GENERALIZATIONS OF GOURSAT'S THEOREM FOR GROUPS

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Abstract. Petrillo's recent article in the College Mathematics Journal explained a theorem of Goursat on the subgroups of a direct product of two groups. In this note, we extend this theorem to commutative rings, and to modules over commutative rings and fields.

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1. Introduction

A recent article by J. Petrillo in the College Mathematics Journal, [4], explained a little-known result of Edouard Goursat that describes subgroups of a direct product of two groups. In this note we will extend the result to apply to the direct product of two commutative rings, and to the direct product of modules over a commutative ring. First we state Goursat's Theorem and outline the proof.

Theorem 1.1. ***Goursat's Theorem*** *If G_1 and G_2 are groups then there exists a bijection between the set S of all subgroups of $G_1 \times G_2$ and the set T of all 5-tuples $(A_1, B_1, A_2, B_2, \varphi)$ where A_i is a subgroup of G_i, B_i is a normal subgroup of A_i, and φ is a group isomorphism mapping A_1/B_1 to A_2/B_2.*

Let G_1 and G_2 be groups. Given a subgroup U of $G_1 \times G_2$ there is a unique 5-tuple $(A_{U_1}, B_{U_1}, A_{U_2}, B_{U_2}, \varphi_U)$ where $B_{U_i} \trianglelefteq A_{U_i}$ are subgroups of G_i and $\varphi_U : A_{U_1}/B_{U_1} \to A_{U_2}/B_{U_2}$ is a group isomorphism. This 5-tuple is formed as follows.

Define the projection maps $\pi_i : G_1 \times G_2 \to G_i$ by

$$\pi_i : (g_1, g_2) \mapsto g_i.$$

Since the π_i are group homomorphisms, then the images of the π_i restricted to U are subgroups of G_i. Furthermore, the image of the kernel of π_i restricted to U is a normal subgroup of G_j with $j \neq i$.

Thus we have that with

$$\begin{aligned} A_{U_1} &= Im(\pi_1|_U) = \{g_1 \in G_1 \mid (g_1, g_2) \in U \text{ for some } g_2 \in G_2\}\,; \\ B_{U_1} &= \pi_1(Ker(\pi_2|_U)) = \{g_1 \in G_1 \mid (g_1, e_{G_2}) \in U\}\,; \\ A_{U_2} &= Im(\pi_2|_U) = \{g_2 \in G_2 \mid (g_1, g_2) \in U \text{ for some } g_1 \in G_1\}\,; \\ B_{U_2} &= \pi_2(Ker(\pi_1|_U)) = \{g_2 \in G_2 \mid (e_{G_1}, g_2) \in U\}\,. \end{aligned}$$

and with $\varphi_U : A_{U_1}/B_{U_1} \to A_{U_2}/B_{U_2}$ defined by

$$\varphi_U(a_1 B_{U_1}) = a_2 B_{U_2} \text{ when } (a_1, a_2) \in U;$$

then φ_U is an isomorphism and so the 5-tuple $(A_{U_1}, B_{U_1}, A_{U_2}, B_{U_2}, \varphi_U)$ lies in T.

Conversely, given $(A_1, B_1, A_2, B_2, \varphi) \in T$ then the set U_φ defined as

$$U_\varphi = \{(a_1, a_2) \in A_1 \times A_2 \mid \varphi(a_1 B_1) = a_2 B_2\}$$

is a subgroup of $G_1 \times G_2$.

Define $\alpha : S \to T$ by $\alpha(U) = (A_{U_1}, B_{U_1}, A_{U_2}, B_{U_2}, \varphi_U)$ and then α is a bijection with inverse $\beta : T \to S; (A_1, B_1, A_2, B_2, \varphi) \mapsto U_\varphi$.

The proof of this theorem is outlined as a series of exercises in [4]; full details can be found in [2].

We are now going to generalize Goursat's Theorem to direct products of commutative rings and to direct products of modules over commutative rings.

