An Exploration of the Cantor Set

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Introduction

Georg Cantor (1845-1918), a mathematician best known for his work in set theory, first introduced the set that became known as the Cantor ternary set in the footnote to a statement saying that perfect sets do not need to be everywhere dense. This footnote gave an example of an infinte, perfect set that is not everywhere dense in any interval, a set he defined as real numbers of the form

$$x = \frac{c_1}{3} + ... + \frac{c_v}{3^v} + ... \tag{1.1}$$

where c_v is 0 or 2 for each positive integer v.[1]

In this paper, we attempt to study the Cantor ternary set from the perspective of fractals and Hausdorff dimension and make our own investigations of other general Cantor sets we will construct and their dimensions. Cantor sets can be understood geometrically by imagining the continuous removal of a set portion of a shape in such a way that at every stage of removal, the chucks of the shape that remain each have the same percentage removed from their centers. If this removal process continues infinitely, then the tiny bits of the shape that remain form a Cantor set. The dimension of such a set is not an integer value. In effect, it has a "fractional" dimension, making it by definition a **fractal**. Points are understood to have a dimension of 0 and lines have a dimension of 1, but the dimension of a fractal set could lie anywhere inbetween integer values, depending on how the set is formed.

The term "Cantor set" is most often used to refer to what is known as the Cantor ternary set, which is constructed as follows:

Let I be the interval $[0, 1]$. Divide I into thirds. Remove the open interval that makes up the middle third, that is, $(\frac{1}{3}, \frac{2}{3})$, and let A_1 be the remaining set. Then

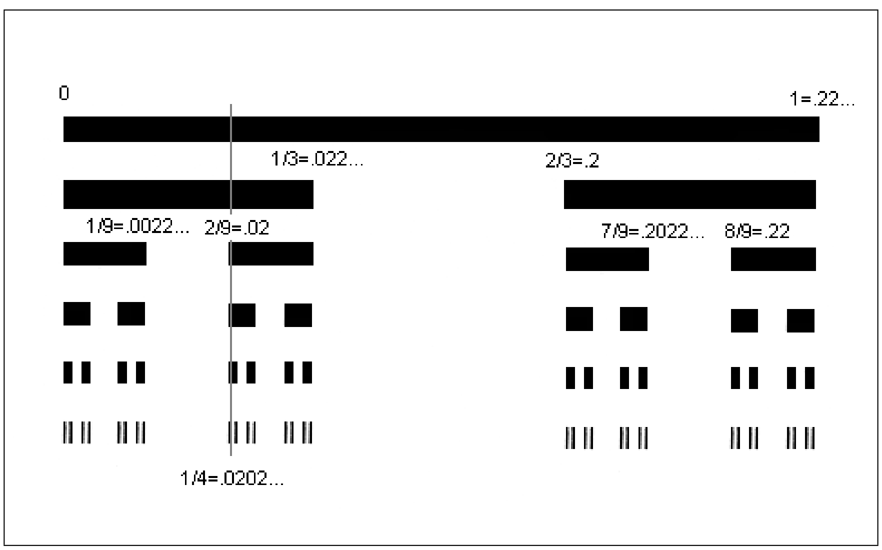
$$A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]. \tag{1.2}$$

Continue by removing the open middle third interval from each of the two closed inervals in A_1 and call the remaining set A_2. So,

$$A_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]. \tag{1.3}$$

Continue in this fashion at each step k for $k \in \mathbb{N}$, removing the open middle third interval from each of the closed intervals in A_k and calling the remaining set A_{k+1}. For each $k \in \mathbb{N}$, A_k is the union of 2^k closed intervals each of length 3^{-k}.

Definition (1.4): The Cantor ternary set, which we denote C_3, is the "limiting set" of this process, i.e. $C_3 = \bigcap_{k=1}^{\infty} A_k$. [2]



Five stages of removal of the Cantor Ternary Set

Each element of C_3 can be written in a ternary (base 3) expansion of only 0s and 2s. At every level of removal, every number with a ternary expansion involving a 1 in a particular place is removed. At the first stage of removal, for instance, any number remaining would have the digit $c_1 = 0$ or 2 where $x = 0.c_1c_2c_3...$, since if $x \in [0, \frac{1}{3}]$, $c_1 = 0$ and if $x \in [\frac{2}{3}, 1]$, $c_1 = 2$. Repeating this argument for each level of removal, it can be shown that if x remains after removal n, c_n is 0 or 2.

