An Exploration of the Application of the Banzhaf Power Index to Weighted Voting Systems

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1. Introduction

In the United States Electoral College, each state is allocated as many votes as it has members of Congress. In many companies, stockholders’ votes are based on the number of shares they own. These are two examples of weighted voting systems. A weighted voting system (WVS) has \( n \) players, with the \( i^{th} \) player denoted \( P_i \). Player \( P_i \) has \( v_i \) votes, and \( v_1 \geq v_2 \geq \ldots \geq v_n > 0 \) [3]. The quota, \( q \), is the number of votes needed to pass a measure. This research focuses on how changing a quota in a WVS of up to five players changes the power of the players. This is directly applicable to any organization that uses a WVS. Specifically, this research looks at what happens to a WVS with fixed \( v_i \)'s as the quota is increased from its minimum to its maximum.

There are two restrictions on the quota in a valid WVS, and they determine \( q \)'s extrema. First, the quota has to be strictly greater than half of the sum total of votes, giving us

\[
q_{\text{min}} = \left\lfloor \frac{\sum_{i=1}^{n} v_i}{2} \right\rfloor + 1.
\]

Second, no player can have veto power. The sum of the votes of every player except one must equal or exceed the quota, giving us

\[
\sum_{i=1}^{n} v_i \leq q - 1.
\]

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1 The electoral college includes all fifty states plus the District of Columbia. Most states put all their electoral college votes towards the candidate who received the most votes from that state. The exceptions are Maine and Nebraska, which award two votes to the state-wide winner and one vote to the winner of each congressional district. Thus the players in the U.S. Electoral College are the fifty states, the District of Columbia, and the congressional districts of Maine and Nebraska. [4]
\[ q_{\text{max}} = v_2 + v_3 + \ldots + v_n. \]

These restrictions tell us \( v_2 + v_3 + \ldots + v_n \geq q > \left(\frac{1}{2}\right)(v_1 + v_2 + \ldots + v_n). \) Thus \( q > \frac{1}{2}v_1 + \frac{1}{2}(v_2 + v_3 + \ldots + v_n) \geq \frac{1}{2}v_1 + \frac{1}{2}q, \) and therefore \( q > v_1. \) This means that no single player has enough votes to pass a measure alone.

Power is determined by the Banzhaf Power Index. In a 1965 article in the *Rutgers Law Review*, John Banzhaf used the idea of a WVS to create a formal measure of a player’s power [1]. He showed that three of the six members of the Nassau County Board of Supervisors had effectively no power, even though their votes were proportional to the population of their municipalities. To understand the Banzhaf Power Index it is first necessary to understand coalitions.

In a WVS, players form a coalition by all voting the same way. If their vote total equals or exceeds the quota, the coalition is a *winning coalition*. If the players’ votes sum to less than the quota, it is a *losing coalition*. The complement of every winning coalition is a losing coalition; the complement of a losing coalition is not necessarily a winning coalition [3].

A WVS is denoted \([q; v_1, v_2, \ldots, v_n]\. \) Consider the example \([103; 76, 51, 42, 36]\. \) The winning coalitions are \( \{P_1, P_2\}, \{P_1, P_3\}, \{P_1, P_4\}, \{P_1, P_2, P_3\}, \{P_1, P_2, P_4\}, \{P_1, P_3, P_4\}, \{P_2, P_3, P_4\}, \) and \( \{P_1, P_2, P_3, P_4\}. \) The losing coalitions are \( \{P_2, P_3\}, \{P_2, P_4\}, \) and \( \{P_3, P_4\}. \) In this example, the quota happens to be \( q_{\text{min}} = 103; \) any choice of quota between 103 and \( q_{\text{max}} = 129 \) is valid for these values of \( v_1. \) Intuitively, a player matters to a winning coalition if that coalition would lose if the player left.
The Banzhaf Power Index is calculated as follows. A player $P_i$ is *critical* to a winning coalition if the coalition would become a losing coalition if $P_i$ left [3]. In the example [103; 76, 51, 42, 36] above, $P_1$ is critical to the winning coalition $\{P_1, P_2, P_3\}$ because $\{P_2, P_3\}$ is a losing coalition. A player $P_i$’s Banzhaf Power Index $B(P_i)$ is the ratio of the number of times that $P_i$ is critical to the total number of critical instances in the WVS. The example [103; 76, 51, 42, 36] has 12 critical instances, shown in Table 1.