2. Preliminaries

We assume that the reader is familiar with elementary group theory. A good reference for this material is [5].

Definition 2.1. A **commutative ring** R is a set with two operations, addition $+$ and multiplication written $\cdot$ or by concatenation such that $(R,+)$ is an abelian group with identity written 0, multiplication is associative, commutative, and has an identity element $1 \neq 0$, and the distributivity property

$$a(b+c) = ab + ac, \text{ holds for every } a, b, c \in R.$$

Definition 2.2. An **ideal** in a commutative ring R is a subgroup $(I,+)$ of $(R,+)$ such that if $a \in I$ and $r \in R$, then $ra \in I$. An ideal I different from R or $\{0\}$ is called a **proper ideal**.

Definition 2.3. If R_1 and R_2 are commutative rings, a **ring homomorphism** is a function $f : R_1 \to R_2$ such that f is a group homomorphism from $(R_1, +_1)$ to $(R_2, +_2)$ and

(i) $f(1) = 1$;
(ii) $f(aa') = f(a)f(a')$ for all $a, a' \in R_1$.

If f is an bijection then it is called a **ring isomorphism** and we write $R_1 \cong R_2$.

Definition 2.4. If $f : R_1 \to R_2$ is a ring homomorphism, then its **kernel** is

$$\text{Ker } f = \{a \in R_1 \mid f(a) = 0\}.$$

Note that Ker f is an ideal in R_1. The **image** of f is

$$\text{Im } f = \{r \in R_2 \mid r = f(a) \text{ for some } a \in R_1\},$$

where Im f is a subring of R_2. [5, p. 248]

Definition 2.5. Let R be a commutative ring. An **R-module** is a set M together with maps $M \times M \to M$, $(m,n) \mapsto m+n$, and $R \times M \to M$, $(r,m) \mapsto rm$, called addition and scalar multiplication such that $(M,+)$ is an abelian group and for all $r, s \in R$ and $m, n \in M$:

(a) $(r+s)m = rm + sm$,
(b) $(rs)m = r(sm)$,
(c) $r(m+n) = rm + rn$.

An **R-submodule** N of M is a subset of M that is also an R-module with the same addition and scalar multiplication. [1, p. 318]

For example, R is itself an $R-$module under the addition in R and with multiplication in R as scalar multiplication. Ideals in R are submodules of the R-module R.

Definition 2.6. Let R be a ring and let M and N be R-modules. A map $\xi : M \to N$ is an **R-module homomorphism** if ξ is a homomorphism of abelian groups from $(M,+)$ to $(N,+)$ and $\xi(rx) = r\xi(x)$, for all $r \in R$, and $x \in M$. We call ξ an **R-module isomorphism** if ξ is a bijection. [1, p. 326]

Definition 2.7. Let R be a ring, M be an R-module and N be a submodule of M. The (additive, abelian) quotient group M/N can be made into an R-module by defining an action of R on M/N by

$$r(x+N) = (rx) + N, \text{ for all } r \in R, x+N \in M/N.$$

The natural projection map $\pi : M \to M/N$ defined by $\pi(x) = x+N$ is an R-module homomorphism with kernel N.

Remark 2.1. The notation $\mathbb{Z}_n$ will be used to denote $\mathbb{Z}/n\mathbb{Z}$, the integers modulo n.

Definition 2.8. A **field** F is a commutative ring with $1 \neq 0$ in which every nonzero element a is a multiplicative unit; that is, there is $a^{-1} \in F$ with $a^{-1}a = 1$. [5, p. 230]

For example, the complex numbers, the real numbers and the rational numbers are fields. The integers form a commutative ring but not a field.

Remark 2.2. If R is a field then an R-module is called a vector space.

