Perfect Distance Stars mod m

Xianrui Meng
xianruimeng@gmail.com
November, 2009

Department of Mathematics, Computer Science, and Statistics
Bloomsburg University of Pennsylvania

Abstract

A *perfect distance tree* is a weighted tree with *n* vertices in which the set of distances between vertices is $\{1,2,3,\ldots,\binom{n}{2}\}$. We define a weighted tree with n vertices to be a *perfect distance tree mod m* if the distances $\{1,2,3,\ldots,\binom{n}{2}\}$(mod m) can be obtained. In this paper, we find that every weighted star with $m+1$ vertices, where *m* is odd, labeled with 0, 1, 2, 3… m-1 is a perfect distance tree mod m. The stars obtained from this star by removing the edge labeled 0 or by changing the weight 0 to another weight are also perfect distance trees mod m. By combining stars, we show that every star with $km+j$ vertices can be labeled to be a perfect distance tree mod *m*, where *m* is odd, $k \geq 1$ and $-1 \leq j \leq 4$. Finally, we show that certain twin-stars (trees of diameter 3) can be labeled as perfect distance trees mod *m*.

1. Introduction

A *graph* consists of a finite set of *vertices* and a set of unordered pairs of distinct vertices called *edges*. Vertices *x* and *y* are *adjacent* if {x, y} is an edge. A *path* is a finite sequence of distinct vertices $x_0, x_1, \ldots, x_n$ and edges $a_1, a_2, \ldots, a_n$, where the endpoints of a_i are x_{i-1} and x_i for each *i*. A *tree* is a graph in which there is a unique path between any two distinct vertices. A *star* is a tree with a center vertex adjacent to all the other vertices. A *weighted tree* is a tree in which each edge is labeled with a positive integer, called the *weight* of the edge. The *distance* between two vertices in a weighted tree is the sum of the weights on the edges of the unique path connecting the pair. Since each pair of vertices determines a distance, there are a total of $\binom{n}{2}$ distances in a tree with n vertices. If all of these distances are distinct, we call the tree a *distinct distance tree*. We refer to a distinct distance tree on *n* vertices as a *perfect distance tree* if the set of distances $\{1,2,3,\ldots,\binom{n}{2}\}$ can be achieved. In the next section we will define a *perfect distance tree mod m*. The goal of this paper is to show that various trees that are labeled appropriately are perfect distance trees mod m. Most of the trees we will consider are stars.

A major result on perfect distance trees, known as Taylor's Condition, states that a tree with n vertices cannot be labeled as a perfect distance tree unless $n = k^2$ or $n = k^2+2$ for some integer k, see [5] or [8]. It has been proved that there is a perfect distance tree for n = 2, 3, 4, 6. Also, by computer search, Shen Lin showed there is no perfect distance tree with n = 9, see [9]. Moreover, Calhoun, Ferland, Lister and Polhill showed that for n = 11, 16 there is no perfect distance tree, see [2]. We still do not know whether there exists a perfect distance tree for n = 18. Figure 1 shows the only perfect distance trees on 6 or fewer vertices.

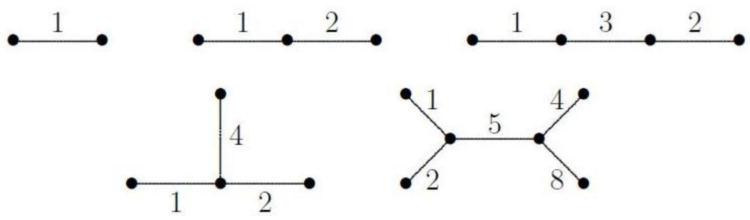


Figure 1: All Known Perfect Distance Trees

2. Perfect Distance Tree mod m

Definition: A tree with n vertices is a *perfect distance tree mod m* if the distances $\{1,2,3,\dots,\binom{n}{2}\}(\text{mod m})$ can be obtained. We use the notation $\{1,2,3,\dots,\binom{n}{2}\}(\text{mod m})$ to indicate the multi-set {1 mod m, 2 mod m, 3 mod m, …, $\binom{n}{2}$ mod m}.