<table>
<thead>
<tr>
<th>Winning Coalitions</th>
<th>Critical Players</th>
</tr>
</thead>
<tbody>
<tr>
<td>${P_1, P_2, P_3, P_4}$</td>
<td>None</td>
</tr>
<tr>
<td>${P_1, P_2, P_3}$</td>
<td>$P_1$</td>
</tr>
<tr>
<td>${P_2, P_3, P_4}$</td>
<td>$P_1$</td>
</tr>
<tr>
<td>${P_1, P_2}$</td>
<td>$P_1, P_2$</td>
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<tr>
<td>${P_1, P_3}$</td>
<td>$P_1, P_3$</td>
</tr>
<tr>
<td>${P_1, P_4}$</td>
<td>$P_1, P_4$</td>
</tr>
</tbody>
</table>

Table 1: This shows the winning coalitions and the corresponding critical instances for WVS = [103; 76, 51, 42, 36].

Player $P_1$ has six critical instances and each of the other players has two. Thus the Banzhaf Power Distribution (the list of the individual power indices) for this WVS is $(1/2, 1/6, 1/6, 1/6)$. Notice that $P_2, P_3,$ and $P_4$ have the same power, even though they don’t have the same number of votes.

2. **Banzhaf Power Distributions for Three, Four, and Five Player WVS**

The possible Banzhaf Power Distributions for WVS with three, four, and five players have previously been determined and are described below [2, 3].
The simplest weighted voting system has only three players. To avoid either giving a player veto power or allowing a single player to pass a measure, each set of two players must be a winning coalition. There are three different two player winning coalitions yielding six critical instances; the Banzhaf Power Distribution is \((\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\).

With four players, there are five different Banzhaf Power Distributions. They are listed below in the form \((B(P_1), B(P_2), B(P_3), B(P_4))\) followed by an example of a WVS \([q; v_1, v_2, v_3, v_4]\) with that power distribution.

(a): \((\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})\) \([3; 1, 1, 1, 1]\]
(b): \((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)\) \([4; 2, 2, 2, 1]\]
(c): \((\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})\) \([3; 2, 1, 1, 1]\]
(d): \((\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})\) \([4; 2, 2, 1, 1]\]
(e): \((\frac{5}{12}, \frac{1}{4}, \frac{1}{6}, \frac{1}{12})\) \([5; 3, 2, 2, 1]\]

The Banzhaf Power Distribution of a four-player WVS is determined by its winning two-player coalitions. In case (a), none of the two player coalitions win. In case (d), the only winning two player coalition is \(\{P_1, P_2\}\). Case (e) has two winning two player coalitions, \(\{P_1, P_2\}\) and \(\{P_1, P_3\}\). There are two possibilities for three winning two player coalitions: case (c) has \(\{P_1, P_2\}\), \(\{P_1, P_3\}\), and \(\{P_1, P_4\}\) win, while case (b) has \(\{P_1, P_2\}\), \(\{P_1, P_3\}\), and \(\{P_2, P_3\}\) win. Since \(\{P_1, P_4\}\) and \(\{P_2, P_3\}\) are complements, they cannot simultaneously win. For clarity, here is a list of the winning coalitions for each case, excluding the trivial winning coalition \(\{P_1, P_2, P_3, P_4\}\):

(a) \(\{P_1, P_2, P_3\}\), \(\{P_1, P_2, P_4\}\), \(\{P_1, P_3, P_4\}\), \(\{P_2, P_3, P_4\}\)
(b) \(\{P_1, P_2\}\), \(\{P_1, P_3\}\), \(\{P_2, P_3\}\), \(\{P_1, P_2, P_3\}\), \(\{P_1, P_2, P_4\}\), \(\{P_1, P_3, P_4\}\), \(\{P_1, P_3, P_4\}\), \(\{P_2, P_3, P_4\}\)
In a five player WVS, there are 35 different possible Banzhaf Power Distributions. For the complete list of cases, their corresponding winning coalitions, their Banzhaf Power Distributions, and a simple example of each see Appendix A.

Just as the four player cases were distinguished by the winning two player coalitions, the 35 different five player cases are split into five different parts based on two player winning coalitions. The five different parts are then broken down into individual cases characterized by winning three player coalitions. Part 1 consists of cases (1a) – (15), all of which have no winning two player coalitions. Another case that is associated with Part 1 is the case where no three player coalitions win, case (1b), with BPI = \((\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})\) [2]. Part 2, cases (16) – (25), consists of all of the cases with \(\{P_1, P_2\}\) as the only winning two player coalition. Part 3 is made up of cases (26) – (31); in these cases both \(\{P_1, P_2\}\) and \(\{P_1, P_3\}\) are winning coalitions. Part 4, cases (32) – (34), has three winning two player coalitions. All of the cases in this part have \(\{P_1, P_2\}\) and \(\{P_1, P_3\}\) win, but cases (32) and (34) have \(\{P_1, P_4\}\) as the third winning two player coalition while case (33) has \(\{P_2, P_3\}\) win. Part 5 has one case, (35), with \(\{P_1, P_2\}\), \(\{P_1, P_3\}\), \(\{P_1, P_4\}\) and \(\{P_1, P_5\}\) representing winning coalitions.

