dimensional spaces in differential topology? One answer was given by Palais-Smale[1], for a certain class of functions. More recently, Andreas Floer[2] gave a construction for a completely different class of functions on infinite-dimensional manifolds. Floer introduced new homology theories in differential topology which were infinite dimensional analogues of Morse homology. Since then, Floer homology has led to remarkable advances in our knowledge about 3- and 4-dimensional topology.

But despite its importance, Floer homology is often difficult to compute. One way to get around the difficulty is to trade one variety of Floer homology for another. For example, a recent result of Timothy Perutz[3] confirms the conjecture of Paul Seidel[4].

Conjecture 1.1. *Let $\alpha_1, \ldots, \alpha_n$ be closed curves on a surface Σ. Let $\tau_\alpha \colon \Sigma \to \Sigma$ denote a Dehn twist around α. Then:*

$$HF(\tau_{\alpha_1} \circ \ldots \circ \tau_{\alpha_n}, id) \cong \overrightarrow{HH_*}(\alpha_1, \ldots, \alpha_n)^1,$$

where the left hand side denotes the relative fixed point Floer homology of the diffeomorphism $\tau_{\alpha_1} \circ \ldots \circ \tau_{\alpha_n}$, and the right hand side is the Hochschild homology of the Fukaya category generated by $\alpha_1, \ldots, \alpha_n$.

It is not obvious how to compute $HF(\tau_{\alpha_1} \circ \ldots \circ \tau_{\alpha_n}, id)$, since the theory relies on computations with nonlinear PDE's. However, $\overrightarrow{HH_*}(\alpha_1, \ldots, \alpha_n)$ is algorithmically computable. We have developed a computer program which computes the Hochschild homology $\overrightarrow{HH_*}$ from suitable input data. Here, we present the relevant mathematical background and provide some examples.

We begin in Section 2.1 by briefly reviewing Morse homology. In Section 2.2, we define the Floer complex of two curves α and β on surface, where the basis of the chain complex is the set of intersection points of α and β; the differential is defined using bigons formed by α and β. We will show that this gives a well-defined homology group $HF(\alpha, \beta)$ which is

[1] Technically, this is the Hochschild homology of A with coefficients in its bimodule B. Here, A is the directed subcategory of the (balanced) Fukaya category of Σ generated by $(\alpha_1, \ldots, \alpha_n)$; B is the full subcategory with the same objects; and A is regarded as an algebra over the semisimple ring $(\mathbb{Z}/2)^n$.

invariant under isotopies. In Section 2.3, we build on the idea of bigons to define operations involving polygons. With these higher products as our starting point, we introduce the Fukaya category associated to a sequence of curves. In Section 2.4, we define the Hochschild complex of a Fukaya category and give an explicit formula for the differential using the "polygon products". In Section 3, we describe an algorithm for computing $\overrightarrow{HH_*}$ and provide some examples.

This project was completed in the summer of 2009, with funding from NSF's RTG grant DMS-0739392. We would like to thank our professors Robert Lipshitz and Timothy Perutz, as well as our T.A.'s Jonathan Bloom and Thomas Peters, for their guidance.

2 Background

2.1 Morse homology

In this section, we briefly review the basic concepts of Morse homology.

Definition 1. *A **Morse function** on a surface, (or, more generally manifold) S is a differentiable function $f : S \to \mathbb{R}$ such that the second derivative test does not fail at any critical point of f. At any critical point p of f, the **index** of f is the number of negative eigenvalues of the second derivative maximal to f at p. A **flow line** of f is an integral curve of $-\nabla f$.*

Remark. The number of critical points is **not** independent of f. However, let $C_n(f) = (\mathbb{Z}/2)\langle x_n \rangle$, where x_n is index n critical points of f, and define $\partial_n : C_n(f) \to C_{n-1}(f)$ by

$$\partial_n(p) = \sum_{q \mid ind(q)=n-1} n(p,q)q,$$

where $n(p, q)$ denotes the number of flow lines from p to q. Let $H_n(f) = ker(\partial_n)/im(\partial_{n+1})$. Then, $dim(H_n(f))$ **is** independent of f, for f satisfies same "transversality" conditions. $H_n(f)$ is called the **Morse homology** of f.

2.2 Floer complex of two curves on a surface

Let Σ be a compact, oriented genus g surface, or, more generally, the compliment of finitely many points in such a surface. Let α and β be two simple closed curves that intersect transversally on Σ. Assume α and β are not contractible. By the implicit function theorem, the intersection points of α and β are isolated, and therefore there are only finitely many.

Definition 2. *Let U be the set $\{x + iy \in \mathbb{C} \mid x \geq 0, x^2 + y^2 \leq 1\}$, and let x_- and x_+ be intersection points of α and β on a surface Σ. A **bigon** from x_- to x_+ is an immersion $u : U \to \Sigma$ such $u(i) = x_-$ and $u(-i) = x_+$, such that $u(\partial_\alpha U) \subset \alpha$, and $u(\partial_\beta U) \subset \beta$, where $\partial_\alpha U = U \cap (y\text{-axis})$ and $\partial_\beta U = U \cap \{(x, y) \mid x^2 + y^2 = 1\}$. Furthermore, the immersion* *must preserve orientation, and the corners are convex, which means that the derivative* $D_i u$ *carries the tangent cone to* U*, inside* $T_i\mathbb{C}$*, to a subset of* $T_{x_-}\Sigma$ *that contains no complete line; and the same at* $-i$ *and* x_+*. See Figure 1. Two bigons* u *and* v *on* Σ *are* ***equivalent*** *if there exists a regular homotopy* H *between them; i.e., at every time* $t \in [0,1]$ *of the homotopy* H *between* u *and* v*, the function* $H(x,t)$ *is an immersion.*

Note that if a bigon has curves α on the left and β on the right, the bigon is then orientated from x_- to x_+, as signified by the downward arrow in Figure 1.

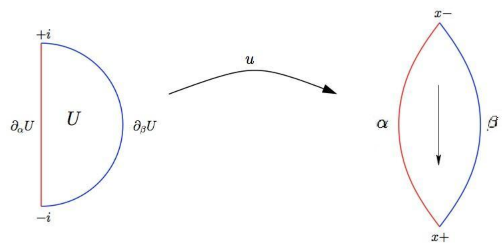


Figure 1: A bigon from x_- to x_+

Recall that an **ungraded chain complex** (C,∂) is a vector space C and an endomorphism ∂: $C \to C$, such that $\partial \circ \partial = 0$. The **homology group** of C is defined as the quotient group $H(C) = \ker \partial / \operatorname{im} \partial$. Now we are ready to define the **Floer complex** of α and β.