Theorem 1: If $m_1|m_2$ and T is a perfect distance tree mod m_2, then T is a perfect distance tree mod m_1, where m_2 is odd and $m_2|n$.

Proof: Since T is a perfect distance tree mod m_2, T can be labeled to achieve the distances $\{1,2,3,\dots,\binom{n}{2}\}(\text{mod } m_2)$. In the case m_2 is odd and $m_2|n$, we have $\binom{n}{2} \equiv 0$ (mod m_2), which means that in the multi-set $\{1,2,3,\dots,\binom{n}{2}\}(\text{mod } m_2)$ the number of 0's, 1's, 2's, …, m_2-1's are equal. And since $m_1|m_2$, the multi-set {0, 1, 2, …, m_2 -1}(mod m_1) has equally many 0's, 1's, 2's, …, m_1 -1's. So $\{1,2,3,\dots,\binom{n}{2}\}(\text{mod } m_1)$ has equally many 0's, 1's, 2's, …, m_1-1's, which makes T a perfect distance tree mod m_1. ■

For simplicity, the previous theorem was stated in the special case that m_2 is odd and $m_2|n$. However, the theorem holds generally. We omit the proof.

3. Perfect Distance Star mod m

We call a weighted star a *perfect distance star mod m* if this star is a perfect distance tree mod m.

Theorem 2: Every star with m+1 vertices, where m is odd, labeled with 0, 1, 2, 3… m-1 is a perfect distance star mod m.

Proof: All the distances we can get from the star are the single edges' weights and all the sums of any two edges' weights. Below is the table that shows all the sums of any two edges' weights:

0 + 0	1 + 0	… …	(m – 1) + 0
0 + 1	1 + 1	… …	(m – 1) + 1
…	…	… …	…
0 + (m – 1)	1 + (m – 1)	… …	(m – 1) + (m – 1)

Since we are working mod m, we are adding weights in the group $\mathbf{Z}_m$. So every column gives us the distances 0, 1, 2 … m-1 (mod m). There are total of m many 0's, 1's, … ,m-1's (mod m) in the table. Each column is the edge's weight plus all the other edges' weights. Since we can't get a distance by adding an edge to itself, the diagonal of the table above is eliminated, which gives us m-1 many 0's, 1's … (m-1)'s. This follows because the diagonal contains {2*0, 2*1… 2*(m-1)}(mod m) = {0, 1, 2 … (m-1)}(mod m), since m is odd. However, each distance appears twice in the table because a + b = b + a (a, b are the weights of those edges). So we have $\frac{m-1}{2}$ many 0's, 1's … (m-1)'s without counting the single edges. Including the single edges, there are 1 + $\frac{m-1}{2}$ many 0's, 1's … (m-1)'s. And the multi-set {1, 2 …$\binom{m+1}{2}$}(mod m) can be achieved from the star. Hence the star is a perfect distance tree mod m. ■

Corollary 3: Every star with m vertices, where m is odd, labeled with 1, 2, 3… m-1 is a perfect distance star mod m.

Proof: The star is obtained from the one from Theorem 2 by removing the edge labeled zero. The missing distances are the single edge's weight 0 and 0+1, 0+2 … 0+(m-1), which is 0, 1, 2 … m-1. So after removing the edge there are $\frac{m+1}{2}$ -1 many 0's, 1's … (m-1)'s. Then {1, 2, 3 … $\binom{m}{2}$}(mod m) can be obtained. Hence, the star is a perfect distance tree mod m. ■

So the star with m vertices labeled with 1, 2, 3 … m-1, where m is odd, is a perfect distance star mod m. We refer such a star as a PDS_m.

Corollary 4: Every star with m+1 vertices, where m is odd, labeled with 1, 2, 3… m-1, x ($0 \leq x \leq m-1$), is a perfect distance star mod m.