### 3. Changing the Quota

How does the choice of quota affect the Banzhaf Power Distribution? We can see how different quotas affect Banzhaf Power Distribution by sweeping the quota from its minimum to its
maximum and fixing $v_i$. The next few sections show the different outcomes of increasing the quota.

### 3.1 Four Player WVS

First, an example. The WVS $[103; 76, 51, 42, 36]$ is in case (c) at the minimum quota of 103 because $\{P_1, P_4\}$ is a winning coalition. If the quota is raised to $q = 113$, the coalition $\{P_1, P_4\}$ will lose and the WVS will move to case (e). If the quota is raised by an additional six, to 119, the only winning two player coalition is $\{P_1, P_2\}$, giving case (d). Finally, the WVS is in case (a) when $q = 127$.

In general, if the WVS has $v_2 + v_3 > v_1 + v_4$ with $v_2 + v_3 > q_{min}$, it will start in case (b), and follow the path (b) $\rightarrow$ (e) $\rightarrow$ (d) $\rightarrow$ (a). On the other hand, if the WVS has $v_1 + v_4 < v_2 + v_3$ with $v_1 + v_4 < q_{min}$, it will start in case (c) and follow the path (c) $\rightarrow$ (e) $\rightarrow$ (d) $\rightarrow$ (a).

The paths that can be followed in the four player WVS are illustrated by the *path diagram* in Figure 1. The edges indicate the kinds of case changes that are possible as the quota is increased for a fixed choice of $v_i$’s. For example, if a WVS is in case (d), an increase in the quota may move it to case (a), but will not move it to cases (c), (b), or (e).
A WVS will not necessarily be in case (b) or (c) at minimum quota, nor will it necessarily be in case (a) at maximum quota. For example, a WVS with $v_1 > v_3 + v_4$ will not be in case (a) at maximum quota because that inequality implies that $v_1 + v_2 > v_2 + v_3 + v_4$. This means that $\{P_1, P_2\}$ is a winning coalition at the maximum quota of $v_2 + v_3 + v_4$. The case in which a WVS is at the maximum quota is called a stopping case; for a four player WVS the stopping cases are case (a) and case (d). This idea will be discussed in further detail in Section 4.

3.2 Changing the Quota for a Five Player WVS

Changing the quota in the five player WVS yields much more complicated paths. The path diagrams are separated based on the different parts: Part 1, Part 2, and Part 3-5. Figure 2 shows the path diagram for the cases in Part 1 under the additional assumption that no two players have
the same number of votes. We’ll describe the reasoning behind the top portion of the diagram; the rest is similar. In Part 1, a WVS starting in case (1a) has ten winning three player coalitions. The only case with nine winning three player coalitions is case (2), so case (1) has a path to case (2). Similarly, case (2) has a path to case (3) because case (3) is the only case with eight winning three player coalitions in Part 1. After case (3), there are two cases with seven winning three player coalitions. If \( v_2 + v_3 > v_1 + v_4 \), the WVS will follow the path on the right from case (3) to case (5) because in case (5) \( \{P_1, P_4, P_5\} \) loses rather than \( \{P_2, P_3, P_5\} \). If the WVS had the initial condition that \( v_1 + v_4 > v_2 + v_3 \), then the WVS would follow the left hand path from (3) to case (4). If \( v_1 + v_4 = v_2 + v_3 \), then the WVS would lose two winning three player coalitions with one increase of the quota and the WVS would follow the middle path from case (3) to case (7) which has both \( \{P_1, P_4, P_5\} \) and \( \{P_2, P_3, P_5\} \) lose.

All of the following path diagrams have paths that are labeled with equalities or inequalities; the numbers 1 through 5 represent \( v_1 \) through \( v_5 \). If a path says \( 23 > 14 \), then \( v_2 + v_3 > v_1 + v_4 \). If a path is not labeled, that is the path that the WVS must take when the quota is increased.
Figure 2: This is the 5 player WVS path diagram for cases (1)-(15) and (1b). This does not include any special cases.
The following figure, Figure 3, is the path diagram for Part 1, with special cases considered where there is one set of equality. For simplicity, there is the assumption that, for example, there will not be $v_1 = v_2$ and $v_3 = v_4$, or any other paths with two sets of equalities.