Definition 3. *The* ***Floer complex*** *of* α *and* β *is*

$$CF(\alpha, \beta) = (\mathbb{Z}/2)^{\alpha \cap \beta};$$

that is, the vector space over $\mathbb{Z}/2$ *generated by the intersection points of* α *and* β*. The differential* $\partial : CF(\alpha,\beta) \to CF(\alpha,\beta)$ *is given by:*

$$\partial\langle x_- \rangle = \sum_{x_+ \in \alpha\cap\beta} n(x_-, x_+)\langle x_+ \rangle,$$

where $n(x_-, x_+)$ *denotes the number of equivalence classes of bigons from* x_- *to* x_+*,* $\operatorname{mod} 2$*. (The number of bigons is finite.)*

Proposition 2.1. $\partial^2 = 0$

Proof. Let x_1, y, w denote intersection points of α and β, and u_1 and u_2 be bigons from w to x_1 and x_1 to y, respectively. We will explain that, to prove $\partial^2 = 0$, it suffices to find an intersection point x_2, distinct from x_1, and a pair of bigons v_1, connecting w to x_2, and v_2 connecting x_2 to y. (See Figure 2.) After this, we will show that there is such an intersection point x_2.

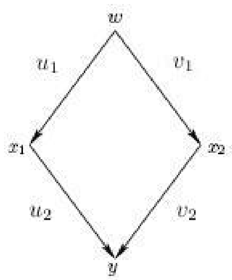


Figure 2: Floer Complex. The left branch from w to y represents u_1 and u_2; the right represents v_1 and v_2.

To see why it suffices to find a pair of matching bigons, first note that we are working over $\mathbb{Z}/2$ and hence $y+y=2y=0$. The two y terms from $\partial(x_1)$ and $\partial(x_2)$ become canceling terms in $\partial^2(w)$. Now, we consider the four possible pictures of the intersection point on the surface.(See Figure 4.) Let us denote the domain of u_1 as U_1, and of u_2 as U_2. From the definition of a bigon the domains U_1 and U_2, and the respective images can be represented as Figure 3.

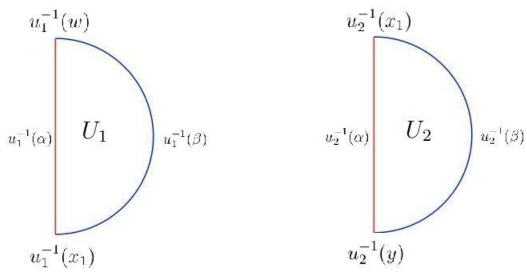
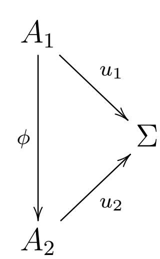


Figure 3: Two bigons.

Without loss of generality, assume that near $x_1 \in \Sigma$ the curves α and β intersect as in Figure 4 (a). There is an arc A_1 on $\partial_\beta U_1$ starting at $-i$ and an arc A_2 on $\partial_\beta U_2$ starting at $+i$ such that $u_1 \mid_{A_1} = u_2 \mid_{A_2}$ up to reparametrization. That is, there exists a homeomorphism $\phi : A_1 \to A_2$ such that $\phi(i) = -i$ and $u_2 \circ \phi = u_1$, i.e., the following diagram commutes:



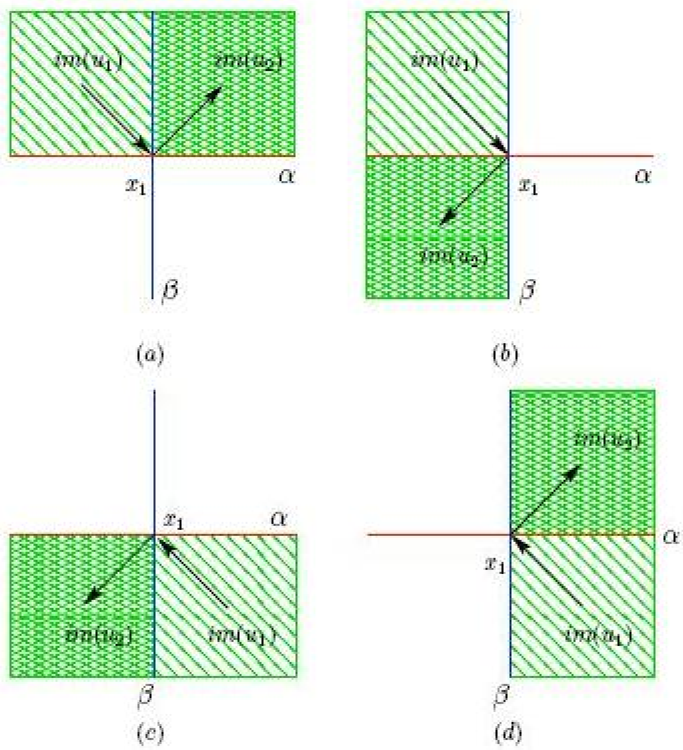


Figure 4: Four possible pictures of Σ near x_1.

Choose A_1 and A_2 to be as long as possible. Define the quotient space

$$D = (U_1 \cup U_2)/\{t \sim \phi(t)\, \text{for}\, t \in A_1\}.$$

Let $u = u_1 \amalg u_2$ to be the map on D that agrees with u_1 on U_1 and u_2 on U_2. In this way, each bigon is asociated to another unique bigon, and thus we only need to consider one pair at a time.

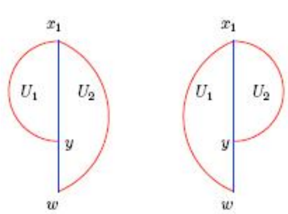


Figure 5: Two possible pictures of D. Since α is simple, y and w are distinct.

Now, we proceed to construct two bigons that cancel u_1 and u_2 by providing another intersection point x_2. Because α is smooth and simple and u_1 and u_2 are immersions, $u^{-1}(\alpha)$ intersects $u_1^{-1}(\beta)$ at y as in Figure 5. It then travels through the interior of D and eventually leaves the interior, intersecting $u_1^{-1}(\beta)$ again at a point x_2 between y and w as in Figure 6.

