

# THE SIGMA INVARIANTS OF THE LAMPLIGHTER GROUPS

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ABSTRACT. We compute the Bieri-Neumann-Strebel-Renz geometric invariants,  $\Sigma^n$ , of the lamplighter groups  $L_m$  by using the Diestel-Leader graph  $DL(m, m)$  to represent the Cayley graph of  $L_m$ .

**Keywords:** Sigma invariants, lamplighter groups

## 1. INTRODUCTION

The purpose of this paper is to calculate the Bieri-Neumann-Strebel-Renz geometric invariants  $\Sigma^n$  of the lamplighter groups  $L_m \cong \mathbb{Z}_m \wr \mathbb{Z}$ . Given an infinite, finitely generated group, one may wish to know certain finiteness properties (e.g. finitely generated or finitely presented) of normal subgroups with abelian quotients. The geometric invariants of interest detect these properties, and are particularly interesting since their computation generally involves geometric methods by studying a group through the space on which it acts, while the information they provide is algebraic in nature.

Recall that the lamplighter groups as a wreath product can be rewritten as  $(\bigoplus_{i=-\infty}^{\infty} A_i) \rtimes \mathbb{Z}$  where each  $A_i \cong \mathbb{Z}_m$ . A presentation of the lamplighter groups is

$$L_m \cong \langle a, t \mid a^m = 1, [a^{t^i}, a^{t^j}] = 1 \rangle$$

where  $a^t$  denotes the conjugate  $t^{-1}at$ . However, it is difficult to obtain an intuitive geometric interpretation of the Cayley graph resulting from this presentation, so we instead consider the generating set  $\{t, ta, \dots, ta^{m-1}\}$  for  $L_m$ . This allows us to use the Diestel-Leader graph  $DL(m, m)$  as a representative for the Cayley graph of  $L_m$  as shown in [4]. We now give an alternate description of the lamplighter groups  $L_m$  and present the construction of the Cayley graph of  $L_m$  in § 4.

There is a convenient way of visualizing the group elements of the lamplighter groups  $L_m$ . Each element can be thought of as a configuration of light bulbs where each light bulb is in one of  $m$  states and placed at an integer point on the real line as part of a bi-infinite string of bulbs, and a cursor which points to the light bulb of interest. The generator  $a$  acts on this picture by changing the state of the bulb, and the generator  $t$  and its inverse shift the cursor right and left, respectively. For example, in  $L_2$ , there are only two states for each bulb: either on or off. Figure 1 shows an element of  $L_2$  using this interpretation.

An element of  $L_m$  is then a set of instructions for our cursor to traverse the bi-infinite string of light bulbs and illuminate a finite collection of light bulbs to one of the  $m$  states. The generator  $ta^k$  for some  $0 \leq k < m$  shifts the cursor to the right and illuminates the bulb to

state  $(k + h) \bmod m$  if it was in state  $h$  to begin with. This generating set then causes us to form words of  $L_m$  in a step-light, step-light fashion.

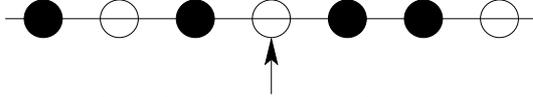


FIGURE 1. The element  $(ta)^{-1}t^{-1}(ta)^{-1}t^3(ta)^2t^{-2}$  in  $L_2$ . The shaded circles represent a bulb in the “on” state, while the clear bulbs represent a bulb in the “off” state. The cursor is at the origin of the real line.

Recall the definition of a Cayley graph as a geometric space on which a group acts. Let  $G$  be a group with generating set  $S$ , and let  $g_1, g_2 \in G$ . A *Cayley graph* of the group  $G$  is composed of a set of vertices  $V := G$  and a set of directed edges  $E := \{(g_1, g_2) \mid \exists \text{ a generator } x \in S \text{ st } g_1x = g_2\}$  where  $g_1$  is the initial point of the edge and  $g_2$  is the terminal point. By directing the edges, the Cayley graph is endowed with an orientation which aids in the process of locating group elements. However, the geometric invariants of interest here describe connectivity properties of a group’s Cayley graph and therefore we need not make a distinction between an oriented or non-oriented Cayley graph. The Cayley graph of a group is a convenient way of realizing a group geometrically, and thus will be integral in our computation of  $\Sigma^n(L_m)$ .

The following simplified definition of  $\Sigma^1(G)$ , the Bieri-Neumann-Strebel geometric invariant of a group  $G$  from [1], is a special case of  $\Sigma^n(G)$  when  $n = 1$ . The complete definition of  $\Sigma^n(G)$  will be given in § 2.

Let  $\Gamma$  be the Cayley graph of a finitely generated group  $G$ . Denote by  $G'$  the commutator subgroup of  $G$ . Consider the real vector space  $\text{Hom}(G, \mathbb{R}) \cong \mathbb{R}^n$  which is defined as the set of homomorphisms from  $G$  into  $\mathbb{R}$ , the group of additive reals. The dimension of  $\text{Hom}(G, \mathbb{R})$  is the  $\mathbb{Z}$ -rank of the abelianization  $G/G'$  of  $G$ , and thus there exists an epimorphism  $\phi : G \rightarrow \mathbb{Z}^n$ . A *height function* is a  $G$ -map  $h : \Gamma \rightarrow \mathbb{R}^n$  defined by  $h(g) := \phi(g)$  for all vertices  $g \in \Gamma$ , extending linearly on edges.