Figure 3: This is the 5 player WVS path diagram for Part 1, case 1 through case (1b), that includes all special cases. All paths are marked.
The paths that a WVS can follow in the other parts have another feature to consider. In a case in Part 2, (16) – (25), when the quota is increased, the WVS can go to a case still in Part 2, or the quota increase can cause \( \{P_1, P_2\} \) to lose, putting the new WVS in Part 1. The dotted paths in Figure 4 symbolize ways that a case in Part 2 can travel to a case in Part 1. It is also necessary to consider special cases such as \( v_2 = v_4 + v_5 \) in addition to individual \( v_i \)'s being equal. Figure 3 shows the possible paths a WVS can follow as its’ quota is increased in Part 2.

Figure 4: This is the 5 player WVS path diagram for Part 2 that includes all special cases and paths down to Part 1. All paths are marked.
Figure 5 represents all of the paths that can be travelled by increasing the quota in Parts 3-5, cases (26) – (35). The dotted lines symbolize quota changes that can put the case in Part 2 or Part 1.

This conclusively shows all of the different pathways a five player WVS can follow as the quota increases to its maximum.
4. Banzhaf Power Distributions that Represent Maximum and Minimum Quota

Since the research focuses on what happens to a WVS with fixed $v_i$'s as the quota is increased from its minimum to its maximum, the extrema are significant. If a case is at the maximum quota for some WVS, it is called a stopping case. If a case is at the minimum quota for some WVS, it is called a starting case. For example, the WVS $[117; 90, 42, 40, 37, 23]$ follows the path $(32) \rightarrow (34) \rightarrow (30) \rightarrow (20) \rightarrow (6)$ as the quota increases to $q_{max}$, 142. This means case (6) is a stopping case and case (32) is a starting case.

**Theorem 4.1**: A five player Banzhaf Power Distribution is stopping case if and only if $\{P_2, P_3, P_4\}$ is a losing coalition.

Proof:

Suppose that case (x) is a stopping case. Assume that case (x) has $\{P_2, P_3, P_4\}$ as a winning coalition, so $v_2 + v_3 + v_4 \geq q_{max}$. In a five player WVS, $q_{max} = v_2 + v_3 + v_4 + v_5$, giving us $v_2 + v_3 + v_4 \geq v_2 + v_3 + v_4 + v_5$. This yields the contradiction that $0 \geq v_5$. Thus if case (x) is a stopping case, then $\{P_2, P_3, P_4\}$ is a losing coalition.

Conversely, every case in which $\{P_2, P_3, P_4\}$ is a losing coalition is a stopping case. We demonstrate this by listing all cases with $\{P_2, P_3, P_4\}$ as losing coalition and providing a corresponding WVS at the maximum quota for each.

Thus all these cases are stopping cases.

Case (6): $[117; 90, 42, 40, 37, 23]$

Case (8): $[213; 110, 60, 55, 50, 48]$

Case (10): $[140; 70, 65, 40, 30, 5]$
Case (12): [242; 100, 99, 50, 48, 45]
Case (13): [197; 100, 80, 65, 42, 10]
Case (14): [180; 80, 79, 60, 33, 20]
Case (15): 142; 65, 41, 38, 35, 29
Case (1b): [145; 54, 46, 39, 32, 28]
Case (20): [5; 3, 2, 1, 1, 1]
Case (23): [130; 100, 70, 30, 20, 10]
Case (24): [120; 70, 65, 31, 19, 5]
Case (25): [6; 3, 3, 1, 1, 1]
Case (30): [6; 4, 2, 2, 1, 1]
Case (31): [8; 5, 3, 3, 1, 1]
Case (34): [7; 5, 2, 2, 2, 1]
Case (35): [7; 5, 2, 2, 2, 1].

**Theorem 4.2:** A five player Banzhaf Power Distribution is a starting case if and only if \( \{ P_1, P_4, P_5 \} \) or \( \{ P_2, P_3, P_5 \} \) is a winning coalition.

Proof: Suppose that case (x) is a starting case. Assume that \( \{ P_1, P_4, P_5 \} \) and \( \{ P_2, P_3, P_5 \} \) are both losing coalitions. This means that there is some WVS with \( v_1 + v_4 + v_5 < q_{min} \) and \( v_2 + v_3 + v_5 < q_{min} \). Since \( q_{min} \) is defined to be strictly greater than half the sum of the \( v_i \)'s, we have

\[
v_1 + v_4 + v_5 \leq \frac{1}{2} (v_1 + v_2 + v_3 + v_4 + v_5)
\]

and
\[ v_2 + v_3 + v_4 \leq \frac{1}{2} (v_1 + v_2 + v_3 + v_4 + v_5). \]

These inequalities can be added to get
\[ v_1 + v_4 + v_5 + v_2 + v_3 + v_5 \leq 2\left[\frac{1}{2} (v_1 + v_2 + v_3 + v_4 + v_5)\right], \text{ giving} \]
\[ v_1 + v_2 + v_3 + v_4 + 2v_5 \leq v_1 + v_2 + v_3 + v_4 + v_5. \]
This implies that \( 2v_5 \leq v_5 \), which can never happen because \( v_5 > 0 \). Thus our assumption was wrong and \( \{P_1, P_4, P_5\} \) and \( \{P_2, P_3, P_5\} \) cannot both lose in a starting case.