The Cantor ternary set is interesting in mathematics because of the apparent paradoxes of it. By the way it is constructed, an infinite number of intervals whose total length is 1 is removed from an interval of length 1, so the set cannot contain any interval of non-zero length. Yet the set does contain an infinite number of points, and, in fact, it has the cardinality of the full interval $[0, 1]$. So the Cantor set contains as many points as the set it is carved out of, but contains no intervals and is nowhere dense, meaning that the set has no points that are completely surrounded by other points of the set. We know that the set contains an infinite number of points because the endpoints of each closed interval will always remain in the set, but the Cantor set actually contains more than just the endpoints of the closed intervals A_k. In fact, $\frac{1}{4} \in C$ but is not an endpoint of any of the intervals in any of the sets A_k. We can write $\frac{1}{4}$ as $0.0202\overline{02}...$ in ternary expansion. At the k^{th} stage of removal, any new endpoint has a form of either a 2 in the $3^{-(k-1)}$ ternary place which repeats infinitely or terminates at the $3^{-k^{th}}$ ternary place. Since $\frac{1}{4} = 0.0202\overline{02}$, it does not follow the pattern of the endpoints. Therefore, $\frac{1}{4}$ is not an endpoint of the Cantor set, and there are actually infinitely many points like that.

Properties of the Cantor set

Certain easily proven properties of the Cantor ternary set, when they are pieced together, help to show the special nature of Cantor sets. The Cantor ternary set, and all general Cantor sets, have uncountably many elements, contain no intervals, and are compact, perfect, and nowhere dense.

C_3 has uncountably many elements.

We will show that the Cantor ternary set has uncountable many elements by contradiction. Let $C_3 = \{x \in [0, 1) : x \text{ has a ternary expansion involving only zeros and twos}\}$. Suppose C_3 is countable. Then there exists $f : \mathbb{N} \rightarrow^{1-1}_{onto} C_3$ by the definition of countablility. Let $x_n = f(n)$ for all $n \in \mathbb{N}$. So $C_3 = \{x_1, x_2, x_3, ..., x_n, ...\}$ where:

$$
\begin{aligned}
x_1 &= 0.c_{1_1}c_{1_2}c_{1_3}...\\
x_2 &= 0.c_{2_1}c_{2_2}c_{2_3}...\\
&\vdots\\
x_n &= 0.c_{n_1}c_{n_2}c_{n_3}...\\
&\vdots
\end{aligned}
\tag{2.1}
$$

where c_{n_m} is either 0 or 2 for all n, m. Define $c = 0.c_1c_2c_3...$ by

$$
c_1 = \begin{Bmatrix} 2 \text{ if } c_{1_1} = 0 \\ 0 \text{ if } c_{1_1} = 2 \end{Bmatrix}, c_2 = \begin{Bmatrix} 2 \text{ if } c_{2_2} = 0 \\ 0 \text{ if } c_{2_2} = 2 \end{Bmatrix}, ..., c_n = \begin{Bmatrix} 2 \text{ if } c_{n_n} = 0 \\ 0 \text{ if } c_{n_n} = 2 \end{Bmatrix}, ... \tag{2.2}
$$

Clearly, $c \in C_3$. But, $c \neq x_n$ for any n, since c and x_n differ in the $3^{-n^{th}}$ place. This is a contradiction. Therefore, C_3 is uncountable.

C_3 contains no intervals.

We will show that the length of the complement of the Cantor set C_3 is equal to 1, hence C_3 contains no intervals. At the k^{th} stage, we are removing 2^{k-1} intervals from the previous set of intervals, and each one has a length of $\frac{1}{3^k}$. The length of the complement within $[0, 1]$ after an infinite number of removals is:

$$
\sum_{k=1}^{\infty} 2^{k-1}(\frac{1}{3^k}) = \frac{1}{3}\sum_{k=1}^{\infty}(\frac{2}{3})^{k-1} = \frac{1}{3}\sum_{k=0}^{\infty}(\frac{2}{3})^k = \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = 1. \tag{2.3}
$$

Thus, we are removing a length of 1 from the unit interval $[0, 1]$ which has a length of 1. Therefore, the Cantor set must have a length of 0, which means it has no intervals.