Consider two geodesic rays  $\gamma, \gamma' : [0, \infty) \rightarrow \mathbb{R}^n$ . These geodesic rays are said to be *equivalent*, or  $\gamma \sim \gamma'$ , if and only if for all  $t \in [0, \infty)$ ,  $\|\gamma(t) - \gamma'(t)\| = \|\gamma(0) - \gamma'(0)\|$ . Define  $\mathcal{R}$  to be the set of all geodesic rays in  $\mathbb{R}^n$ . We will denote  $\mathcal{R}/\sim$  by  $\partial_\infty \mathbb{R}^n$ , which is the *boundary of  $\mathbb{R}^n$  at infinity*.

Let  $s \in [0, \infty)$ . The point  $\gamma(s)$  is the point on  $\gamma$  that corresponds to  $s$ . The hyperplane of dimension  $n - 1$  orthogonal to  $\gamma$  through the point  $\gamma(s)$  separates  $\mathbb{R}^n$  into two *half-spaces*. Denote by  $H_s(\gamma)$  the half-space that contains  $\gamma([s, \infty))$ . We also define  $\Gamma_s(\gamma)$  to be the largest subgraph of  $\Gamma$  containing  $h^{-1}(H_s(\gamma))$  (or equivalently, the induced subgraph on the vertices of  $h^{-1}(H_s(\gamma))$ ).

Let  $e \in \partial_\infty \mathbb{R}^n$  and let  $\gamma$  be a geodesic ray which determines  $e$ . We say a point  $e \in \partial_\infty \mathbb{R}^n$  is *controlled 0-connected*, or  $CC^0$ , if and only if for all  $s \geq 0$ ,  $\Gamma_s(\gamma)$  is path-connected. Thus, the *Bieri-Neumann-Strebel geometric invariant of  $G$*  is

$$\Sigma^1(G) := \{e \in \partial_\infty \mathbb{R}^n \mid e \text{ is } CC^0\}.$$

There is an interesting implication of this definition worth noting. In [1], it was shown that the commutator subgroup of a group  $G$  is finitely generated if and only if  $\Sigma^1(G) = \partial_\infty \mathbb{R}^n$  where  $n$  is the  $\mathbb{Z}$ -rank of the abelianization  $G/G'$  of  $G$ . There are similar algebraic results that are determined by the higher-dimensional analogues of this invariant which we present in § 2.

The author would like to thank Nic Koban for his advice and encouragement throughout the writing of this paper.

## 2. THE SIGMA INVARIANTS OF A GROUP

Of interest to us in this paper are the Bieri-Neumann-Strebel-Renz geometric invariants of a group  $G$ , denoted  $\Sigma^n(G)$ , as defined in [1] for  $n = 1$  and in [2] for  $n \geq 2$ . In § 1 we presented a simplified definition of  $\Sigma^1(G)$ , so now we present a definition of the higher-dimensional analogues of this invariant.

Let  $X$  be a  $n$ -dimensional,  $(n - 1)$ -connected CW-complex on which a finitely generated group  $G$  acts freely where  $G \backslash X$  is a finite complex. The group action of  $G$  on  $X$  is to permute cells by homeomorphisms. Let  $\text{Hom}(G, \mathbb{R}) \cong \mathbb{R}^n$  for some  $n \in \mathbb{Z}_+$ . Further, let  $e \in \partial_\infty \mathbb{R}^n$  and let  $\gamma$  be a geodesic ray defining  $e$ . Since the  $\mathbb{Z}$ -rank of the abelianization  $G/G'$  of  $G$  is  $n$ , there exists an epimorphism  $\psi : G \rightarrow \mathbb{Z}^n$ , and therefore a cocompact action  $\eta$  of  $G$  on  $\mathbb{R}^n$  by translations. Let  $h : X \rightarrow \mathbb{R}^n$  be a  $G$ -map and for all  $s \in \mathbb{R}$ , let  $X_s(\gamma)$  denote the largest subcomplex of  $X$  containing  $h^{-1}(H_s(\gamma))$ . We say the action  $\eta$  is *controlled  $(n - 1)$ -connected*, or  $CC^{n-1}$ , *in the direction of  $e$*  if and only if for all  $s \in \mathbb{R}$  and for all  $-1 \leq q \leq n - 1$ , every continuous function from the  $q$ -sphere to  $X_s(\gamma)$  can be extended to a continuous function from the  $(q + 1)$ -ball to  $X_s(\gamma)$ . Therefore we define the *Bieri-Neumann-Strebel-Renz geometric invariants of  $G$*  to be

$$\Sigma^n(G) := \{e \in \partial_\infty \mathbb{R}^n \mid \eta \text{ is } CC^{n-1} \text{ in the direction of } e\}.$$

It is clear from this definition that  $\Sigma^1(G) \supseteq \Sigma^2(G) \supseteq \dots \supseteq \Sigma^n(G)$ . In § 1 we noted that in [1], it was shown that the commutator subgroup  $G'$  of a group  $G$  is finitely generated if and only if  $\Sigma^1(G) = \partial_\infty \mathbb{R}^n$ . Additionally,  $G'$  is finitely presented if and only if  $\Sigma^2(G) = \partial_\infty \mathbb{R}^n$ . In general,  $G'$  is of type  $F_n$  if and only if  $\Sigma^n(G) = \partial_\infty \mathbb{R}^n$ . For a detailed explanation on what it means for a group to be of type  $F_n$ , see [3].