3. Goursat's Theorem for the direct product of commutative rings

Let R_1 and R_2 be commutative rings, with multiplicative identities denoted 1_1 and 1_2. It is easy to check that the direct product $R_1 \times R_2$ is also a commutative ring with component-wise operations $\cdot_i$ and $+_i$. We show that there exists a bijection between $\mathcal{S}$ (set of subrings of $R_1 \times R_2$) and $\mathcal{T}$, the set of 5-tuples $(A_1, B_1, A_2, B_2, \varphi)$ where A_i is a subring of R_i, B_i is an ideal in A_i, and $\varphi : A_1/B_2 \to A_2/B_2$ is a ring isomorphism.

Proposition 3.1. Given $V \in \mathcal{S}$, we define A_{V_i} and B_{V_i} as in Section 1 by letting $G_i = R_i$ and $e_{G_i} = 0$. Define $\varphi_V : A_{V_1}/B_{V_1} \to A_{V_2}/B_{V_2}$ by

$$\varphi_V(a_1 + B_{V_1}) = a_2 + B_{V_2} \text{ when } (a_1, a_2) \in V.$$

Then the 5-tuple $(A_{V_1}, B_{V_1}, A_{V_2}, B_{V_2}, \varphi_V)$ lies in $\mathcal{T}$.

Proof. Since the projection π_i is a ring epimorphism, A_{V_i} is the image of $\pi_i|_V$ making A_{V_i} a subring of R_i. For $j \neq i$, the kernel of $\pi_j|_V$ is $(R_1 \times 0) \cap V$ or $(0 \times R_2) \cap V$ so that B_{V_i} is an ideal of A_{V_i}.

By Goursat's Theorem for groups, Theorem 1.1, φ_V is a group isomorphism and therefore a bijection. It remains to show that the ring structure is respected.

Since V is a subring, $(1, 1) \in V$ so then $\varphi_V(1 + B_{V_1}) = 1 + B_{V_2}$ and φ maps the multiplicative unit in A_{V_1}/B_{V_1} to the multiplicative unit in A_{V_2}/B_{V_2}. For the multiplicative property, note that if (a_1, a_2), and (a'_1, a'_2) are in V, then $(a_1a'_1, a_2a'_2) \in V$. Thus,

$$\varphi_V(a_1+B_{V_1})\varphi_V(a'_1+B_{V_1}) = (a_2+B_{V_2})(a'_2+B_{V_2}) = a_2a'_2+B_{V_2} = \varphi_V(a_1a'_1+B_{V_1}).$$

Therefore, we can construct the 5-tuple $(A_{V_1}, B_{V_1}, A_{V_2}, B_{V_2}, \varphi_V)$ from a given subring V of $R_1 \times R_2$.

□

Proposition 3.2. Given a 5-tuple $(A_1, B_1, A_2, B_2, \varphi) \in \mathcal{T}$, the set V_φ defined by $V_\varphi = \{(a_1, a_2) \in R_1 \times R_2 \mid \varphi(a_1 + B_1) = a_2 + B_2\}$ is a subring of $R_1 \times R_2$.

Proof. From Goursat's Theorem, we already know that $(V_\varphi, +)$ is a subgroup of $(R_1 \times R_2, +)$ so we just need to prove it is also a ring. Since φ is a ring isomorphism, then $\varphi(1+B_1) = 1+B_2$ and thus $(1, 1) \in V_\varphi$. Also, V_φ is closed under multiplication because if $(a_1, a_2), (a'_1, a'_2) \in V_\varphi$, then $\varphi(a_1+B_1)\varphi(a'_1+B_1) = (a_2+B_2)(a'_2+B_2) = a_2a'_2 + B_2$ hence $(a_1a'_1, a_2a'_2) \in V_\varphi$.

Therefore V_φ is a subring of $R_1 \times R_2$. □

Theorem 3.1. ***Goursat's Theorem for Commutative Rings***
If R_1 and R_2 are commutative rings then there exists a bijection between the set $\mathcal{S}$ (subrings of $R_1 \times R_2$) and $\mathcal{T}$, the set of all 5-tuples $(A_1, B_1, A_2, B_2, \varphi)$ where B_i is an ideal of A_i, A_i is a subring of R_i, and $\varphi : A_1/B_1 \to A_2/B_2$ is a ring isomorphism.