C_3 is compact.

Using the Heine-Borel Theorem, which states that a subset of $\mathbb{R}$ is compact iff it is closed and bounded, it can be shown rather easily that C_3 is compact. C_3 is the intersection of a collection of closed sets, so C_3 itself is closed. Since each A_k is within the interval $[0, 1]$, C_3, as the intersection of the sets A_k, is bounded. Hence, since C_3 is closed and bounded, C_3 is compact.

So far we have that the Cantor set is a subset of the interval $[0, 1]$ that has uncountably many elements yet contains no intervals. It has the cardinality of the real numbers, yet it has zero length.

C_3 is perfect.

A set is considered perfect if the set is closed and all the points of the set are limit points of the set. Since C_3 is compact, it is necessarily closed. For each endpoint in the set C_3 there will always exist another point in the set within a deleted neighborhood of some radius $\varepsilon > 0$ on one side of that point since the remaining intervals at each step are being divided into infinitely small subintervals and since the real numbers are infinitely dense. Likewise, for each nonendpoint in the set there will always exist another point in the set within a deleted neighborhood of some radius $\varepsilon > 0$ on both sides of that point. Hence, there must exist a deleted neighborhood of some radius $\varepsilon > 0$ around each point of the set C_3 for which the intersection of that deleted neighborhood and the set is nonempty. Therefore, each point in the set is a limit point of the set, and since the set is closed, the set C_3 is perfect.

C_3 is nowhere dense.

A set is considered nowhere dense if the interior of the closure of the set is empty, that is, when you add the limit points of the set, there are still no intervals. The closure of a set is the union of the set with the set of its limit points, so since every point in the set C_3 is a limit point of the set the closure of C_3 is simply the set itself. Now, the interior of the set must be empty since no interval of points remains in the set. At the infinite level of removal, if there did exist an interval of points, the middle third section of that interval will be removed and the removal would continue on an infinitely small scale, ultimately removing anything between two points. Hence the set C_3 is nowhere dense.

Dimension of the Cantor Set

The Cantor set seems to be merely a collection of isolated points, so should intuitively have a dimension of zero, as any random collection of isolated points would have. Under the concept of topological dimension, a point has a dimension of 0, a line has a dimension of 1, a plane has a dimension of 2, a cube has a dimension of 3, and the Euclidean space $\mathbb{R}^n$ has a dimension of n. The topological dimension of a space equals the number of real number parameters that are necessary to describe different points in a space. In this sense, the topological dimension of the Cantor set is 0. But utilizing a different definition of dimension, such as Hausdorff dimension, allows us to see the fractional dimension of the Cantor set while still maintaining the integer dimensions of points, lines, and planes.

Hausdorff Dimension

The dimension of any subset E of the real numbers $\mathbb{R}^n$ can be measured using Hausdorff dimension. In simple terms, Hausdorff dimension involves covering the set E with δ-covers. If you take the sum of the diameters of all δ-covers of E raised to the power of s, the infimum (greatest lower bound) of that sum is denoted $H^s_\delta(E)$. If we take the supremum (least upper bound) of all possible values of $H^s_\delta(E)$, we get the Hausdorff s-dimensional measure of E. Finally, the Hausdorff dimension of E is obtained by taking the smallest real number value of s for which the Hausdorff s-dimensional measure of E is 0, that is, the smallest value of s (s being the power to which we raised the diameters of the δ-covers of E) for which we are still able to cover the set. (For a more technical mathematical definition of Hausdorff dimension, see K.J. Falconer's book, *The Geometry of Fractal Sets.*)

Hausdorff dimension generalizes the concept of dimension of a vector space in such a way that points have Hausdorff dimension 0, lines have Hausdorff dimension 1, etc. but in general the Hausdorff dimension of a set is not necessarily integer-valued. Fractals are defined as sets whose Hausdorff dimension is greater than its topological dimension, with the Hausdorff dimension of fractals specifically non-integer.