## 3. THE SIGMA INVARIANTS OF VARIOUS GROUPS

In this section we present known results of the computation of the Bieri-Neumann-Strebel geometric invariant  $\Sigma^1$  of a few groups. The computation of  $\Sigma^1$  of these groups is relatively simple in order to familiarize the reader with the process of calculating these invariants. Despite the ease with which  $\Sigma^1$  is computed for these examples, these invariants are, in general, difficult to calculate and therefore they are not known for many classes of groups.

### Example 1.

We begin by considering the free abelian group  $\mathbb{Z} \oplus \mathbb{Z} \cong \langle a, b \mid ab = ba \rangle$ . A Cayley graph of  $\mathbb{Z} \oplus \mathbb{Z}$  is shown in Figure 2.

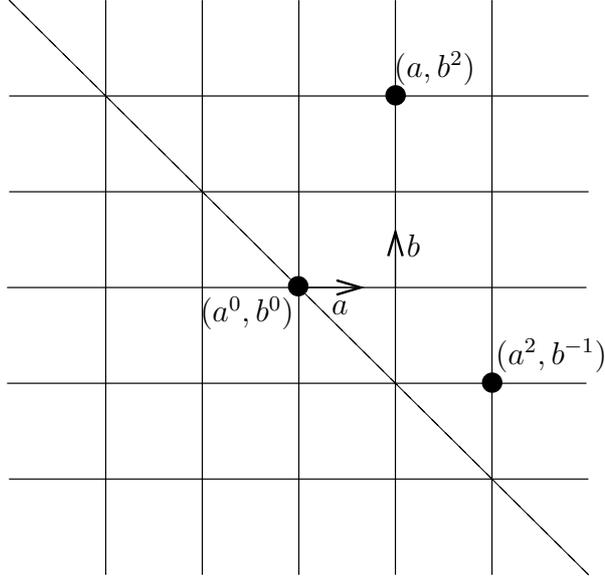


FIGURE 2. A finite portion of the Cayley graph of  $\mathbb{Z} \oplus \mathbb{Z} \cong \langle a, b \mid ab = ba \rangle$ .

Here we consider each group element as an ordered pair  $(a^i, b^j)$  for some  $i, j \in \mathbb{Z}$ . The identity element is denoted as  $(a^0, b^0)$ . Since  $\text{Hom}(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{R}) \cong \mathbb{R}^2$ , we define our height function  $h : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{R}^2$  by  $h((a^i, b^j)) := (i, j)$ . Since there is a one-to-one correspondence between points on the boundary at infinity of the plane and the unit circle, the choice of geodesic ray that will determine the half-spaces is arbitrary. Suppose we choose the line  $y = -x$  in  $\mathbb{R}^2$  to separate the plane into two half-spaces which is displayed in Figure 2 as the dashed line.

Say we choose two elements  $(a, b^2)$  and  $(a^2, b^{-1})$  which map under the height function to the ordered pairs  $(1, 2)$  and  $(2, -1)$ , respectively. We readily verify that both elements are mapped to the same half-space. It is not difficult to see that one can find a path in the Cayley graph of  $\mathbb{Z} \oplus \mathbb{Z}$  between these two points that lies entirely within the same half-space. Due to the symmetry of the Cayley graph, a similar argument holds for any pair of points that lie in the other half-space. Therefore we say that  $\Sigma^1(\mathbb{Z} \oplus \mathbb{Z}) = \partial_\infty \mathbb{R}^2$ .

### Example 2.

The next group under consideration is the free group on two generators which we henceforth denote by  $F_2$ . Recall that  $F_2$  has presentation  $F_2 \cong \langle a, b \mid \emptyset \rangle$ . Figure 3 depicts the Cayley graph of  $F_2$  where the horizontal edges are labeled by the generator  $a$  while the vertical edges are labeled by the generator  $b$ . The identity element is labeled as  $e$  since it can be thought of as the empty word. Since there are no relations it is clear that  $\text{Hom}(F_2, \mathbb{R}) \cong \mathbb{R}^2$ . Given a word  $g = a^{i_1} b^{j_1} a^{i_2} b^{j_2} \dots a^{i_n} b^{j_n} \in F_2$  where  $i_k, j_k \in \mathbb{Z}$  for  $1 \leq k \leq n$ , define the height function  $h : F_2 \rightarrow \mathbb{R}^2$  by  $h(g) := (\sum_{k=1}^n i_k, \sum_{k=1}^n j_k)$ . As in the previous example, the choice of geodesic ray is arbitrary, so in this example we choose the line  $y = x$  in  $\mathbb{R}^2$  to divide the plane into half-spaces.

Consider the two elements  $b^2$  and  $ab^2$  which map under the height function to the ordered pairs  $(0, 2)$  and  $(1, 2)$ , respectively. Both of these points in  $\mathbb{R}^2$  lie above the line  $y = x$  and therefore their pre-images under the height function are in the same half-space. Since  $F_2$  is a tree, there is a unique geodesic path between these two elements in the Cayley graph. However, this path passes through the element  $a$  which maps to  $(0, 1) \in \mathbb{R}^2$ . The point  $(0, 1)$  lies underneath the line  $y = x$ , and therefore the element  $a$  is not in the same half-space as  $b^2$  and  $ab^2$ . In fact, given any pair of group elements that map to the same half-space, it is not difficult to verify that the unique geodesic path between them in the Cayley graph contains at least one group element that maps to the other half-space. Therefore,  $\Sigma^1(F_2) = \emptyset$ .