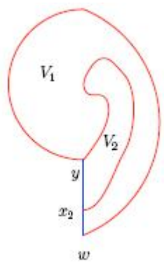


Figure 6: The two new cancelling bigons.

This component of the preimage $u^{-1}(\alpha)$ cuts D into disks V_1 and V_2. The restriction $u|_{V_1}$ and $u|_{V_2}$ are bigons between w and x_2, and x_2 and y. Thus, we have found the canceling bigons. This concludes the proof. □

The following is a simple example of a Floer complex of two curves.

Example 1. *Let α and β be two simple closed curves on the torus as shown in Figure 7. Since $\alpha \cap \beta = \{x_1, x_2\}$, $CF(\alpha, \beta)$ can be represented by $\mathbb{Z}/2_{(x_1)} \oplus \mathbb{Z}/2_{(x_2)}$. U_1 and U_2 are both images of immersed bigons from x_1 to x_2, so $n(x_1, x_2) \equiv 0$ mod 2. There are no bigons originating from x_2. Thus $\partial(x_1) = n(x_1, x_2)\langle x_2 \rangle = 0$, and $\partial(x_2) = 0$.*

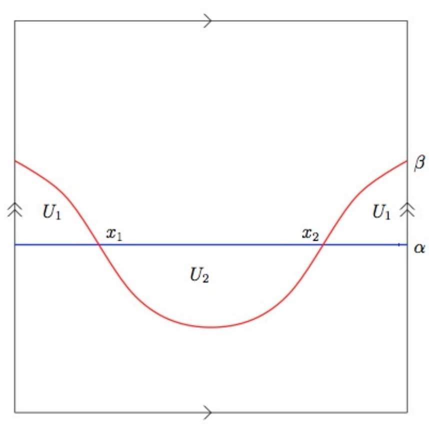


Figure 7: An example of a Floer complex over $\mathbb{Z}/2$, spanned by x_1 and x_2.

Thus we have shown that $(CF(\alpha, \beta), \partial)$ is indeed a chain complex. The homology group of $(CF(\alpha, \beta), \partial)$ is called the **Floer homology** of α and β and is denoted by $HF(\alpha, \beta)$. Next we show that the homology defined this way is invariant under isotopies.

Theorem 2.2. *If α and β are non-isotopic and β and β' are isotopic, then $HF(\alpha, \beta) \cong HF(\alpha, \beta')$.*

In order to prove the theorem, we first introduce an algorithm that simplifies a given chain complex to compute homology. Suppose C_* is a finite dimensional chain complex over $\mathbb{Z}/2$ with basis $\{x_j\}_{j=1,2,3...,n}$ and a nontrivial differential. Since we are working over $\mathbb{Z}/2$, the differential takes the form

$$\partial_C x_j = x_{i_1} + x_{i_2} + x_{i_3} + \ldots + x_{i_m}$$

where $x_{i_k} \in \{x_j\}$. By abuse of notation, we also denote $\partial_C x_j$ as the set $\{x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_m}\}$. By re-numbering the basis elements x_j's, we can assume that

$$\partial_C x_1 = x_2 + \ldots.$$

Let D_* be the chain complex generated by $\{x_j\}_{j\geq 3}$ with differential:

$$\partial_D x_i = \begin{cases} \partial_C x_i & \text{if } x_1, x_2 \notin \partial_C x_i \\ \partial_C x_i + \partial_C x_1 & \text{if } x_2 \in \partial_C x_i \\ \partial_C x_i - \partial_C x_1 & \text{if } x_1 \in \partial_C x_i \end{cases}$$

Then, one can check that $H(D_*) \cong H(C_*)$. In this way, we have kept the homology invariant while reducing the number of generators by two. We refer to this procedure as the **arrow canceling move**, which can be visualized as in Figure 8.

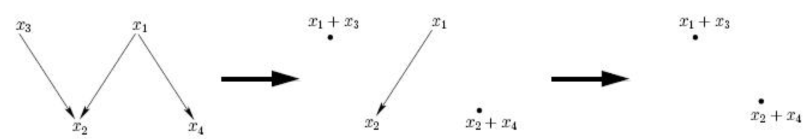


Figure 8: Arrow cancelling move

We will also assume the following standard result from transversality theory.

Lemma 2.3. *Any isotopy between two curves β and β' can be broken up into a sequence of "finger moves" as illustrated in Figure 9 and isotopies not changing the intersection pattern with α.*

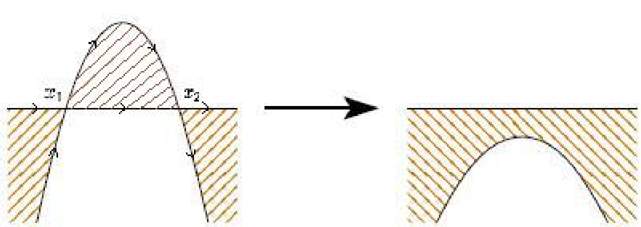


Figure 9: Finger move

Proof of Theorem 3.2. Since, isotopies which do not change the intersection pattern with α do not change the combinatorics of the resulting complexes, by Lemma 2.3, it suffices to show that $HF(\alpha, \beta)$ is invariant under a "finger move". In Figure 9, three shaded regions refer to three bigons which represent the three arrows in the first diagram of Figure 8. As we execute a finger move as Figure 9, bigons vanish in the local sense and this result precisely refers to the last diagram of Figure 8 with no arrow. Therefore, a finger move exactly corresponds to an "arrow canceling move" on the chain complex. This concludes the proof. □

Another important result in the theory is the following corollary.

Corollary 2.4. *If α and β are non-isotopic essential curves in Σ then*

$$HF(\alpha, \beta) \cong (\mathbb{Z}/2)^{I(\alpha,\beta)}$$

where $I(\alpha, \beta)$ is the minimal geometric intersection number of α and β.

The proof of the corollary uses the "bigon criterion" which can be found e.g. in B. Farb and D. Margalit's "A primer on mapping class groups"[5].