There is a shortcut for computing the Hausdorff dimension of sets that are self-similar in nature. Self-similarity is a property that many fractals have, but not all, and it occurs when a set is exactly similar to a part of itself. In a sense, zooming in on a smaller and smaller range of a self-similar set reveals a smaller copy of the same set. Mathematically, to be self-similar, the mappings that create a set A down to the infinite level must be a finite collection of similitudes, that is, the mappings ϕ_i that generate the set on each level preserve the geometry of the set, such that the set A is invariant with respect to the set of mappings ϕ_i, and there must exist a positive real number s such that H^s of A (the Hausdorff s-dimensional measure of A) is positive but H^s of the intersection of two different mappings ϕ_1 and ϕ_2 of A is zero.

The Cantor set is self-similar since its mappings, all variations on $\frac{1}{3}x$, preserve the geometry of the set and there is a positive Hausdorff s-dimensional measure of the set but not of the intersection of two different mappings since each pair of mappings of the set has a non-empty intersection. The Hausdorff dimension of a self-similar set can be found by using the following theorem:

Theorem 1: Let $\{\Phi_i\}_{i=1}^k$ be a collection of similitudes such that $E \subseteq \mathbb{R}^n$ is invariant with respect to $\{\Phi_i\}_{i=1}^k$. If $\{\Phi_i\}_{i=1}^k$ satisfies the open set condition and r_i is the ratio of the i-th similitude Φ_i, then the Hausdorff dimension of E is equal to the unique positive number s for which $\sum_{i=1}^{k}(r_i)^s = 1$. [**3**]

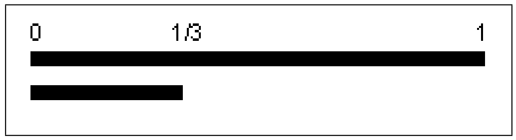
The computation of the Hausdorff dimension of the Cantor ternary set, C_3, follows very easily from Theorem 1.

Proposition: The dimension of the Cantor ternary set C_3 is $d = \frac{\log 2}{\log 3}$.

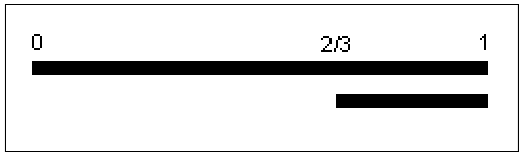
Proof: Let $\phi_1(x)$ and $\phi_2(x)$ be defined as:

$$\phi_1(x) = \frac{1}{3}x$$

$$\phi_2(x) = \frac{1}{3}x + \frac{2}{3} \tag{3.1}$$



$\phi_1(x)$, mapping $[0,1]$ to $[0,1/3]$



$\phi_2(x)$, mapping $[0,1]$ to $[2/3,1]$

Notice that $C_3 = \bigcup_{i=1}^{2} \Phi_i(C_3)$. Also, $\{\Phi_i\}_{i=1}^{2}$ satisfies the open set condition for $W = (0,1)$. Applying the theorem with $r_1 = \frac{1}{3}$ and $r_2 = \frac{1}{3}$, we need to find s such that

$$\sum_{i=1}^{2} (r_i)^s = 1.$$

$$2(\frac{1}{3})^s = 1 \text{ iff } s = \frac{\log 2}{\log 3}. \tag{3.2}$$

$$\dim(C_3) = \frac{\log 2}{\log 3}. \quad \blacksquare$$

[3].

General Cantor Sets

Up to this point, our discussion of the Cantor set has been limited to what is known as the Cantor ternary set, defined in the Introduction. We will now discuss some generalizations. My further investigations were motivated by a curiosity as to what would happen to the dimension of the set if the removal process was defined differently. We will consider three different general methods of removal from the interval $[0,1]$ that depend upon a natural number k. As will be shown, all three methods of removal are equivalent when $k = 3$, yielding the Cantor ternary set.

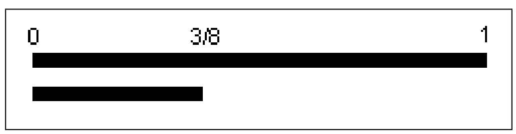
1.) Method C: Let $\{C_k\}$ be the collection of sets defined in terms of k, for $k \geq 2$, in which each set in the sequence is formed by the repetitive removal of $\frac{1}{k}$ of each remaining interval, removing an open interval from the center of each closed interval, starting with the interval $[0, 1]$. In this way, the size of the closed intervals remaining on either side of the open interval removed will be $(\frac{1}{2} - \frac{1}{2k})$.