Proof: As we proved above, if a star is a PDS_m, then there are $\frac{m-1}{2}$ 0's, 1's, 2's ….

m-1's distances in the star. Now we add another edge labeled with x ($0 \leq x \leq m-1$) to the star to obtain a star with m+1 vertices. We get extra distances {x, x+1, x+2 …

x+m-1}(mod m) ≡ {0, 1, 2 …. m-1}(mod m). Then there are ($\frac{m-1}{2}$ + 1=$\frac{m+1}{2}$) 0's, 1's,

2's …. (m-1)'s. So the new star with m+1 vertices is still a perfect distance star mod m. ■

Note that the star in Theorem 2 is a special case of Corollary 4.

Corollary 5: Every star with m-1 vertices, where m is odd, can be labeled to be a perfect distance star mod m.

Proof: Suppose we remove one edge labeled with x from a PDS_m and this new star is a perfect distance star mod m.
Since m is odd,

$$\binom{m-1}{2} = \frac{(m-1)(m-2)}{2} = \frac{m(m-3)}{2} + 1 \equiv 1 \pmod{m}$$

We must have an extra distance 1 to form a perfect distance star. When we remove the edge labeled with x, the distances x+0, x+1, x+2…x+x-1, x+x+1… x+m-1 are removed. So the only distance that is not removed is x+x. Since we must have an extra distance 1 for the new star, we must have x+x ≡ 1 (mod m).
So x ≡ 2^{-1}(mod m). Since m is odd, we can always find an x satisfying x ≡ 2^{-1}(mod m). ■

Corollary 6: Every star with m-2 vertices, where m is odd and m≥ 3, can be labeled to be a perfect distance star mod m.

Proof: From Corollary 5, we know that after we removed the edge labeled with *x* from a PDS_m, where x ≡ 2^{-1}(mod m), the resulting star is still a perfect distance tree mod m. Now suppose we remove the edge labeled *x* and also remove another edge labeled *y* from a PDS_m. We will show that we can choose *y* so the resulting star is also a perfect distance star mod m.

$$\binom{m-2}{2} = \frac{(m-2)(m-3)}{2} = \frac{m(m-5)}{2} + 3 \equiv 3 \pmod{m}$$, since *m* is odd.

We must have extra distances of 1, 2, and 3 to obtain a perfect distance star.
The distances we did not remove are x+x, y+y, and x+y. From corollary 2, we know that x ≡ 2^{-1}(mod m), since x+x ≡1 (mod m). So let y+y ≡ 3(mod m) and x+y ≡ 2 (mod m).

i.e. $\begin{cases} 2y \equiv 3 \pmod{m} \\ y + x \equiv 2 \pmod{m} \end{cases}$ y ≡ 3×2^{-1}(mod m)

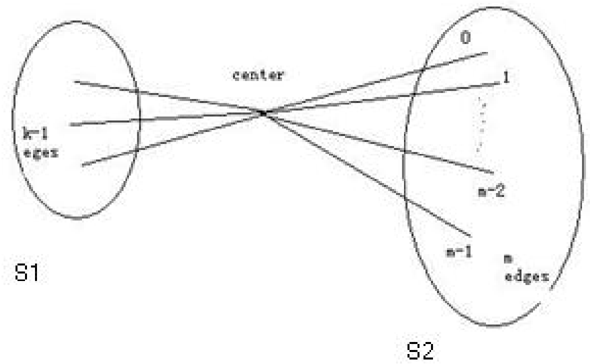
(Note: from the first congruence, $y \equiv 3 \times 2^{-1}(\text{mod m})$. Plug this into the second congruence, we have $3\times 2^{-1} + 2^{-1} \equiv 4 \times 2^{-1}(\text{mod m}) \equiv 2(\text{mod m})$. So y also satisfies the second congruence.) ■

And since m is odd, we can always find a y satisfying y ≡ 3× 2^{-1}(mod m).
Hence, every star with m-2 vertices, where m is odd and m $\geq$ 3, can be labeled to be a perfect distance star mod m.

Theorem 7: If we start with a perfect distance star mod *m*, where *m* is odd, and add *m* edges labeled with 0, 1, 2 … *m*-1, the result is a perfect distance star mod *m*.