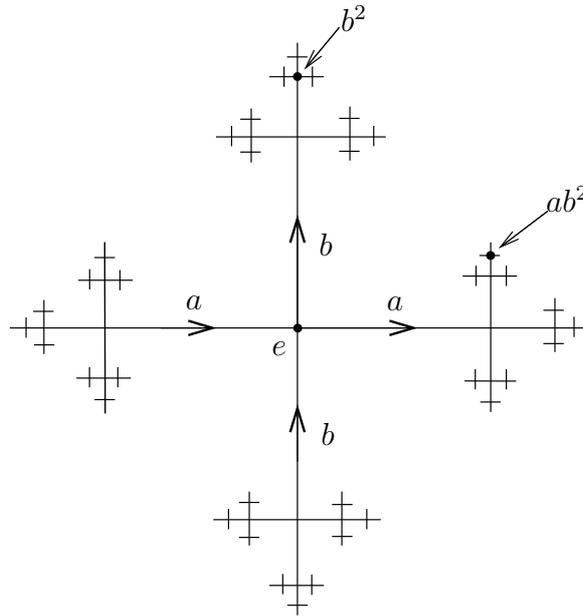


FIGURE 3. A finite portion of the Cayley graph of  $F_2 \cong \langle a, b \mid \emptyset \rangle$ .

### Example 3.

We conclude this section by considering one of the Baumslag-Solitar groups,

$$BS(1, 2) \cong \langle x, t \mid t^{-1}xt = x^2 \rangle.$$

A finite piece of the Cayley graph of  $BS(1, 2)$  is shown in Figure 4. Notice that the Cayley graph branches like a tree. By examining the one relation in the group, we notice that all homomorphisms from  $BS(1, 2)$  to the additive reals takes the generator  $x$  to the number 0. We can map the generator  $t$  to any number we wish, so  $\text{Hom}(BS(1, 2), \mathbb{R}) \cong \mathbb{R}$ . We define the height function to be  $h(g) := \exp(g)$  where  $\exp(g)$  denotes the exponential sum of all occurrences of the generator  $t$  for some element  $g \in BS(1, 2)$ . For example, given the element  $g = t^2xt^{-1}x^{-1}$ ,  $\exp(g) = 1$ . The numbers in Figure 4 indicate the exponential sum of the  $t$  generators in the elements at that particular height in the graph. This leads to a natural way of understanding where the pre-images  $h^{-1}([0, \infty))$  and  $h^{-1}((-\infty, 0])$  are located in the Cayley graph. All elements that map to the interval  $[0, \infty)$  lie on and above the height in the graph labeled  $t = 0$  while all elements that map to  $(-\infty, 0]$  lie on and below the height

$t = 0$  in the graph. It is fairly easy to see that given any two points above height  $t = 0$  in the graph, one can always find a path between them by climbing up further in the graph to ensure that no point in the path will map to the interval  $(-\infty, 0]$ . However, given any two points below the height  $t = 0$  in the graph, one cannot connect them with a path that lies entirely within  $h^{-1}((-\infty, 0])$  since the graph is similar to a rooted tree. Therefore, while  $\infty \in \Sigma^1(BS(1, 2))$ ,  $-\infty \notin \Sigma^1(BS(1, 2))$ , and so we say  $\Sigma^1(BS(1, 2)) = \{\infty\}$ .

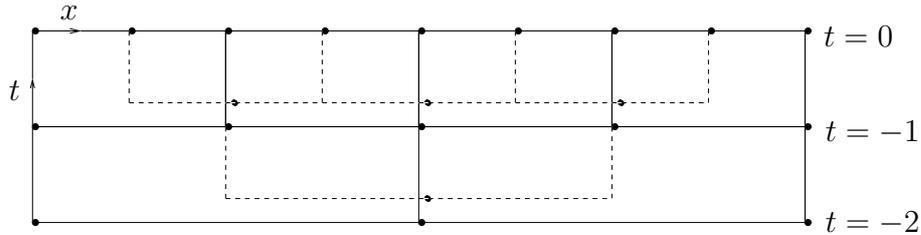


FIGURE 4. A finite portion of a Cayley graph of  $BS(1, 2)$ . The dotted lines indicate that they are behind the solid lines.

#### 4. THE CAYLEY GRAPH OF $L_m$

The method of describing the lamplighter groups in terms of a bi-infinite string of light bulbs with a finite collection illuminated to a particular state and a cursor pointing to the light bulb of interest is merely an interpretation of the group elements and as such we can not construct a Cayley graph of  $L_m$  using this geometric interpretation. Instead, we introduce Diestel-Leader graphs as a way of realizing the Cayley graph of  $L_m$  with respect to the generating set  $\{t, ta, \dots, ta^{m-1}\}$  as in [4]. It should be noted that while the Cayley graph does depend on the choice of generating set,  $\Sigma^n(G)$  is invariant under the choice of a particular generating set.