Since each of the sets C_k are self-similar sets, we can use Theorem 1 to find the Hausdorff dimension of each of the sets, which we will do in general for any $k \geq 2$.

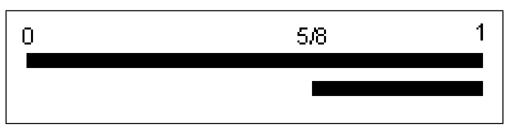
Let $\phi_1(x)$ and $\phi_2(x)$ be defined as:

$$\phi_1(x) = (\frac{1}{2} - \frac{1}{2k})x$$

$$\phi_2(x) = (\frac{1}{2} - \frac{1}{2k})x + \frac{1}{2} + \frac{1}{2k} \tag{4.1}$$



$\phi_1(x)$ when $k = 4$, mapping $[0, 1]$ to $[0, 3/8]$



$\phi_2(x)$, with $k = 4$, mapping $[0, 1]$ to $[5/8, 1]$

Notice that $C_k = \bigcup_{i=1}^{2} \phi_i(C_k)$. Applying the theorem for self-similar sets with $r_1 = (\frac{1}{2} - \frac{1}{2k})$ and $r_2 = (\frac{1}{2} - \frac{1}{2k})$, we need to find s such that

$$\sum_{i=1}^{2} (r_i)^s = 1. \tag{4.2}$$

So,

$$2(\frac{1}{2} - \frac{1}{2k})^s = 1 \text{ iff } s = \frac{\log \frac{1}{2}}{\log(\frac{1}{2} - \frac{1}{2k})}. \tag{4.3}$$

Thus,

$$\dim(C_k) = \frac{\log \frac{1}{2}}{\log(\frac{1}{2} - \frac{1}{2k})}. \quad \blacksquare \tag{4.4}$$

2.) Method D: Let $\{D_k\}$ be the collection of sets defined in terms of k, for $k \geq 3$, in which each set in the sequence is formed by the repetitive removal of $(1 - \frac{2}{k})$ of each remaining interval, removing an open interval from the center of each closed interval starting with $[0, 1]$, with intervals of size $\frac{1}{k}$ remaining on each side. Notice that with this method of removal, we are varying the length of the side intervals in terms of k then removing the interval inbetween, whereas in the first method of removal, method C, you are varying the length of the center interval in terms of k.

Again, since each of the sets D_k are self-similar sets, we can use Theorem 1 to find the Hausdorff dimension of each of the sets, which we will do in general for any $k \geq 3$.

Let $\phi_1(x)$ and $\phi_2(x)$ be defined as:

$$\begin{aligned} \phi_1(x) &= \frac{1}{k}x \\ \phi_2(x) &= \frac{1}{k}x + 1 - \frac{1}{k} \end{aligned} \tag{4.5}$$

Notice that $D_k = \bigcup_{i=1}^{2} \phi_i(D_k)$. Applying the theorem for self-similar sets with $r_1 = \frac{1}{k}$ and $r_2 = \frac{1}{k}$, we need to find s such that

$$\sum_{i=1}^{2} (r_i)^s = 1. \tag{4.2}$$

So,

$$2(\frac{1}{k})^s = 1 \text{ iff } s = \frac{\log 2}{\log k}. \tag{4.6}$$

Thus,

$$\dim(D_k) = \frac{\log 2}{\log k}. \quad \blacksquare \tag{4.7}$$

3.) Method E: Let $\{E_k\}$ be the collection of sets defined in terms of k, for $k \geq 3$, in which each set in the sequence is formed by the repetitive removal of $\frac{1}{k}$ of each remaining interval, removing only alternating open intervals from each closed interval, starting with $[0,1]$, when each closed interval is divided into k subintervals. This method of removal leads to two similar but distinctly different cases. When k is odd, $\frac{k-1}{2}$ alternating sections are removed from each closed interval, leaving each interval of size $\frac{1}{k}$. When k is even, $(\frac{k}{2} - 1)$ alternating sections are removed from each closed interval, leaving one interval of size $\frac{1}{k}$ on the left end and two full intervals each of size $\frac{1}{k}$ adjacent to each other on the right end.