2.3 Fukaya category of a surface

A **closed surface** is a compact, connected 2-manifold without boundary. Let $S = (\alpha_1, \dots, \alpha_n)$ be a sequence of n simple closed curves on a closed surface Σ such that α_i and α_j intersects transversely for all $i \neq j$. Let

$$\overrightarrow{CF}(\alpha_i, \alpha_j) = \begin{cases} CF(\alpha_i, \alpha_j) & \text{if } i < j \\ \mathbb{Z}/2 & \text{if } i = j \\ 0 & \text{if } i > j \end{cases}$$

Here, $\overrightarrow{CF}(\alpha_i, \alpha_j)$ is called the *directed Floer vector space* of α_i and α_j. We will use x_{ij} to denote an element of $\overrightarrow{CF}(a_i, a_j)$, i.e., an intersection point of α_i and α_j.

Definition 4. *A* ***polygon*** *from the intersection points $x_{i_{d-1}i_d}, \dots, x_{i_0 i_1}$ to the point $x_{i_0 i_d}$ is the immersion of a $(d+1)$-sided polygon into the surface Σ with image as shown in Figure 10. It is orientation-preserving with convex corners. Two polygons are* ***equivalent*** *if there exists a regular homotopy between them.*

The polygon gives rise to a multiplication from $\overrightarrow{CF}(\alpha_{i_{d-1}}, \alpha_{i_d}) \otimes \overrightarrow{CF}(\alpha_{i_{d-2}}, \alpha_{i_{d-1}}) \otimes \dots \otimes \overrightarrow{CF}(\alpha_{i_0}, \alpha_{i_1})$ to $\overrightarrow{CF}(\alpha_{i_0}, \alpha_{i_d})$, where we think of the intersection points $x_{i_{d-1}i_d}, \dots, x_{i_0 i_1}$ as the multiplication inputs and $x_{i_0 i_d}$ as the multiplication output. We now proceed to lift our discussion to a category-theoretic level; this will help us define $\overrightarrow{HH}_*$ in the next section.

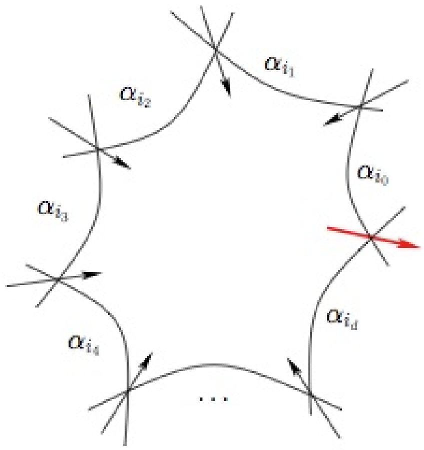


Figure 10: The arrows represent the intersection points. Those arrows pointing into the polygon denote the multiplication's inputs; the one pointing out denotes the multiplication's output.

Definition 5. *An A_∞ **category** is:*

- *A collection of objects $ob\mathscr{C}$,*
- *For a_i, $a_j \in ob\mathscr{C}$ a vector space $Mor(a_i, a_j)$;*
- *For $a_1, \ldots, a_k \in ob\mathscr{C}$ a composition map $\mu^k : Mor(a_{k-1}, a_k) \otimes \ldots \otimes Mor(a_1, a_2) \to Mor(a_1, a_k)$ satisfying the A_∞ relation:*

$$\sum_{\substack{0 \le j \le d \\ 0 \le i \le d \\ 0 \le i+j \le d}} \mu^d \left(x_{d(d+1)}, \ldots, x_{(i+j+1)(i+j+2)}, \mu^j(x_{(i+j)(i+j+1)}, \ldots, x_{(i+1)(i+2)}), x_{i(i+1)}, \ldots, x_{12}\right) = 0$$

for any sequence of elements $x_{12}, \ldots, x_{(k-1)k}$ where $x_{i(i+1)} \in Mor(x_i, x_{i+1})$. Note that the vector spaces are over $\mathbb{Z}/2$.

Definition 6. *The **directed Fukaya category** associated to $(\alpha_1, \ldots, \alpha_n)$ is an A_∞ category with objects $\{\alpha_1, \ldots, \alpha_n\}$ and $Mor(\alpha_i, \alpha_j) = \overrightarrow{CF}(\alpha_i, \alpha_j)$. Define the composition maps*

$$\mu^d : \overrightarrow{CF}(\alpha_{i_{d-1}}, \alpha_{i_d}) \otimes \overrightarrow{CF}(\alpha_{i_{d-2}}, \alpha_{i_{d-1}}) \otimes \ldots \otimes \overrightarrow{CF}(\alpha_{i_0}, \alpha_{i_1}) \to \overrightarrow{CF}(\alpha_{i_0}, \alpha_{i_d})$$

where $i_0 > i_1 > \ldots > i_d$, as the sum of the outputs of the polygons with the inputs $x_{i_{d-1}i_d}, \ldots, x_{i_0 i_1}$, i.e.

$$\mu^d(x_{i_{d-1}i_d}, \ldots, x_{i_0 i_1}) = \sum_{x_{i_0 i_d} \in \alpha_{i_0} \cap \alpha_{i_d}} n(x_{i_{d-1}i_d}, \ldots, x_{i_0 i_1}) x_{i_0 i_d}$$

where $n(x_{i_{d-1}i_d}, \ldots, x_{i_0 i_1})$ *counts the number of equivalence classes of polygons from* $x_{i_{d-1}i_d}, \ldots, x_{i_0 i_1}$ *to* $x_{i_0 i_d}$*, mod* 2*.*

Remark. It can be checked that the A_∞ relations hold in this case. We assume that d, the number of the inputs of a composition map, is less than or equal to n. In general, $n(x_{i_{d-1}i_d}, \ldots, x_{i_0 i_1})$ is not finite; to ensure finiteness, assume that the surface is hyperbolic (i.e. $\chi < 0$) and that $(\alpha_1, \ldots, \alpha_n)$ is a *balanced* sequence of curves, in the sense of Paul Seidel[6]. Also note that the indices of the curves are strictly decreasing, since by definition $\overrightarrow{CF}(\alpha_i, \alpha_j) = 0$ for $i < j$. We also remark that this condition is met on hyperbolic surfaces; i.e., $\chi < 0$.