Conversely, every case in which either \( \{P_1, P_4, P_5\} \) or \( \{P_2, P_3, P_5\} \) is a winning coalition is a starting case. Here is a list of all such cases: (1), (2), (3), (4), (5), (6), (16), (17), (18), (19), (20), (26), (27), (28), (30), (32), (33), (34), and (35). The corresponding examples of weighted voting systems in Appendix A are all at minimum quota, showing that these are starting cases.

As a result from these theorems, cases (6), (20), (30), (34) and (35) all appear as stopping and starting cases. They all have \( \{P_1, P_4, P_5\} \) win and \( \{P_2, P_3, P_4\} \) lose. The following cases are not starting or stopping cases: (7), (9), (11), (21), and (22). None of these cases have \( \{P_1, P_4, P_5\} \) or \( \{P_2, P_3, P_5\} \) win, and they all have \( \{P_2, P_3, P_4\} \) as a winning coalition.

**Theorem 4.3**: If a five player WVS satisfies \( v_1 < v_4 + v_5 \), then case (1b) is its stopping case.

**Proof**: Suppose we have a five player WVS satisfying \( v_1 < v_4 + v_5 \). The maximum quota is \( (v_2 + v_3 + v_4 + v_5) \). If \( v_1 < v_4 + v_5 \), then \( v_1 + v_2 + v_3 < v_2 + v_3 + v_4 + v_5 \), thus making the three player coalition with the largest collective
number of votes a losing coalition at \( q_{\text{max}} \). Clearly no other three player coalitions can win if \( \{P_1, P_2, P_3\} \) loses. Therefore only the four player coalitions are winning coalitions, putting the system in case (1b) at the maximum quota. ■

The above theorem implies that if a WVS starts in case (1a), which has the property \( v_1 > v_4 + v_5 \), its stopping case will be (1b). A WVS with the inequality \( v_1 \geq v_4 + v_5 \) must stop in a case above (1b) on figure 4’s path diagram. Similarly, a WVS with the condition \( v_1 \geq v_3 + v_5 \) cannot stop in case (15) because case (15) has \( \{P_1, P_2, P_4\} \) lose which gives the contradiction that \( v_1 + v_2 + v_4 < v_2 + v_3 + v_4 + v_5 \). Thus any WVS with \( v_1 \geq v_3 + v_5 \) cannot have a stopping case below case (14) in figure 3.

5. Connection Between the Four and Five Player WVS

In the four player WVS, there were five different Banzhaf Power Distributions possible for valid WVS. Of those five, case (b) has the property that BPI = \( \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0 \right) \). Note that the fourth player has no power and the other three have the BPI of the three player case [3]. Since there are five different possible power distributions for a four player WVS, there are five cases in the 5 player WVS that have B(\( P_5 \)) = 0. These five cases each have a BPI for \( P_1, \ldots, P_4 \) that correspond to cases (a) – (e)’s BPIs in the four player WVS [2]. The conditions to have a 5 player WVS with B(\( P_5 \)) = 0 are: there can be no two player winning coalitions with \( P_5 \), all of the three player coalitions without \( P_5 \) must win, and if \( P_5 \) is present in a three player winning coalition \( \{P_i, P_j, P_5\} \), then \( \{P_i, P_j\} \) must also be a winning coalition. The cases with B(\( P_5 \)) = 0 are (33), (32), (29), (22), and (11) and their BPI’s are determined by which two player coalitions win. The path
diagram, Figure 6, portrays how to travel between these cases. It is a reproduction of the path
diagram for the four player WVS (Figure 1).

Since all of the cases with \( B(P_5) = 0 \) are in separate Parts, a WVS cannot directly travel from
one case with \( B(P_5) = 0 \) to another case with \( B(P_5) = 0 \). Note that each of the in-between
cases has \( B(P_5) = 1/25 \), which is the smallest nonzero BPI for \( P_5 \). Presumably there are thirty-
five different cases in a six player power distribution with \( B(P_6) = 0 \) that correspond to all of the
five player cases in Appendix A.
6. Further Research

We would like to know more about how \( P_1 \)'s power changes in a five player WVS as the quota is increased to its maximum. \( P_1 \) has a special interest because they are in the most winning coalitions. See Appendix 2 for some examples. It appears as though \( P_1 \)'s power must increase before it decreases for a WVS from the Part 1 path diagram. Note that it is not always a smooth path of increasing and then decreasing power; the cases that correspond to the dips in the graphs are the same cases that cannot be a starting or a stopping case. From the graphs and the listed Banzhaf Power Distributions in Appendix 1, \( P_1 \)'s power never goes below 1/5.