Let  $T_1$  be a regular  $(m+1)$ -valent tree and let  $T_2$  be a regular  $(n+1)$ -valent tree for positive integers  $m \leq n$ . The trees are oriented as shown in Figure 5 so that  $T_1$  has  $m$  outgoing edges and  $T_2$  has  $n$  outgoing edges. If we fix a distinguished point at infinity as a basepoint in each tree we may define a height function  $\mathfrak{h}_i : V(T_i) \rightarrow \mathbb{Z}$  for  $i = 1, 2$ , where  $V(T_i)$  denotes the vertex set of each tree. This is the *Busemann function* as noted in [5] and thus we will adopt the notation for this function to distinguish it from our  $G$ -map. The *Diestel-Leader graph*  $DL(m, n)$  is composed of the vertex set which is the subset of  $V(T_1 \times T_2)$  where  $\mathfrak{h}_1 + \mathfrak{h}_2 = 0$  and the edge set such that there is an edge between two elements  $(a, B), (c, D) \in DL(m, n)$  where  $a, c \in T_1$  and  $B, D \in T_2$  if and only if there is an edge between  $a, c$  in  $T_1$  and an edge between  $B, D$  in  $T_2$ . By placing the basepoints of each tree at opposite ends of a page, the Diestel-Leader graph takes a form as shown in Figure 5 where the integers on the left of the diagram denote the height in  $T_1$  while the integers on the right indicate the height in  $T_2$ .

We will only concern ourselves with the Diestel-Leader graph  $DL(m, m)$  which represents the Cayley graph of  $L_m$  with respect to the generating set  $\{t, ta, \dots, ta^{m-1}\}$ . Denote a group element  $g \in L_m$  by an ordered pair  $(x, X)$  where  $x \in T_1$  and  $X \in T_2$  such that  $\mathfrak{h}_1(x) + \mathfrak{h}_2(X) = 0$ . It will be convenient to adopt the convention of denoting vertices in  $T_1$  as lower case letters and vertices in  $T_2$  as capital letters as in [4].

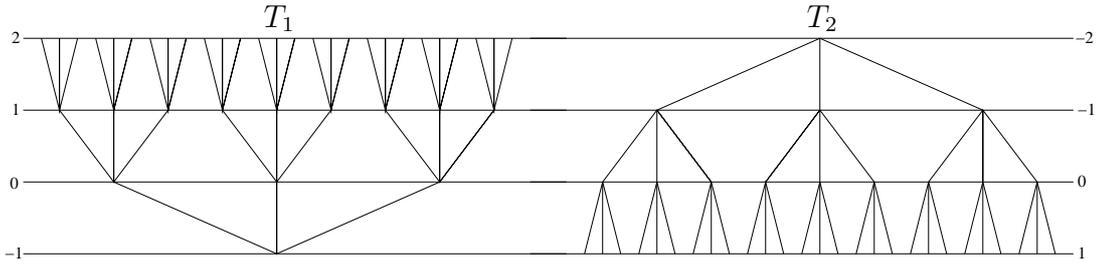


FIGURE 5. A finite portion of the Diestel-Leader graph  $DL(3,3)$  which is representative of the Cayley graph of  $L_3$ . The numbers on the leftmost and rightmost sides of the figure indicate the height in each tree as determined by the Busemann function.

It is suggested that the reader keep in mind that  $DL(m,m)$  merely represents the Cayley graph of  $L_m$  to the extent that a vertex in the Cayley graph of  $L_m$  corresponds to a pair of points in  $DL(m,m)$ . In [5], Woess notes that it is useful to imagine a spring that connects the two vertices in  $DL(m,m)$  together in order to recognize that, when taken together, they represent a vertex of the Cayley graph of  $L_m$ . The spring may stretch infinitely far since each tree extends indefinitely in either direction, but the spring must remain horizontal at all times in order to ensure that  $\mathfrak{h}_1(x) + \mathfrak{h}_2(X) = 0$  for some  $g = (x, X)$ . Additionally, the negative heights are useful in order to give an idea as to where the identity element of  $L_m$  is located in  $DL(m,m)$  since our  $G$ -map will map the identity element to a height of zero, and other elements to either positive or negative integers, as we will see.

In § 1 we saw the action of  $L_m$  on the lamplighter picture of group elements. Here we discuss the action of  $L_m$  on  $DL(m,m)$ . The action of  $t$  on  $DL(m,m)$  translates up in height in  $T_1$  and down in height in  $T_2$  so that  $\mathfrak{h}_1 + \mathfrak{h}_2 = 0$ . If  $(x, X)$  is a point in  $DL(m,m)$ , then the vertices adjacent to  $x$  in  $T_1$  will be denoted  $x_0, x_1, \dots, x_{m-1}$  where  $\mathfrak{h}_1(x_i) = \mathfrak{h}_1(x) + 1$  for  $0 \leq i \leq m-1$ . The action of  $a$  on  $DL(m,m)$  is to cause a cyclic rotation among these adjacent vertices so that  $a \cdot (x_i, X) = (x_{(i+1) \bmod m}, X)$ . Elements of the form  $a^{tk}$  cause a similar cyclic rotation on vertices in  $T_2$ .

Let us take a moment to trace out a path in  $DL(3,3)$  from the identity element which leads to the group element  $t(ta)t^{-1}(ta^2)^{-1} \in L_3$ . We will trace out the path to this element by separately examining the path traveled in each tree in  $DL(3,3)$ . Please refer to Figures 6 and 7 for the following explanation.