Example 2. *Let us provide an example of a composition map. Let* α_1*,* α_2*, and* α_3 *be three simple closed curves on a punctured torus as shown in Figure 11. There are no bigons since each pair of curves intersect in exactly one point.* V_1 *and* V_2 *are both triangles from* x_{32} *and* x_{21} *to* x_{13}*. Thus* $n(x_{32}, x_{21}, x_{13}) \equiv 0$ *mod* 2*, and* $\mu^2(x_{32}, x_{21}) = 0$*.*

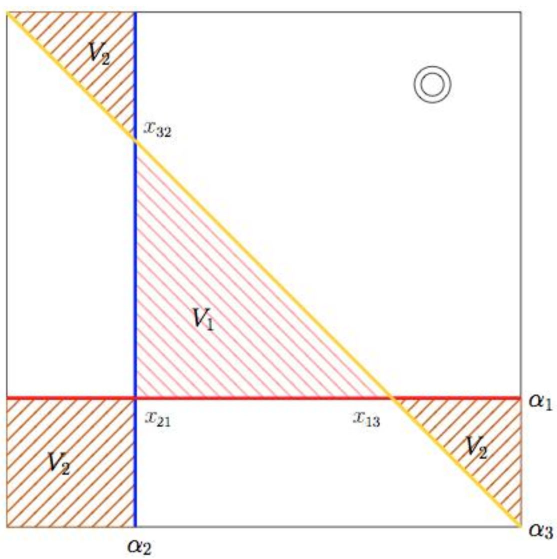


Figure 11: Three simple closed curves on a punctured torus.

2.4 Directed Hochschild complex of a sequence of curves

Finally, we define the directed Hochschild complex of a sequence of curves.

Definition 7. *Let* A *denote the directed Fukaya category associated to* $(\alpha_1, \ldots, \alpha_n)$*. The* *vector space of the* ***directed Hochschild complex*** *of* $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ *for* A *is:*

$$\overrightarrow{HC}(A; \alpha_1, \ldots, \alpha_n) = \bigoplus_{\substack{n \geq r > 0 \\ i_0 > i_1 > \cdots > i_r}} \overrightarrow{CF}(\alpha_{i_r}, \alpha_{i_0}) \otimes \big[\overrightarrow{CF}(\alpha_{i_{r-1}}, \alpha_{i_r}) \otimes \cdots \cdots \otimes \overrightarrow{CF}(\alpha_{i_0}, \alpha_{i_1})\big]$$

For convenience, we denote: $b \otimes a_r \otimes \ldots \otimes a_1$ *by*

$$b[a_r \mid \ldots \mid a_1],$$

where $a_j \in \overrightarrow{CF}(\alpha_{i_{j-1}}, \alpha_{i_j})$ *and* $b \in \overrightarrow{CF}(\alpha_{i_n}, \alpha_{i_0})$.

There are some cases where some perturbed curves are considered. For a curve α_i, we will use α_i' to denote a perturbation of α_i such that α_i intersects α_i' exactly twice. We define the composition maps including the perturbations of curves as below.

Definition 8. *Define the composition maps*

$$\mu^d : \overrightarrow{CF}(\alpha_{i_{d-1}}, \alpha_{i_0}') \otimes \overrightarrow{CF}(\alpha_{i_{d-2}}, \alpha_{i_{d-1}}) \otimes \ldots \otimes \overrightarrow{CF}(\alpha_{i_0}, \alpha_{i_1}) \to \overrightarrow{CF}(\alpha_{i_0}, \alpha_{i_0}')$$

where $i_0 > i_1 > \ldots > i_{d-1}$ *and* $i_d = i_0$ *as in the similar way to Definition 6 with* α_{i_d} *replaced by* α_{i_0}'.

Example 3. *Let us provide an example of a surface with some perturbed curves. Let* α_1, α_2, *and* α_3 *be the same curves as in Example 2. The perturbed curves are labeled with primes and represented by the thinner lines in Figure 12. As shown above there are no bigons formed by the non-perturbed curves. There are, however, bigons formed by each* α_i *and* α_i'. *For example,* U_1 *and* U_2 *are both bigons from* t_1 *to* s_1, *so* $n(t_1, s_1) \equiv 0 \bmod 2$. *(Figure 13 (a).) Similarly, there are exactly two bigons from* t_2 *to* s_2 *and two bigons from* t_3 *to* s_3 *as this is always the case for the two intersection points of a perturbation.*

Now, let us provide the definition of the differential here.

Definition 9. *The* ***differential***

$$D : \overrightarrow{HC}_*(A; \alpha_1, \ldots, \alpha_n) \to \overrightarrow{HC}_*(A; \alpha_1, \ldots, \alpha_n)$$

is defined as below:

$$D(b[a_r \mid \ldots \mid a_1]) = \sum_{i+j \leq r} b[a_r \mid \ldots \mid \mu^j(a_{i+j}, \ldots, a_{i+1}) \mid \ldots \mid a_1] + \sum_{i+j \leq r} \mu^{r-j+1}(a_i, \ldots, a_1, b, a_r, \ldots, a_{i+j+1})[a_{i+j} \mid \ldots \ldots \mid a_{i+1}].$$

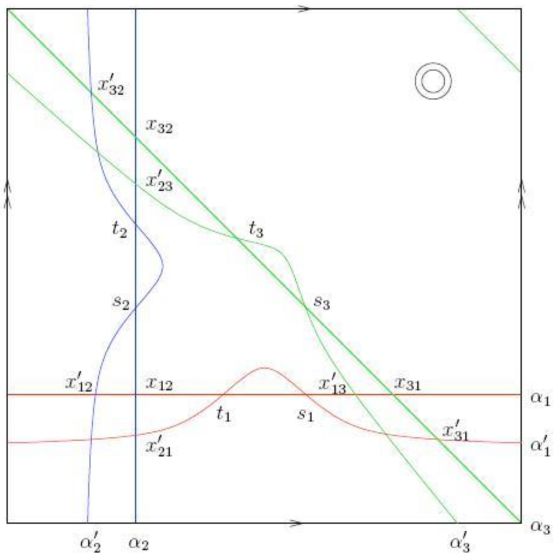


Figure 12: Three simple closed curves and their perturbed curves on a punctured torus.

The fact that $D^2 = 0$ *follows from the fact that* m *and* μ *satisfy the* A_∞ *relations. The Hochschild homology* $\overrightarrow{HH_*}$ *is the homology of* $\overrightarrow{HC_*}$ *with the differential* D.