Proof. Define the mapping $\alpha : \mathcal{S} \to \mathcal{T}$ by: $\alpha(V) = (A_{V_1}, B_{V_1}, A_{V_2}, B_{V_2}, \varphi_V)$; and $\beta : \mathcal{T} \to \mathcal{S}$ by: $\beta(A_1, B_1, A_2, B_2, \varphi) = V_\varphi$.

By Proposition 3, α maps $\mathcal{S}$ to $\mathcal{T}$, by Proposition 3.2, β maps $\mathcal{T}$ to $\mathcal{S}$ and by Goursat's Theorem for groups, α and β are inverse bijections. □

Example 3.1. Subrings of $\mathbb{Z}_2 \times \mathbb{Z}_6$. Table 1 lists the subrings A_1 and A_2, their ideals B_1 and B_2 and the quotients A_1/B_1 and A_2/B_2 that can be formed if $R_1 = \mathbb{Z}_2$ and $R_2 = \mathbb{Z}_6$.

TABLE 1. $R_1 = \mathbb{Z}_2$ and $R_2 = \mathbb{Z}_6$

Subrings of $\mathbb{Z}_2$				Subrings of $\mathbb{Z}_6$			
A_1	B_1	A_1/B_1	Order	A_2	B_2	A_2/B_2	Order
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2/\mathbb{Z}_2$	1	$\mathbb{Z}_6$	$\mathbb{Z}_6$	$\mathbb{Z}_6/\mathbb{Z}_6$	1
$\mathbb{Z}_2$	$\{0\}$	$\mathbb{Z}_2/\{0\}$	2	$\mathbb{Z}_6$	$\langle[2]\rangle$	$\mathbb{Z}_6/\langle[2]\rangle$	2
				$\mathbb{Z}_6$	$\langle[3]\rangle$	$\mathbb{Z}_6/\langle[3]\rangle$	3

The only subring of the form $A_1 \times A_2$ is $\mathbb{Z}_2 \times \mathbb{Z}_6$. Applying Theorem 3.1 the isomorphism $\varphi : \mathbb{Z}_2/\{0\} \to \mathbb{Z}_6/\langle[2]\rangle$ can be used to form the subring

$$U_\varphi = \{(0,0), (0,2), (0,4), (1,1), (1,3), (1,5)\}.$$

Note that this is the only nontrivial subring of $\mathbb{Z}_2 \times \mathbb{Z}_6$.

4. Goursat's Theorem for the direct product of modules

Let R be a commutative ring and M_1, M_2 be R-modules. The direct product $M_1 \times M_2$ is also an R-module with component-wise addition and scalar multiplication given by $r(m_1, m_2) = (rm_1, rm_2)$. Define the projection $\pi_i : M_1 \times M_2 \to M_i$ by $(m_1, m_2) \mapsto m_i$. It is easy to see that this is an R-module homomorphism onto M_i with the kernel of π_i isomorphic to M_j, $j \neq i$.

We apply Goursat's Theorem for groups to find all R-submodules of $M_1 \times M_2$.

Theorem 4.1. *Let M_1 and M_2 be R-modules. There is a one to one correspondence between submodules N of the R-module $M_1 \times M_2$ and 5-tuples $(W_1, S_1, W_2, S_2, \varphi)$ where $S_i \subseteq W_i$ are submodules of M_i and φ is an R-module isomorphism from W_1/S_1 to W_2/S_2.*

Proof. Let N be an R-submodule of $M_1 \times M_2$ and let $(W_1, S_1, W_2, S_2, \varphi : W_1/S_1 \to W_2/S_2)$ be the tuple associated to the abelian group $(N, +)$ by Goursat's Theorem. The image of $\pi_i(N) = W_i$ so the subgroups $(W_i, +)$ of $(M_i, +)$ are R-submodules. The kernel J_i of the map π_i restricted to N is $(0 \times M_j) \cap N$ and $S_j = \pi_j(J_i)$ is a submodule of W_j. That is $S_i \subseteq W_i$ are submodules of M_i. It remains to show that the isomorphism φ from Goursat's Theorem for groups preserves scalar multiplication. Recall that $\varphi(w_1 + S_1) = w_2 + S_2$ if and only if $(w_1, w_2) \in N$. But then $(rw_1, rw_2) \in N$ for any $r \in R$ and thus

$$\varphi(rw_1 + S_1) = rw_2 + S_2 = r\varphi(w_1 + S_1)$$

as required.