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References


Appendix A: 5 Player Cases


1. \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_1, P_3, P_5\}; \{P_1, P_4, P_5\}; \{P_1, P_5, P_4\}; \{P_1, P_5, P_3\}; \{P_2, P_3, P_4\}; \{P_2, P_3, P_5\}; \{P_2, P_4, P_5\}; \{P_3, P_4, P_5\}; \{P_2, P_5, P_3\}; \{P_1, P_2, P_3, P_4\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_4, P_5\}; \{P_1, P_2, P_5, P_3\}; \{P_1, P_3, P_4, P_5\}; \{P_1, P_3, P_5, P_4\}; \{P_2, P_3, P_4, P_5\};
   a. Example: [3; 1, 1, 1, 1, 1]
   b. BPI= (1/5, 1/5, 1/5, 1/5, 1/5)

2. \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_1, P_3, P_5\}; \{P_1, P_4, P_5\}; \{P_2, P_3, P_4\}; \{P_2, P_3, P_5\}; \{P_2, P_4, P_5\}; \{P_3, P_4, P_5\}; \{P_2, P_5, P_3\}; \{P_1, P_2, P_3, P_4\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_4, P_5\}; \{P_1, P_2, P_5, P_3\}; \{P_1, P_3, P_4, P_5\}; \{P_1, P_3, P_5, P_4\}; \{P_2, P_3, P_4, P_5\};
   a. Example: [7; 3, 3, 2, 2, 2]

3. \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_1, P_3, P_5\}; \{P_1, P_4, P_5\}; \{P_2, P_3, P_4\}; \{P_2, P_3, P_5\}; \{P_2, P_4, P_5\}; \{P_3, P_4, P_5\}; \{P_2, P_5, P_3\}; \{P_1, P_2, P_3, P_4\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_4, P_5\}; \{P_1, P_2, P_5, P_3\}; \{P_1, P_3, P_4, P_5\}; \{P_1, P_3, P_5, P_4\}; \{P_2, P_3, P_4, P_5\};
   a. Example: [8; 4, 3, 3, 2, 2]
   b. BPI= (2/7, 3/14, 3/14, 1/7, 1/7)

4. \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_1, P_3, P_5\}; \{P_1, P_4, P_5\}; \{P_2, P_3, P_4\}; \{P_2, P_3, P_5\}; \{P_2, P_4, P_5\}; \{P_3, P_4, P_5\}; \{P_2, P_5, P_3\}; \{P_1, P_2, P_3, P_4\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_4, P_5\}; \{P_1, P_2, P_5, P_3\}; \{P_1, P_3, P_4, P_5\}; \{P_1, P_3, P_5, P_4\}; \{P_2, P_3, P_4, P_5\};
   a. Example: [6; 3, 2, 2, 2, 1]
   b. BPI = (1/3, 5/27, 5/27, 5/27, 1/9)

5. \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_1, P_3, P_5\}; \{P_1, P_4, P_5\}; \{P_2, P_3, P_4\}; \{P_2, P_3, P_5\}; \{P_2, P_4, P_5\}; \{P_1, P_2, P_3, P_4\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_4, P_5\}; \{P_2, P_3, P_4, P_5\}; \{P_1, P_2, P_5, P_3\}; \{P_1, P_3, P_4, P_5\}; \{P_1, P_3, P_5, P_4\}; \{P_2, P_3, P_4, P_5\};
   a. Example: [5; 2, 2, 2, 1, 1]
   b. BPI = (7/27, 7/27, 7/27, 1/9, 1/9)

6. \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_1, P_3, P_5\}; \{P_1, P_4, P_5\}; \{P_2, P_3, P_4\}; \{P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_4, P_5\}; \{P_1, P_2, P_5, P_3\}; \{P_1, P_3, P_4, P_5\}; \{P_1, P_3, P_5, P_4\};
   a. Example: [4; 2, 1, 1, 1, 1]
   b. BPI = (5/13, 2/13, 2/13, 2/13, 2/13)

7. \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_1, P_3, P_5\}; \{P_2, P_3, P_4\}; \{P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_4, P_5\}; \{P_1, P_2, P_5, P_3\}; \{P_1, P_3, P_4, P_5\}; \{P_1, P_3, P_5, P_4\};
   a. Example: [8; 4, 3, 3, 2, 1]
   b. BPI = (4/13, 3/13, 3/13, 2/13, 1/13)