Since different mappings will be required in order to generate the sets, each of the two cases yields sets with different Hausdorff dimensions. In the case where k is odd, using Theorem 1 to find the Hausdorff dimension of these self-similar sets, we let $\phi_1(x)$, $\phi_2(x)$, ... , $\phi_{\frac{k+1}{2}}(x)$ be defined as:

$$\begin{aligned} \phi_1(x) &= \frac{1}{k}x \\ \phi_2(x) &= \frac{1}{k}x + \frac{2}{k} \\ &, ..., \\ \phi_{\frac{k+1}{2}}(x) &= \frac{1}{k}x + \frac{k-1}{k} \end{aligned} \tag{4.8}$$

Note that the number of ϕ_i mappings needed is determined by the value of the natural number k, with $\frac{k+1}{2}$ mappings needed when k is odd. Applying the theorem for self-similar sets with $r_1 = \frac{1}{k}$, $r_2 = \frac{1}{k}$, ... , $r_{\frac{k+1}{2}} = \frac{1}{k}$, we need to find s such that

$$\sum_{i=1}^{\frac{k+1}{2}} (r_i)^s = 1. \tag{4.2}$$

So,

$$(\frac{k+1}{2})(\frac{1}{k})^s = 1 \text{ iff } s = \frac{\log \frac{k+1}{2}}{\log k}. \tag{4.9}$$

Thus, when k is odd,

$$\dim(E_k) = \frac{\log \frac{k+1}{2}}{\log k}. \quad \blacksquare \tag{4.10}$$

In the case where k is even, we let $\phi_1(x)$, $\phi_2(x)$, ... , $\phi_{\frac{k+1}{2}}(x)$ be defined as:

$$\phi_1(x) = \frac{1}{k}x$$

$$\phi_\alpha(x) = \frac{1}{k}x + \frac{2}{k} \tag{4.11}$$

$$,...,$$

$$\phi_{\frac{k}{2}}(x) = \frac{2}{k}x + \frac{k-2}{k}$$

for $\alpha = 1, 2, ..., (\frac{k}{2} - 1)$. Note that the number of ϕ_i mappings is determined by k, with $\frac{k}{2}$ mappings needed when k is even. Applying the theorem for self-similar sets with $r_1 = \frac{1}{k}$, $r_\alpha = \frac{1}{k}$, $r_{\frac{k}{2}} = \frac{2}{k}$, we need to find s such that

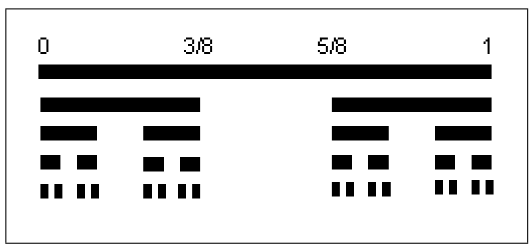
$$\sum_{i=1}^{\frac{k}{2}} (r_i)^s = 1. \tag{4.2}$$

So,

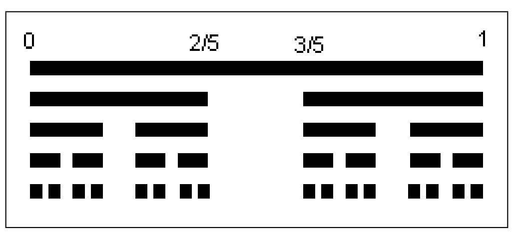
$$(\frac{k}{2}-1)(\frac{1}{k})^s+(\frac{2}{k})^s=1. \tag{4.12}$$

Simplifying this equation to $(\frac{k}{2}-1)=k^s-2^s$, it is not noticeably solvable directly for a general k, but the dimension of each set is equal to the value of s that satisfies the equation for its given k, which can be calculated for $k=4$ using the quadratic formula.

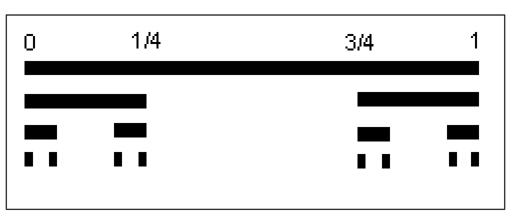
The differences between the sets formed under each method of removal are apparent when $k=4$, and the differences between different values of the natural number k for a given method of removal are apparent by comparing $k=4$ with $k=5$.