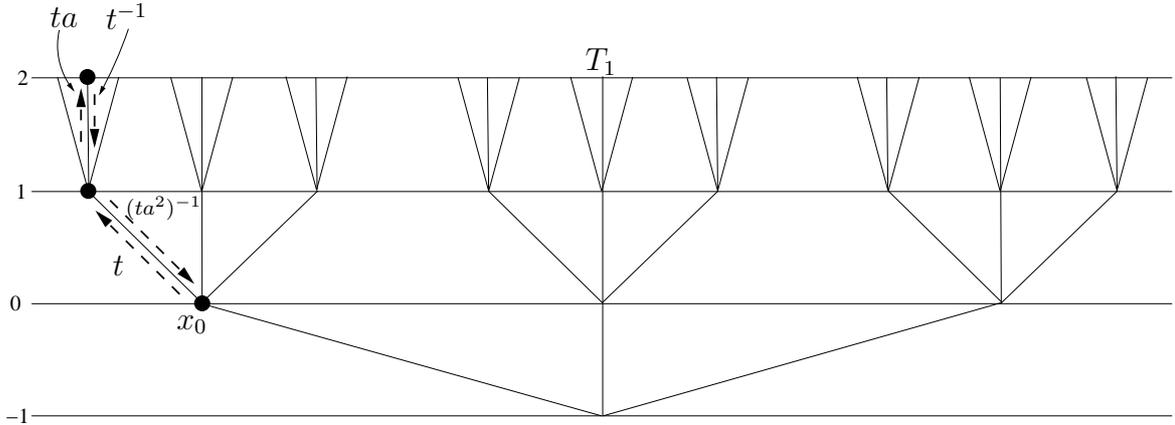


FIGURE 6. The path that the first coordinate of the identity element  $(x_0, X_0)$  of  $L_3$  travels in  $T_1$  of  $DL(3, 3)$  in order to arrive at the element  $t(ta)t^{-1}(ta^2)^{-1}$ .

Beginning at the point  $x_0$  in  $T_1$  of  $DL(3, 3)$ , we first traverse the edge labeled by the generator  $t$  increasing in height in  $T_1$  since the generator has a positive exponent, which is arbitrarily chosen to be the leftmost edge emanating from  $x_0$ . Moving left to right, the next two edges leaving the point  $x_0$  are labeled by  $ta$  and  $ta^2$ , respectively.

The next generator in the element under consideration is  $ta$ , and thus we cyclically rotate over to the right one edge and then traverse it. The last two generators in this element have a negative exponent which means we must move down in height in  $T_1$ . In order to trace out  $t^{-1}$ , we can only travel downward in height through one edge, and are thus forced to travel over the edge that was labeled  $ta$ . Note that in  $T_1$  of  $DL(m, m)$ , there is no way to distinguish between traversing an edge from the set  $\{t^{-1}, (ta)^{-1}, \dots, (ta^{m-1})^{-1}\}$ . As such, when tracing out words in  $L_m$  one must be careful to remember the labelings of edges in  $DL(m, m)$  in order to properly locate group elements. Finally, in a similar fashion as before, we are forced to travel the edge labeled as  $t$  backwards in tracing out the generator  $(ta^2)^{-1}$ .

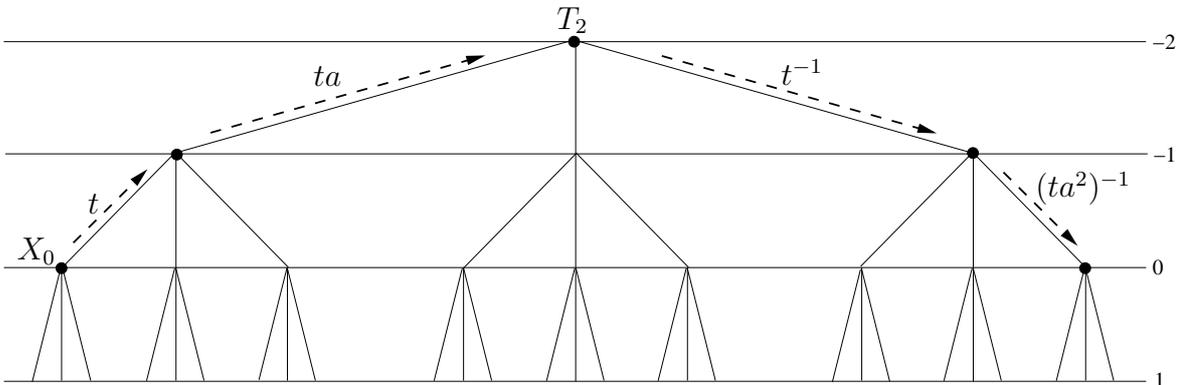


FIGURE 7. The path that the second coordinate of the identity element  $(x_0, X_0)$  of  $L_3$  travels in  $T_2$  of  $DL(3, 3)$  in order to arrive at the element  $t(ta)t^{-1}(ta^2)^{-1}$ .

We must now trace out the path that the point  $X_0$  in  $DL(3, 3)$  takes under the orbit of  $t(ta)t^{-1}(ta^2)^{-1} \in L_3$ . Since the first generator of the word has a positive exponent, we must still move upward in the picture which is the same as decreasing in height in  $T_2$ . Again, there is only one choice for an edge to take, as is the case for the next generator in the word,  $ta$ . The last two generators have a negative exponent which means we must travel up in height in  $T_2$ . Since the previous edge traveled was labeled  $ta$ , then from left to right, the next two edges from height  $-1$  to  $-2$  are labeled as  $ta^2$  and  $t$ , respectively. Therefore we must cyclically rotate over two edges in order to traverse the edge  $t$  backwards which corresponds to tracing out the generator  $t^{-1}$ .