8. \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_1, P_3, P_5\}; \{P_2, P_3, P_4\}; \{P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_4, P_5\}; \{P_1, P_2, P_5, P_3\}; \{P_1, P_3, P_4, P_5\}; \{P_1, P_3, P_5, P_4\};
   a. Example: [6; 3, 2, 2, 1, 1]
   b. BPI = (9/25, 1/5, 1/5, 3/25, 3/25)
9. \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_2, P_3, P_4\}; \{P_1, P_3, P_4, P_5\}; \{P_1, P_2, P_3, P_4\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_4, P_5\}; \{P_2, P_3, P_4, P_5\}
   a. Example: [7; 3, 3, 2, 2, 1]
   b. BPI = (7/25, 7/25, 1/5, 1/5, 1/25)

10. \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_3, P_4\}; \{P_2, P_3, P_4, P_5\}; \{P_1, P_3, P_4, P_5\}; \{P_1, P_2, P_4, P_5\}; \{P_1, P_2, P_3, P_4\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4, P_5\}
   a. Example: [8; 4, 3, 2, 2, 1]
   b. BPI = (1/3, 1/4, 1/6, 1/6, 1/12)

11. \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_3, P_4\}; \{P_2, P_3, P_4\}; \{P_1, P_3, P_4, P_5\}; \{P_1, P_2, P_4, P_5\}; \{P_1, P_2, P_3, P_4\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4, P_5\}
   a. Example: [6; 2, 2, 2, 2, 1]
   b. BPI = (1/4, 1/4, 1/4, 1/4, 0)

12. \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_3, P_4\}; \{P_2, P_3, P_4, P_5\}; \{P_1, P_3, P_4, P_5\}; \{P_1, P_2, P_4, P_5\}; \{P_1, P_2, P_3, P_4\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4, P_5\}
   a. Example: [5; 2, 2, 2, 2, 1]
   b. BPI = (7/23, 7/23, 2/21, 2/21, 2/23)

13. \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_3, P_4\}; \{P_2, P_3, P_4, P_5\}; \{P_1, P_3, P_4, P_5\}; \{P_1, P_2, P_4, P_5\}; \{P_1, P_2, P_3, P_4\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4, P_5\}
   a. Example: [7; 3, 2, 2, 2, 1]

14. \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_2, P_3, P_4, P_5\}; \{P_1, P_3, P_4, P_5\}; \{P_2, P_3, P_4, P_5\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4\}
   a. Example: [8; 3, 3, 2, 2, 1]
   b. BPI = (3/11, 3/11, 2/11, 2/11, 1/11)

15. \{P_1, P_2, P_3\}; \{P_2, P_3, P_4, P_5\}; \{P_1, P_3, P_4, P_5\}; \{P_2, P_3, P_4, P_5\}; \{P_1, P_3, P_4, P_5\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4\}
   a. Example: [6; 2, 2, 2, 2, 1]
   b. BPI = (5/21, 5/21, 5/21, 1/7, 1/7)

16. \{P_1, P_2\}; \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_1, P_3, P_5\}; \{P_1, P_4, P_5\}; \{P_1, P_2, P_3, P_4\}; \{P_2, P_3, P_4\}; \{P_2, P_3, P_5\}; \{P_2, P_4, P_5\}; \{P_1, P_2, P_4, P_5\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4\}
   a. Example: [4; 2, 2, 2, 2, 1]
   b. BPI = (2/7, 2/7, 1/7, 1/7, 1/7)

17. \{P_1, P_2\}; \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_1, P_3, P_5\}; \{P_1, P_4, P_5\}; \{P_2, P_3, P_4\}; \{P_2, P_3, P_5\}; \{P_2, P_4, P_5\}; \{P_1, P_2, P_4, P_5\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4\}
   a. Example: [9; 5, 4, 3, 2, 2]
   b. BPI = (1/3, 7/27, 5/27, 1/9, 1/9)

18. \{P_1, P_2\}; \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_1, P_3, P_5\}; \{P_1, P_4, P_5\}; \{P_2, P_3, P_4\}; \{P_2, P_3, P_5\}; \{P_2, P_4, P_5\}; \{P_1, P_2, P_4, P_5\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4\}
   a. Example: [8; 5, 4, 2, 2, 1]
b. $\text{BPI} = (5/13, 3/13, 2/13, 2/13, 1/13)$

19. $\{P_1, P_2\}; \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_2, P_3, P_4\}$
   a. Example: $[7; 4, 4, 2, 1, 1]$
   b. $\text{BPI} = (8/25, 8/25, 1/5, 2/25, 2/25)$

20. $\{P_1, P_2\}; \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_2, P_3, P_4\}$
   a. Example: $[5; 3, 2, 1, 1, 1]$

21. $\{P_1, P_2\}; \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_2, P_3, P_4\}$
   a. Example: $[9; 5, 4, 3, 2, 1]$
   b. $\text{BPI} = (9/25, 7/25, 1/5, 3/25, 1/25)$