C_4: dimension ≈ 0.7067

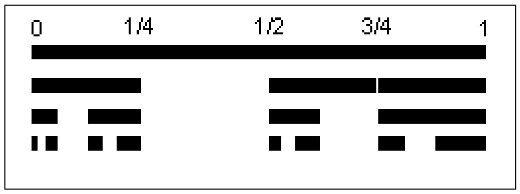


C_5: dimension ≈ 0.7565

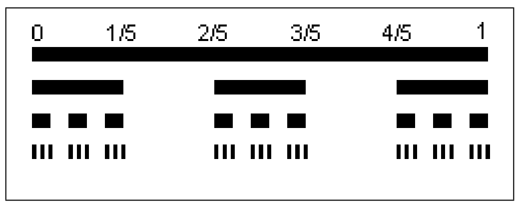


D_4: dimension $=0.5$

D_5: dimension ≈ 0.4307



E_4: dimension ≈ 0.6942



E_5: dimension ≈ 0.6826

Note**:** $C_3 = D_3 = E_3$. The set C_3 is formed by the repetitive removal of an open interval of length $\frac{1}{3}$ from the center of each closed interval, starting with the interval $[0, 1]$ which leaves closed intervals of size $\frac{2}{3}$ on each side. Since the set D_3 is formed by the repetitive removal of an open interval of length $(1 - \frac{2}{3}) = \frac{1}{3}$ from the center of each closed interval starting with $[0, 1]$, with intervals of length $\frac{1}{3}$ remaining on each side, it is equivalent to the set C_3. Also, since the set E_3 is formed by dividing the interval $[0, 1]$ into 3 subintervals and removing alternating sections, which is only the center section, with intervals of length $\frac{1}{3}$ on each end, it is equivalent to both C_3 and D_3. Hence, $C_3 = D_3 = E_3$.

Also note that the calculations of dimension for each set yield the same dimension, which must be the case since the three sets are the same. So,

$$\dim(C_3) = \frac{\log \frac{1}{2}}{\log(\frac{1}{2} - \frac{1}{6})} = \frac{\log 2}{\log 3}. \quad \dim(D_3) = \frac{\log 2}{\log 3}. \quad \dim(E_3) = \frac{\log \frac{3+1}{2}}{\log 3} = \frac{\log 2}{\log 3} \tag{4.13}$$

Thus, $\dim(C_3) = \dim(D_3) = \dim(E_3)$.

In each method of removal, what happens to the dimension as k approaches infinity?

In method C, $\dim(C_k) = \frac{\log \frac{1}{2}}{\log(\frac{1}{2} - \frac{1}{2k})}$. So,

$$\lim_{k\to\infty} \dim(C_k) = \lim_{k\to\infty} \left(\frac{\log \frac{1}{2}}{\log(\frac{1}{2} - \frac{1}{2k})} \right) = \frac{\log \frac{1}{2}}{\log \frac{1}{2}} = 1. \tag{4.14}$$

Hence, when $[0, 1]$ is divided into 3 subintervals, the smaller the length of the interval $\frac{1}{k}$ removed, the closer the Hausdorff dimension gets to 1.

In method D, $\dim(D_k) = \frac{\log 2}{\log k}$. So,

$$\lim_{k\to\infty} \dim(D_k) = \lim_{k\to\infty} \left(\frac{\log 2}{\log k} \right) = 0. \tag{4.15}$$

Hence, when $[0, 1]$ is divided into 3 subintervals, the smaller the length of the side intervals $\frac{1}{k}$, that is, the larger the length of the interval $(1 - \frac{2}{k})$ removed, the closer the Hausdorff dimension gets to 0.