Conversely we show that the subgroup N of $(M_1 \times M_2, +)$ constructed from a 5-tuple $(W_1, S_1, W_2, S_2, \varphi : W_1/S_1 \to W_2/S_2)$ is an R-submodule of $M_1 \times M_2$. Let

$(w_1, w_2) \in N$ if and only if $\varphi(w_1 + S_1) = w_2 + S_2$. Then since φ preserves scalar multiplication, $\varphi(rw_1 + S_1) = rw_2 + S_2$ and $(rw_1, rw_2) \in N$. Thus N is closed under componentwise scalar multiplication and the proof of the theorem is complete. □

Recall that if the commutative ring is a field k then a k-module is called a vector space.

Corollary 4.1. For $V_1 \times V_2$ a direct product of k-vector spaces, the subspaces of $V_1 \times V_2$ are in one to one correspondence with the 5-tuples $(W_1, S_1, W_2, S_2, \varphi)$ where S_i is a subspace of W_i, W_i is a subspace of V_i and φ is an isomorphism as defined in Theorem 4.1.

Example 4.1. Using the notation from Theorem 4.1 let $R = \mathbb{Z}_{18}$, then $M_1 \times M_2 = \mathbb{Z}_{18} \times \mathbb{Z}_{18}$ is an R-module. Let $W_1 = W_2 = \langle [2] \rangle$, $S_1 = S_2 = \langle [6] \rangle$, and define $\varphi : W_1/S_1 \to W_2/S_2$ to be the identity map. Then

$$N_\varphi = \langle [6] \rangle \times \langle [6] \rangle \cup ([2] + \langle [6] \rangle \times [2] + \langle [6] \rangle) \cup ([4] + \langle [6] \rangle \times [4] + \langle [6] \rangle),$$

or more explicitly,

$$\begin{aligned} N_\varphi \quad = \quad & \{(0,0),(0,6),(0,12),(6,0),(6,6),(6,12),(12,0),(12,6),(12,12) \\ & (2,2),(2,8),(2,14),(8,2),(8,8),(8,14),(14,2),(14,8),(14,14) \\ & (4,4),(4,10),(4,16),(10,4),(10,10),(10,16),(16,4),(16,10),(16,16)\}. \end{aligned}$$

Note that $S_1 \times S_2 \subseteq N_\varphi \subseteq W_1 \times W_2$.

Example 4.2. Let $R = \mathbb{Z}[x]$ the polynomials over $\mathbb{Z}$. If $M_1 = \langle x \rangle$, (polynomials with a zero constant term) and $M_2 = \langle x^2 \rangle$ (polynomials where each term has degree 2 or higher) then $M_1 \times M_2$ is an R-module.

Using the notation from Theorem 4.1, let $W_i = M_i$, $S_i = \{0\}$, and define $\varphi : W_1 \to W_2$ by $p(x) \mapsto xp(x)$ where $p(x) \in \mathbb{Z}[x]$ has a zero constant term. Clearly φ is a bijection and preserves scalar multiplication.

Then $N_\varphi = \{(p(x), xp(x)) \mid$ for all $p(x) \in \langle x \rangle\}$.

Example 4.3. Let $k = \mathbb{R}$ and $V = \mathbb{R} \times \mathbb{R}$. It is well-known that the only subspaces of $\mathbb{R}^2$ are: $\mathbb{R}^2$ itself, lines through the origin and the point $(0,0)$. We use Goursat's Theorem for vector spaces together with the fact that the only subspaces of $\mathbb{R}$ are $\mathbb{R}$ and $\{0\}$ to verify this.