Proof: Let's first look at a Perfect Distance Star mod *m* (P.D.S. mod *m*) with *k* vertices, call it S_1.

It is a P.D.S. mod *m*, so according to the definition {1, 2, 3… $\binom{k}{2}$}(mod *m*) can be achieved. Now add a star S_2 with *m* edges labeled with 0, 1, 2, 3… m-1 to this P.D.S. mod *m*, call the new star S_3.



The new star S_3 now has k+m vertices. To prove S_3 is a P.D.S. mod *m*, we must show that {1, 2, 3…$\binom{k+m}{2}$}(mod m) can be achieved.

S_1 is perfect distance star, so, for star S_1, {1, 2, 3…$\binom{k}{2}$}(mod m) can be obtained. When S_2 is added to S_1, the extra distances we get are the D(S_2) (mod m) (distances we get in S_2) and D(S_1, S_2)(mod m) (the lengths of paths from S_2 to S_1). Note that, in S_3, $\binom{k+m}{2} = \frac{(k+m)(k+m-1)}{2} = \frac{k^2-k}{2} + \frac{m(m+2k-1)}{2}$. To show S_3 is a perfect distance star mod *m*, we must prove that the extra distances from D(S_2) (mod m) and D(S_1, S_2)(mod m) have equal 0's, 1's, 2's… m-1's. We will show that the number of extra 0's, 1's, 2's, 3's… or m-1's is $\frac{m+2k-1}{2}$ in each case. From Theorem 2 we know S_2 has $\frac{m+1}{2}$ 0's, 1's, 2's, …, m-1's. Now let's look at D(S_1, S_2) (mod m). The distances from S_1 to S_2 are the weights of the edges in S_1 added to the weights of the edges in S_2. One edge in S_1 added to all the edges in S_2 gives us the distances 0, 1, 2, 3… m-1(mod m). In S_1, there are k-1 edges. Hence the distances in D(S_1, S_2) (mod m) are (k-1) 0's, 1's, 2's, 3's… m-1's.

Thus, when we add S_2 to S_1, we get $\frac{m+1}{2}$ + k-1 = $\frac{m+2k-1}{2}$ extras of each of the numbers 0, 1, 2, 3… m-1. ■

Corollary 8: If we start with a perfect distance star mod m, where m is odd, and, for some $k \in \mathbf{Z}^+$, add km edges labeled with k 0's, 1's, 2's … m-1's, the result is a perfect distance star mod m.

Proof: We prove this using mathematical induction based on Theorem 7**.** The base case is just Theorem 7. Now we assume the statement holds for some $k \geq 1$. Thus when we start with a perfect distance star mod m and add km edges which are labeled with k 0's, 1's, 2's … m-1's, the result is a perfect distance star mod m. If we add m more edges which are labeled with 0, 1, 2 … m-1, we will have added a total of $k+1$ 0's, 1's, 2's … m-1's and, by Theorem 7, the result is a perfect distance star mod m. ■

Theorem 9: Every star with km-1, km-2, km, km+1, km+2, km+3 or km + 4 vertices, where $k \in \mathbf{Z}^+$ and m is odd, can be labeled to be a perfect distance star mod m.

Proof : Applying the Corollary 8 to Theorem 2, and Corollaries 3, 4, 5 and 6, every star with km+1, km, km-1, or km-2 vertices, where m is odd and $k \in \mathbf{Z}^+$ can be labeled to be a perfect distance tree mod m. Moreover, from [2] we already know that the star labeled 1, the star labeled 1 and 2, and the star labeled 1, 2, and 4 are perfect distance stars. Using Corollary 8, we can conclude that every star with km+2, km + 3, and km + 4 vertices can be labeled to be a perfect distance star mod m. ■

Conjecture: A perfect distance star mod m with m vertices must be a PDS_m. Any perfect distance star mod m with n vertices, where n>m, must have a PDS_m as a sub-star.

4. Perfect Distance Twin-Star

We refer to the tree, whose diameter is three, as a twin-star.