The last generator in the element under consideration is  $(ta^2)^{-1}$ . The convention is to use the same labelings for edges in each set of branches on a given height. Since the initial edge traversed in this tree was labeled  $t$  and originated at height 0, and the terminal point of this path is at height 0, then each set of branches at this height has the leftmost edge labeled by  $t$ , the middle edge labeled by  $ta$ , and the rightmost edge labeled by  $ta^2$ . Therefore we will travel up in height in  $T_2$  over the rightmost edge in order to trace out the element  $(ta^2)^{-1}$ . The paths we have traced out in both trees together form the entire path one must travel from the identity element in  $DL(3, 3)$  to arrive at the element  $t(ta)t^{-1}(ta^2)^{-1} \in L_3$ .

## 5. THE SIGMA INVARIANTS OF $L_m$

Before computing  $\Sigma^n(L_m)$ , we must first determine the dimension of  $\text{Hom}(L_m, \mathbb{R})$  using the presentation  $L_m \cong \langle a, t | a^m = 1, [a^{t^i}, a^{t^j}] = 1 \rangle$ . Let  $\phi : L_m \rightarrow (\mathbb{R}, +)$  be a homomorphism from  $L_m$  to the group of real numbers under addition. Since  $\phi(a^m) = \phi(1)$ , then  $m \cdot \phi(a) = 0$  which implies that  $a \mapsto 0$ . Further, we have that

$$\begin{aligned} \phi(t^{-i}a^{-1}t^i t^{-j}a^{-1}t^j t^{-i}a t^i t^{-j}a t^j) &= \phi(1) \\ \Rightarrow -i \cdot \phi(t) - \phi(a) + (i-j) \cdot \phi(t) - \phi(a) + (j-i) \cdot \phi(t) + \phi(a) + (i-j) \cdot \phi(t) + \phi(a) + j \cdot \phi(t) &= \phi(1) \\ \Rightarrow (-i + i - j + j - i + i - j + j) \cdot \phi(t) &= 0 \end{aligned}$$

which implies that we may map the generator  $t$  to any real number. Thus,  $\text{Hom}(L_m, \mathbb{R}) \cong \mathbb{R}$ . Define the height function  $h : L_m \rightarrow \mathbb{R}$  as  $h(g) := \exp_t(g)$  where  $\exp_t(g)$  is the exponential sum of all occurrences of the generator  $t$  for all  $g \in L_m$ . For example, if given the word  $g = t(ta)t^{-1}(ta^2)^{-1}(ta)$ , then  $h(g) = 1$  since there are 3 occurrences of  $t$  and 2 occurrences of  $t^{-1}$ .

We now prove a lemma that will be used in the proof of the main theorem.

**Lemma 1.** *Let  $\rho^+$  be a sequence of points  $(x_i, X_i), (x_{i+1}, X_{i+1}), \dots, (x_k, X_k)$  in  $DL(m, m)$  such that the points  $x_i, x_{i+1}, \dots, x_k$  are strictly increasing in height in  $T_1$  and the points  $X_i, X_{i+1}, \dots, X_k$  are strictly decreasing in height in  $T_2$ , and let  $\rho^-$  be a sequence of points  $(y_k, Y_k), (y_{k-1}, Y_{k-1}), \dots, (y_i, Y_i)$  in  $DL(m, m)$  such that the points  $y_k, y_{k-1}, \dots, y_i$  are strictly decreasing in height in  $T_1$  and the points  $Y_k, Y_{k-1}, \dots, Y_i$  are strictly increasing in height in  $T_2$ , where  $(x_j, X_j) \neq (y_j, Y_j)$  for all  $i \leq j < k$ ,  $X_j \neq Y_j$  for all  $i < j < k$ , and  $\mathfrak{h}_1(x_i) = \mathfrak{h}_1(y_i)$ ,  $\mathfrak{h}_2(X_i) = \mathfrak{h}_2(Y_i)$  for all  $i \leq j \leq k$ . There is no path  $\rho = \rho^+ \rho^-$  starting at  $(x_i, X_i)$  and ending at  $(y_i, Y_i)$ .*

*Proof.* Suppose  $\rho^+$  is a sequence of points  $(x_i, X_i), (x_{i+1}, X_{i+1}), \dots, (x_k, X_k)$  in  $DL(m, m)$  such that the points  $x_i, x_{i+1}, \dots, x_k$  are strictly increasing in height in  $T_1$ , the points  $X_i, X_{i+1}, \dots, X_k$  are strictly decreasing in height in  $T_2$ , and suppose  $\rho^-$  is a sequence of points  $(y_k, Y_k), (y_{k-1}, Y_{k-1}), \dots, (y_i, Y_i)$  in  $DL(m, m)$  such that the points  $y_k, y_{k-1}, \dots, y_i$  are strictly decreasing in height in  $T_1$  and the points  $Y_k, Y_{k-1}, \dots, Y_i$  are strictly increasing in height in  $T_2$ . Suppose  $(x_j, X_j) \neq (y_j, Y_j)$  for all  $i \leq j < k$ ,  $X_j \neq Y_j$  for all  $i < j < k$ , and  $\mathfrak{h}_1(x_i) = \mathfrak{h}_1(y_i)$ ,  $\mathfrak{h}_2(X_i) = \mathfrak{h}_2(Y_i)$  for all  $i \leq j \leq k$ . Then the initial point of  $\rho = \rho^+ \rho^-$  and the endpoint of  $\rho$  are distinct group elements that share a common second coordinate  $X_i = Y_i \in T_2$ .