In method E, $\dim(E_k) = \frac{\log \frac{k+1}{2}}{\log k}$ when k is odd. So,

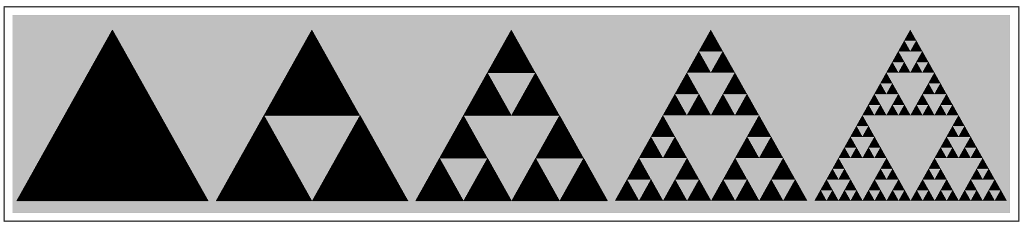
$$\lim_{k\to\infty} \dim(E_k) = \lim_{k\to\infty} \left(\frac{\log \frac{k+1}{2}}{\log k} \right) = 1. \tag{4.16}$$

Hence, when $[0, 1]$ is divided into an odd number of equal length intervals, the larger the number of such intervals, the closer the Hausdorff dimension gets to 1.

Conclusion

The Cantor ternary set and the general Cantor sets are all examples of fractal sets. Their self-similarity allows their Hausdorff dimension to be calculated easily, and each is shown to be non-integer. Many questions for further investigation remain, including expoloring other methods of removal. The ultimate question is whether or not it is possible to find a method of removal that will yield a specific given Hausdorff dimension. Future investigation could also center around what happens to the Hausdorff dimension when two or more Cantor sets of any variety are combined in a union, yielding, for example, a "double" Cantor set. In such a set, is the dimension double the dimension of a single such Cantor set, the same as a single set, or some other dimension entirely? What about three Cantor sets lying end to end?

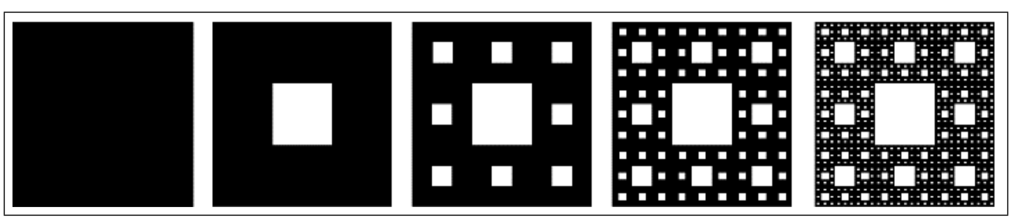
My investigations also included calculating the Hausdorff dimensions (using Theorem 1) of the traditional Sierpinski figures which follow a similar iterative removal process to that used in the Cantor sets. The Sierpinski triangle, formed by the repetitive removal of the center triangle that forms one-fourth of the area of the larger triangle, yields after an infinte number of removals a figure with a Hausdorff dimension of $\frac{\log 3}{\log 2}$.



The first four stages of removal of the Sierpinski triangle

[1]

The Sierpinski square, or Sierpinski carpet, formed by the repetitive removal of the center square that forms one-ninth of the area of the larger square, yields after an infinite number of removals a figure with a Hausdorff dimension of $\frac{\log 8}{\log 3}$.



The first four stages of removal of the Sierpinski square

[2]

Similarly, the Sierpinski cube, formed by the repetitive removal of the center axis of seven cubes that each form one-twenty-seventh of the volume of the larger cube, yields after an infinite number of such removals

[1] Image obtained from http://www.math.cornell.edu/~numb3rs/jrajchgot/505f.html
[2] Image obtained from eom.springer.de/common_img/s130310e.gif.

a figure with a Hausdorff dimension of $\frac{\log 20}{\log 3}$. Their dimensions demonstrate a pattern for these types of figures, that is, the dimension appears to equal $\frac{\log a}{\log b}$, where a is the number of sections (of triangles, squares, cubes, etc.) that remain after one removal and b is the ratio of the side length of the larger figure to that of the next smaller stage. Since the triangle explored was equilateral, further investigation could be done to consider the Hausdorff dimension of isoceles or scalene triangles, varying the lengths of the sides and the sizes of the angles. Similarly, the dimensions of other Sierpinski figures such as a pyramid, a rectangle, or a prism could also be explored, though the removal process may need to be defined differently if the removed sections do not end up being similar in nature.

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