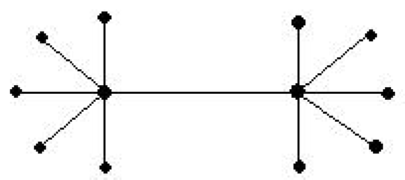
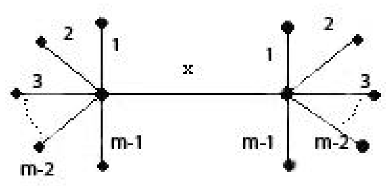


Figure 2: a twin-star

The edge in the center is referred to as *the center edge* of the twin-star, and the two stars without the center edge in this twin-star are called the sub-stars of it.

Theorem 10: If a twin-star with two PDS_m's mod m as its sub-stars, and the center edge is labeled with x, where $0 \leq x \leq m-1$, then it is a perfect distance tree mod m.

Proof:



In this twin-star, we call one of its sub-stars, together with the center-edge $x,\ S_1$. The other sub-star is called S_2. The distances we get in this twin-star are those in the multi-set D(S_1)(mod m) ∩ D(S_2)(mod m) ∩ D(S_1 , S_2)(mod m).

From Theorem 2, we know the distances in S_1 (mod m) are $\frac{m+1}{2}$ 0's, 1's, 2's… m-1's

and S_2 has $\frac{m-1}{2}$ 0's, 1's, 2's… m-1's.

For D(S_1 , S_2)(mod m), the distances we can get are
{x + 1, 1 + x + 1, 1 + x + 2, 1 + x + 3… 1 + x + m-1,
x + 2, 2 + x + 1, 2 + x + 2, 2 + x + 3… 2 + x + m-1,
…
x + m-1, m-1 + x + 1, m-1 + x + 2, m-1 + x + 3… m-1 + x + m-1}(mod m)

For a=1,2, …m-1, we have {x+a, a+x+1, a+x+2, a+x+3… a+x+m-1}(mod m). If we let y=a+x, it becomes {y+0, y+1, y+2, y+3… y+m-1}(mod m) = {0, 1, 2, … m-1}. Since there are m-1 choices for a, D(S_1 , S_2)(mod m) contains m-1 0's, 1's, 2's … m-1's. Hence, there are a total of $\frac{m+1}{2} + \frac{m-1}{2}$+ m-1= 2m-1 0's, 1's, 2's … m-1's in the twin star.

The twin star has 2m vertices. Since $\frac{\binom{2m}{2}}{m} = \frac{2m\times(2m-1)}{2m}$ = 2m-1, the twin-star is a perfect distance tree mod m. ■

5. Acknowledgement

I would like to thank my advisor Dr. William Calhoun, who has published papers in this area. I am extremely lucky to have his guidance and appreciate his help for mentoring me in this independent study and in the preparation of this paper.

References

[1] Brauer, Alfred, A problem of additive number theory and its application in electrical engineering, Journal of the Mitchell Society **61** (1945), 55-66.
[2] Calhoun, W., Ferland, K., Lister, L. and Polhill, J., Minimal distinct distance trees, Journal of Combinatorial Mathematics and Combinatorial Computing **61** (2007),

33-57.

[3] Golomb, S.W., How to number a graph in: R. C. Read, ed., Graph Theory and Computing. Academic Press, New York, 1972, 23-37.

[4] Graham, R. L. and N. J. A. Sloane, On additive bases and harmonious graphs, SIAM J. Alg. Disc. Meth., 1 (1980), 382-404.

[5] Honsberger, Ross, Mathematical Gems. III. The Dolciani Mathematical Expositions, 9. The Mathematical Association of America, Washington, DC, 1985, 56-60.

[6] Leech, John, On the representation of 1, 2, …, n by differences, J. London Math. Soc., **31** (1956), 160-169.

[7] Leech, John, Another tree labeling problem, Amer. Math. Monthly **82** (1975), 923-25.

[8] Taylor, Herbert, Odd path sums in an edge-labeled tree, Mathematics Magazine **50** (1977), 258-59.

[9] Taylor, Herbert, A distinct distance set of 9 nodes in a tree of diameter 36, Discrete Mathematics **93** (1991), 167-168.