Thus  $\rho^+$  is the sequence of generators  $\rho^+ = \{\alpha_i, \alpha_{i+1}, \dots, \alpha_k\}$  such that

$$(x_i, X_i) \prod_{j=i}^k \alpha_j = (x_k, X_k)$$

where each  $\alpha_j \in \{t, ta, \dots, ta^{m-1}\}$ , and  $\rho^-$  is the sequence of generators  $\rho^- = \{\beta_k, \beta_{k-1}, \dots, \beta_i\}$  such that

$$(y_k, Y_k) \prod_{j=k}^i \beta_j = (y_i, Y_i)$$

where each  $\beta_j \in \{t^{-1}, (ta)^{-1}, \dots, (ta^{m-1})^{-1}\}$ .

Since  $\rho^+$  consisted only of generators of positive length, then the orbit of the second coordinate of  $(x_i, X_i)$  under  $\rho^+$  is a unique geodesic path decreasing in height in  $T_2$  ending at  $X_k$ . Since  $T_2$  is a tree, then the only path back to  $X_i = Y_i$  from  $X_k = Y_k$  in  $T_2$  is to retrace the unique geodesic path  $\rho^+$  in reverse. Then the path  $\rho^-$  from  $(y_k, Y_k)$  to  $(y_i, Y_i)$  must be the inverse of  $\rho^+$ .

Thus, if  $\rho^+ = \alpha_i \alpha_{i+1} \dots \alpha_j \dots \alpha_{k-1} \alpha_k$ , then  $\rho^- = \alpha_k^{-1} \alpha_{k-1}^{-1} \dots \alpha_j^{-1} \dots \alpha_{i+1}^{-1} \alpha_i^{-1}$ . Therefore the orbit of  $y_k$  under  $\rho^-$  is a unique geodesic path decreasing in height in  $T_1$  ending at  $x_i$ . Therefore,  $x_j = y_j$  for all  $i \leq j \leq k$ , and thus  $(x_i, X_i) = (y_i, Y_i)$ . This is a contradiction, and therefore there is no path  $\rho = \rho^+ \rho^-$  starting at  $(x_i, X_i)$  and ending at  $(y_i, Y_i)$  when  $(x_i, X_i) \neq (y_i, Y_i)$ .  $\square$

**Theorem 1.**  $\Sigma^1(L_m) = \emptyset$

*Proof.* We proceed by proving the negation of the condition for the point  $\infty \in \partial_\infty \mathbb{R}$  to be an element of  $\Sigma^1(L_m)$ . Due to the symmetry of the Diestel-Leader graph  $DL(m, m)$  which is representative of  $\Gamma$ , the Cayley graph of  $L_m$ , a similar argument showing that  $-\infty \notin \Sigma^1(L_m)$  can be made. It suffices to show that  $\infty \notin \Sigma^1(L_m)$ . Define  $\gamma := [0, \infty)$  and similarly  $\gamma^- := (-\infty, 0]$ . By definition,  $\infty \notin \Sigma^1(L_m)$  if and only if there exists an  $s \geq 0$  such that there exist two elements  $u, v \in \Gamma_s^+(\gamma)$  such that for all paths  $\rho$  from  $u$  to  $v$ ,  $\rho \cap (\Gamma_s^+(\gamma))^c \neq \emptyset$  where  $\Gamma_s^+(\gamma)$  denotes the largest subgraph of  $\Gamma$  containing  $h^{-1}(H_s(\gamma))$  and for all  $g \in \Gamma_s^+(\gamma)$ ,  $\exp_t(g) \geq 0$ . By Lemma 1, no paths exist such that  $\rho$  is entirely contained in  $\Gamma_s^+(\gamma)$ . Consider the identity element of  $L_m$  denoted  $(x_0, X_0)$  and the element  $a = t^{-1}ta$  which we denote  $(y_0, X_0)$ . Since  $\mathfrak{h}_1(x_0) = \mathfrak{h}_1(y_0)$  and  $\mathfrak{h}_1(x_0) + \mathfrak{h}_2(X_0) = 0$ , then the shortest path between the two points contains the element  $t^{-1}$  which lies in  $\Gamma_s^-(\gamma^-)$ , the largest subgraph of  $\Gamma$  containing  $h^{-1}(H_s(\gamma^-))$  and for all  $g' \in \Gamma_s^-(\gamma^-)$ ,  $\exp_t(g') \leq 0$ . We need not consider paths of greater length between these two points as they are all homotopic to

$t^{-1}ta$ . Thus we have shown the existence of two points such that for all paths  $\rho$  from  $(x_0, X_0)$  and  $(y_0, X_0)$ ,  $\rho \cap (\Gamma_s^+(\gamma))^c \neq \emptyset$ .  $\square$

**Corollary 1.**  $\Sigma^n(L_m) = \emptyset$

**Corollary 2.** *The commutator subgroup of  $L_m$  is not finitely generated.*

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