Example 4. *Now, we use this differential to compute the directed Hochschild homology, i.e.,* $\overrightarrow{HH_*}$, *of the curves shown in Figure 12. As noted above, there are exactly two bigons from each* t_i *to* s_i. *Therefore* $n(t_i, s_i) \equiv 0$ *mod 2 for* $i = 1, 2, 3$. *The triangles give rise to the following multiplications:*

$$\begin{array}{lll} \mu_2(x'_{31}, x_{13}) = t_1 & \mu_2(x'_{12}, x_{21}) = t_2 & \mu_2(x'_{32}, x_{23}) = t_2 \\ \mu_2(x'_{13}, x_{31}) = t_3 & \mu_2(x'_{13}, x_{31}) = t_1 & \mu_2(x'_{13}, x_{31}) = t_3 \end{array}$$

Thus, the differentials are given by:

$$\begin{aligned} D[x_{13}[x_{31}]] &= \mu_2(x'_{13}, x_{31}) + \mu_2(x'_{31}, x_{13}) = t_3 + t_1 \\ D[x_{23}[x_{32}]] &= \mu_2(x'_{23}, x_{32}) + \mu_2(x'_{32}, x_{23}) = t_2 + t_3 \\ D[x_{12}[x_{21}]] &= \mu_2(x'_{12}, x_{21}) + \mu_2(x'_{21}, x_{12}) = t_2 + t_1 \end{aligned}$$

Therefore, we have the following directed graph for the Hochschild complex as in Figure 14.

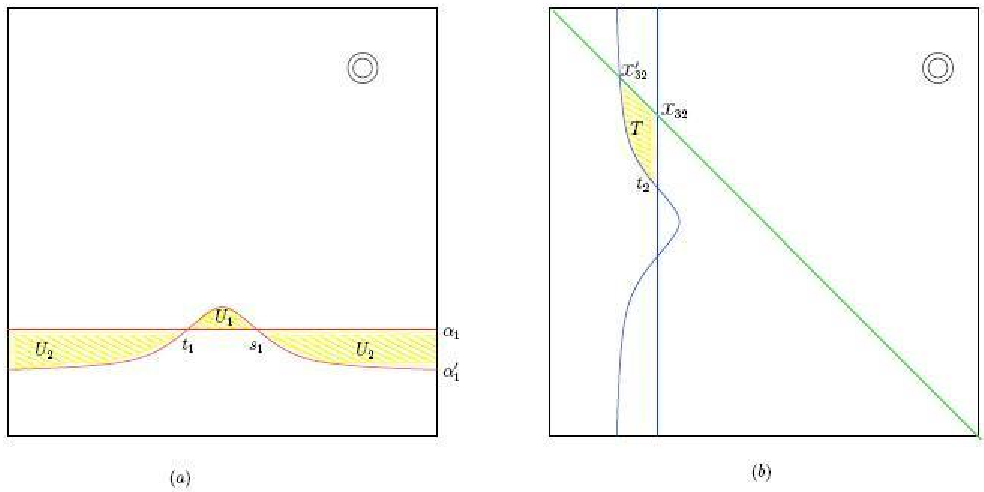


Figure 13: (a) Bigons formed by α_1 and α_1'. (b) A triangle formed by three curves.

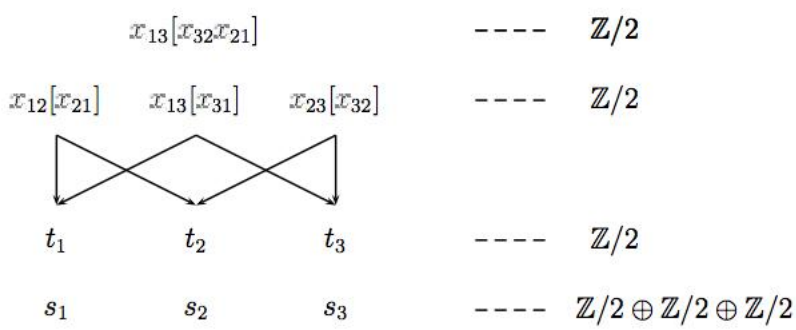


Figure 14: Directed graph

The arrows denote the differential map. After simplifying the diagram we find that the total dimension of the Hochschild homology is 6, i.e.

$$\overrightarrow{HH}_*(A, \alpha_1, \alpha_2, \alpha_3) \cong (\mathbb{Z}/2)^6$$

3 Sage program: explanation and examples

In this section we provide the instructions to use our program: inputs, outputs, and examples. Homology.sage computes the homology of a general digraph of a chain complex over $\mathbb{Z}/2$, represented by a digraph as in Section 3.1. The information for creating a digraph specific to computing the Hochschild homology is provided by FukayaAlg.sage.

Code 1 (Homology.sage). *This code checks whether or not the given digraph represents a complex; i.e., whether $\partial^2 = 0$. If it does, the code computes the dimension of the homology of the complex.*

The inputs are the generators and the arrows of the given digraph. See example 5.

Example 5.

```
sage: egCx=ChainCx({0:[1,2], 1:[3], 2:[3], 3:[]})
sage: egCx.is_complex()
True
sage: egCx.Homology()
Homology is  0  dimensional
```

All the integers in ChainCx(...) represent generators. Then, "0:[1,2]," for example, represents the two arrows starting at 0 and ending at 1 and 2, respectively. Thus, the complex has four generators 0, 1, 2, 3 and four arrows $0 \to 1, 0 \to 2, 1 \to 3, 2 \to 3$.

If the given digraph does not represent a complex, then the code will return an exception. See example 6.

Example 6.

```
sage: ChainCx({0:[1,2], 1:[], 2:[3], 3:[]})
Traceback (click to the left for traceback)
...
Exception: Does not define a chain complex -- d squared not
zero.
```

This case fails since $\partial^2 \neq 0$.

Code 2 (FukayaAlg.sage).
Inputs:

```
1:given_number
2:intersection_dict
3:given_polygons
```

1 is the number of curves. 2 is a dictionary; the keys are the curves in tuples of integers, and the values are the intersection points in strings. 3 is a list of polygons, and each polygon is stored as a list of tuples. See example 7.