22. $\{P_1, P_2\}; \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_2, P_3, P_4\}$
   a. Example: $[10; 5, 5, 3, 3, 1]$
   b. $\text{BPI} = (1/3, 1/3, 1/6, 1/6, 0)$

23. $\{P_1, P_2\}; \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_2, P_3, P_4\}$
   a. Example: $[7; 4, 3, 2, 1, 1]$
   b. $\text{BPI} = (5/12, 1/4, 1/6, 1/12, 1/12)$

24. $\{P_1, P_2\}; \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_2, P_3, P_4\}$
   a. Example: $[9; 5, 3, 2, 2, 1]$
   b. $\text{BPI} = (9/23, 7/23, 3/23, 3/23, 1/23)$

25. $\{P_1, P_2\}; \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_2, P_3, P_4\}$
   a. Example: $[6; 3, 3, 1, 1, 1]$
   b. $\text{BPI} = (4/11, 4/11, 1/11, 1/11, 1/11)$

26. $\{P_1, P_2\}; \{P_1, P_3\}; \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_1, P_3, P_5\}$
   a. Example: $[5; 3, 2, 2, 1, 1]$
   b. $\text{BPI} = (5/13, 3/13, 3/13, 1/13, 1/13)$

27. $\{P_1, P_2\}; \{P_1, P_3\}; \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_1, P_3, P_5\}$
   a. Example: $[8; 5, 3, 3, 2, 1]$
   b. $\text{BPI} = (11/25, 1/5, 1/5, 3/25, 1/25)$
28. \{P_1, P_2\}; \{P_1, P_3\}; \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_2, P_3, P_4\}; \{P_2, P_3, P_5\}; \{P_2, P_3, P_4, P_5\}; \{P_1, P_3, P_4, P_5\}; \{P_1, P_2, P_3, P_4\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4, P_5\}
   a. Example: [7; 4, 3, 1, 1]
   b. BPI = (9/25, 7/25, 7/25, 1/25, 1/25)

29. \{P_1, P_2\}; \{P_1, P_3\}; \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_2, P_3, P_4\}; \{P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4, P_5\}
   a. Example: [10; 6, 4, 4, 2, 1]
   b. BPI = (5/12, 1/4, 1/4, 1/12, 0)

30. \{P_1, P_2\}; \{P_1, P_3\}; \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_1, P_4, P_5\}; \{P_2, P_3, P_4\}; \{P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4, P_5\}
   a. Example: [6; 4, 2, 1, 1]
   b. BPI = (1/2, 1/6, 1/6, 1/12, 1/12)

31. \{P_1, P_2\}; \{P_1, P_3\}; \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_1, P_3, P_5\}; \{P_2, P_3, P_4\}; \{P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4, P_5\}
   a. Example: [8; 5, 3, 3, 1, 1]

32. \{P_1, P_2\}; \{P_1, P_3\}; \{P_1, P_4\}; \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_1, P_3, P_5\}; \{P_2, P_3, P_4\}; \{P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4, P_5\}
   a. Example: [6; 4, 2, 2, 1]
   b. BPI = (1/2, 1/6, 1/6, 1/12, 0)

33. \{P_1, P_2\}; \{P_1, P_3\}; \{P_2, P_3\}; \{P_2, P_4\}; \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_3, P_4\}; \{P_1, P_3, P_5\}; \{P_2, P_3, P_4\}; \{P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4, P_5\}
   a. Example: [6; 3, 3, 1, 1]
   b. BPI = (1/3, 1/3, 1/3, 0, 0)

34. \{P_1, P_2\}; \{P_1, P_3\}; \{P_1, P_4\}; \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_1, P_3, P_5\}; \{P_2, P_3, P_4\}; \{P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4, P_5\}
   a. Example: [7; 5, 2, 2, 1]

35. \{P_1, P_2\}; \{P_1, P_3\}; \{P_1, P_4\}; \{P_1, P_5\}; \{P_1, P_2, P_3\}; \{P_1, P_2, P_4\}; \{P_1, P_2, P_5\}; \{P_1, P_3, P_4\}; \{P_1, P_3, P_5\}; \{P_1, P_4, P_5\}; \{P_1, P_2, P_3, P_4\}; \{P_1, P_2, P_3, P_5\}; \{P_1, P_2, P_3, P_4, P_5\}
   a. Example: [7; 5, 2, 2, 1]
Appendix B: Graphs of $B(P_1)$

WVS $[69, 35, 33, 27, 23, 19]$

Case: 1-2-3-5-7-9-10-12-14-15-1b

WVS $[183, 81, 78, 72, 68, 65]$

Case: 1-2-3-5-7-10-12-14-15-1b