Case 1: Using the notation from Corollary 4.1 let $W_1 = W_2 = \mathbb{R} = S_1 = S_2$. The identity isomorphism $\varphi : \mathbb{R}/\mathbb{R} \to \mathbb{R}/\mathbb{R}$ gives the entire space $\mathbb{R}^2$.

Case 2: Let $W_1 = W_2 = \mathbb{R}$, $S_1 = S_2 = 0$, and define $\varphi : \mathbb{R}/0 \to \mathbb{R}/0$; by $r \mapsto \alpha r$, where α is a nonzero scalar. Then the subspace $U_\alpha = \{(r, \alpha r) \mid r \in \mathbb{R}\}$, which we know is a line through the origin with slope α.

Case 2′: If $W_1 = S_1 = 0$ and $W_2 = S_2 = \mathbb{R}$ then the resulting subspace is the line $\{0\} \times \mathbb{R}$. Similarly the subspace $\mathbb{R} \times \{0\}$ is found when $W_1 = S_1 = \mathbb{R}$ and $W_2 = S_2 = 0$. Note that $\mathbb{R} \times \{0\}$ can also be found using the method in Case 2 by letting $\alpha = 0$.

Case 3: When $W_1 = W_2 = 0 = S_1 = S_2$ the resulting subspace is the origin, $\{(0,0)\}$.

Example 4.4. When $k = \mathbb{R}$ and $V = \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$ the only subspaces of V are lines or planes passing through the origin.

Case 1: If $W_1 = S_1$ and $W_2 = S_2$, then the subspace produced by applying Goursat's theorem for modules is $W_1 \times W_2$ where $W_1 \in \{\mathbb{R}^2,\ \mathbb{R} \times \{0\},\ \{0\} \times \mathbb{R},\ U_\alpha$ with $\alpha \neq 0, \{(0,0)\}\}$ and $W_2 \in \{\mathbb{R}, \{0\}\}$. (Here U_α follows the notation used in Example 4.3.) The subspaces obtained using Goursat's Theorem are: all of $\mathbb{R}^3$, the $x-y$,$y-z$ and $x-z$ planes, the x, y and z axes, lines U_α through the origin, and planes parallel to the z-axis and intersecting the $x-y$ plane in the line U_α.

Case 2: For some $0 \neq \alpha \in \mathbb{R}$, let $W_1 = U_\alpha, S_1 = 0, W_2 = \mathbb{R}$, and $S_2 = 0$. Define $\phi : U_\alpha/0 \to \mathbb{R}/0$; by $(r, \alpha r) + 0 \mapsto \delta r + 0$, where $\delta \in \mathbb{R}$ is a stretch factor.

This mapping gives the subspace V' where every element in the subspace has the form $(r, \alpha r, \delta r)$ for all $r \in \mathbb{R}$. The line $V' \cong \mathbb{R}$ passes through the origin.

Case 3: Let $W_1 = \mathbb{R}^2$ and let $S_1 = U_\alpha$ for some nonzero α. Then the quotient space $\mathbb{R}^2/U_\alpha$ is one dimensional and we may assume that any element in this quotient has the form $(0, s) + U_\alpha$ since $(x, y) + U_\alpha = (x, y) - (x, \alpha x) + U_\alpha$. Define the isomorphism $\psi : \mathbb{R}^2/U_\alpha \to \mathbb{R}/0$ by $(0, s) + U_\alpha \mapsto \gamma s + \{0\}$ where $0 \neq \gamma \in \mathbb{R}$ is a stretch factor. The resulting subspace is a plane V_γ that when cut along $\mathbb{R}^2$ at the origin you get the line U_α. Any point in V_γ has the form $(s + r, s + \alpha r, \gamma s)$ as r and s go through $\mathbb{R}$.

Therefore, as expected, any non-trivial subspace of $\mathbb{R}^3$ will be either a plane or a line passing through the origin.

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