Example 7. *This example corresponds to Figure 15. In general, we suggest drawing a diagram of the curves as above, and labelling the points as follows. For each pair of non-perturbed curves α_i and α_j, $i < j$, label the intersection points between α_i and α_j by $x_{ji}, y_{ji}, \ldots$. Thus, for example, the curves 1 and 3 intersect twice; we label the two intersection points x_{31} and*

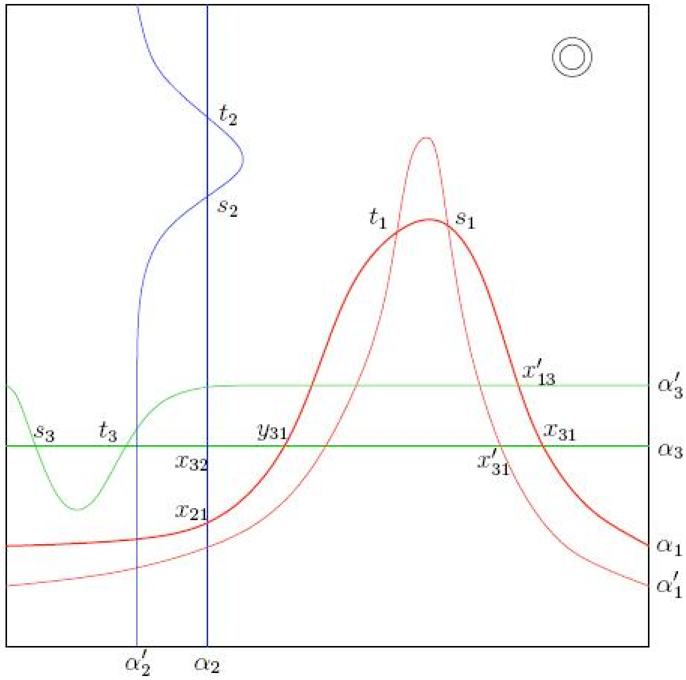


Figure 15: Three simple closed curves and their perturbed curves on a punctured torus.

y_{31}. *Next, the intersection points of one perturbed curve with a different non-perturbed curve have one alphabet letter, but indices written according to the curve that is perturbed; thus* x'_{31} *denotes the intersection point of non-perturbed curve 3 and perturbed curve 1. If a perturbed curve and a different non-perturbed curve intersect in more than one point, the intersections have the same indices but different alphabet letters. For example, the two intersection points of non-perturbed curve 3 and perturbed curve 1 are written as* x'_{31} *and* y'_{31} *Similarly,* x'_{32} *denotes the intersection point of non-perturbed curve 3 and perturbed curve 2. We do not consider the intersection points between two perturbed curves. The last case is the intersection point between a curve and its own perturbation. This is a vertex of a bigon; if the bigon "finger" points right, then the top vertex is given the letter* t*, and the bottom vertex is given the letter* s*. The indices of* t *and* s *are just the curve.*

We now give instructions on how to write each intersection point in our program. In general, notations for the intersection points between two non-perturbed curves are flexible; depending on the choice of matching numbers between adjacent two tuples, we write one of (i,j,'k') and (j,i,'k') for the intersection point x_{ij}*, where 'k' is the alphabet letter denoting the point. However, for the intersection points between one non-perturbed curve and one perturbed curve, we write (i,j,'k') for the intersection point* x'_{ij}*. In this example, we write either (2,1,'b') or (1,2,'b') for* x_{21} *and (2,1,'b') for* x'_{21}*, and we write either (1,3,'c') or (3,1,'c') for* y_{31} *and (3,1,'c') for* y'_{31}*. Observe, however, that the only one of the two choices for the notation of the intersection point between two non-perturbed curves is chosen so that it matches the number of the previous or following tuple. See the following example of inputs.*

```
sage: Tim=FukayaAlg(3,{(1,3):['a','c'],(2,3):['d'],(1,2):['b']},
[[],[[(2,1,'b'),(1,3,'c'),(3,2,'d')],[(3,2,'d'),(2,3,'d'),
(3,3,'t')],[(2,3,'d'),(3,2,'d'),(2,2,'t')],[(2,1,'b'),(1,2,'b'),
(2,2,'t')],[(1,2,'b'),(2,1,'b'),(1,1,'t')],[(3,1,'a'),(1,3,'a'),
(3,3,'t')],[(1,3,'a'),(3,1,'a'),(1,1,'t')],[(3,1,'c'),(1,3,'c'),
(3,3,'t')],[(1,3,'c'),(3,1,'c'),(1,1,'t')]]])
```

The inputs are the number of curves (3 in this case;) the dictionary of curves and intersection points; and the dictionary of the polygons bounded by the curves, arranged in increasing order according to number of vertices.

Thus in the first bracket after the 3, we list the four intersection points. α_i *corresponds to integer* i*, and the intersection points* x_{31} *and* y_{31} *correspond to the strings a and c. Following this dictionary of intersection points, we write [] for 0 bigons. The triangles are listed in the next square bracket; each triangle has its own bracket, with three triples, each labelelled (i,j,'k'), representing the intersection points. The order of the triples should accord with the orientation of the polygon; and they should always be written in counter-clockwise order. For example, to represent the triangle* x_{21}, y_{31}, x_{32}*, write [(2,1,'b'),(1,3,'c'),(3,2,'d')] instead of [(2,1,'b'),(3,2,'d'),(1,3,'c')]. The most important rule is that, between the two adjacent tuples, the ending number of the first tuple should match the starting number of the second tuple. This is the reason that one of the two choices for the notation of the intersection point between two non-perturbed curves is determined. For example, if we try to input the triangle of* y_{31}*,* y'_{31}*, and* t_1*, we first write the fixed tuples for* y'_{31} *and* t_1 *as [,(3,1,'c'),(1,1,'t')] in the bracket and finally choose a notation for* y_{31} *between two choices (1,3,'c') and (3,1,'c'). In this case, we choose (1,3,'c') instead of (3,1,'c') because the ending number of the tuple for* y_{31} *should match the starting number of next tuple (3,1,'c'). Therefore, the representation for the triangle is [(1,3,'c'), (3,1,'c'),(1,1,'t')]. Note that we do not input the triangles which have vertices* s_i*, since the information of the triangles which have vertices* t_i *is enough for the dictionary.*

When we compute the differential of each generator, we just follow the rules different from the rules for inputs. First, if we try to compute $D[x_1[x_2x_3]]$*, we input the tuples like "differential([*x_1*,* x_3*,* x_2*])"; i.e., when we input the tuple of each* x_i*'s, we flip the order of* x_i*'s in the inner bracket. Second, unlike from the rules for inputs of the dictionary of curves and intersection points, we just write (i,j,'a') for every* x_{ij} *in the inputs of differential without any reversing of* i *and* j*. Finally, when we read the result of the differential, an inner bracket refers to a generator which consists of tuples and a comma between inner brackets means adding the generators. The following are examples for the differential.*

```
sage: Tim.differential([(1, 3, 'a'), (2, 1, 'b'), (3, 2, 'd')])
[[(1, 3, 'a'), (3, 1, 'c')]]
sage: Tim.differential([(1, 3, 'c'), (2, 1, 'b'), (3, 2, 'd')])
[[(2, 3, 'd'), (3, 2, 'd')],
 [(1, 3, 'c'), (3, 1, 'c')],
 [(1, 2, 'b'), (2, 1, 'b')]]
```

```
sage: Tim.differential([(1, 2, 'b'), (2, 1, 'b')])
[[(2, 2, 't')], [(1, 1, 't')]]
sage: Tim.differential([(1, 3, 'a'), (3, 1, 'a')])
[[(3, 3, 't')], [(1, 1, 't')]]
```

Since $D[x_{13}[x_{32}x_{21}]] = x_{13}[y_{31}]$*, we input "Tim.differential([(1, 3, 'a'), (2, 1, 'b'), (3, 2, 'd')])" and get [[(1, 3, 'a'), (3, 1, 'c')]]. Similarly, since* $D[y_{13}[x_{32}x_{21}]] = x_{23}[x_{32}] + y_{13}[y_{31}] + x_{12}[x_{21}]$*, we input "Tim.differential([(1, 3, 'c'), (2, 1, 'b'), (3, 2, 'd')])" and get [[(2, 3, 'd'), (3, 2, 'd')], [(1, 3, 'c'), (3, 1, 'c')], [(1, 2, 'b'), (2, 1, 'b')]]. Finally, since* $D[x_{12}[x_{21}]] = t_2 + t_1$*, we input "Tim.differential([(1, 2, 'b'), (2, 1, 'b')])" and get [[(2, 2, 't'), (1, 1, 't')],*

```
sage: G=Tim.GetDigraph()
sage: G
Digraph on 14 vertices
sage: G.edges()
[(((1, 2, 'b'), (2, 1, 'b')), ((1, 1, 't'),), None),
 (((1, 2, 'b'), (2, 1, 'b')), ((2, 2, 't'),), None),
 (((1, 3, 'a'), (2, 1, 'b'), (3, 2, 'd')), ((1, 3, 'a'),
(3, 1, 'c')), None), (((1, 3, 'a'), (3, 1, 'a')),
((1, 1, 't'),), None), (((1, 3, 'a'), (3, 1, 'a')),
((3, 3, 't'),), None), (((1, 3, 'c'), (2, 1, 'b'),
(3, 2, 'd')), ((1, 2, 'b'), (2, 1, 'b')), None),
 (((1, 3, 'c'), (2, 1, 'b'), (3, 2, 'd')), ((1, 3, 'c'),
(3, 1, 'c')), None), (((1, 3, 'c'), (2, 1, 'b'),
(3, 2, 'd')), ((2, 3, 'd'), (3, 2, 'd')), None),
 (((1, 3, 'c'), (3, 1, 'c')), ((1, 1, 't'),), None),
 (((1, 3, 'c'), (3, 1, 'c')), ((3, 3, 't'),), None),
 (((2, 3, 'd'), (3, 2, 'd')), ((2, 2, 't'),), None),
 (((2, 3, 'd'), (3, 2, 'd')), ((3, 3, 't'),), None)]
sage: G.Homology()
Homology is 6
```

Finally, through the "GetDigraph" function, we check that Tim has 14 vertices, and through the "Homology" function, we find that the Hochschild homology of the digraph Tim has dimension 6.

The following is the figure of another example and the corresponding input for the code "Fukaya.sage".

Example 8. *This example corresponds to Figure 16. Since this punctured torus has two copies of the triangle with vertices [(1,3,'c'), (3,2, 'b'), (2,1, 'a')] cancelling each other, there are only 6 triangles each created by two non-perturbed curves and one perturbed curve. Therefore, the corresponding input for the figure above is:*

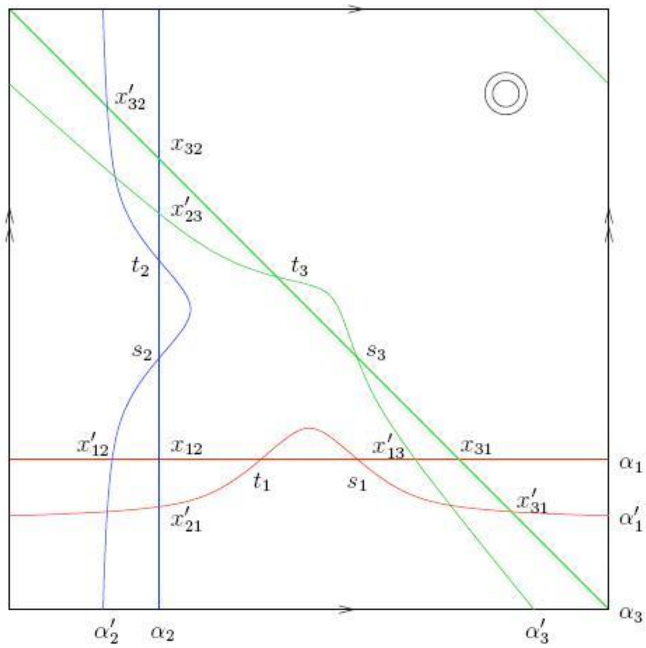


Figure 16: Three simple closed curves and their perturbed curves on a punctured torus.

```
sage: F=FukayaAlg(3,{(1,2):['a'],(2,3):['b'],(1,3):['c']},[[],[[(1,2,'a'),
(2,1,'a'),(1,1,'t')],[(1,3,'c'),(3,1,'c'),(1,1,'t')],[(2,1,'a'),(1,2,'a'),
(2,2,'t')],[(2,3,'b'),(3,2,'b'),(2,2,'t')],[(3,1,'c'),(1,3,'c'),(3,3,'t')],
[(3,2,'b'),(2,3,'b'),(3,3,'t')]]])
```

4 Appendix

4.1 Sage code

We now provide links to to Sage programs. Both include examples (in quotes) alongside the actual code.

Code 3 (Homology.sage)**.** *This code checks whether given chain graph is a complex and calculates the dimension of the homology of a complex.*

The URL:

`http://math.columbia.edu/~topology/programs/Homology.sage`

Code 4 (FukayaAlg.sage)**.** *This code computes the hochschild homology of a given complex.*

The URL:

`http://math.columbia.edu/~topology/programs/FukayaAlg.sage`

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