A stochastic shell model of turbulence: numerical and analytical results

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Abstract

We consider an inviscid shell model of turbulence with the addition of Itô and Stratonovich multiplicative stochastic forcing. Numerical simulations are performed for both models that show dissipation of energy at an algebraic rate. A comparison between the Itô and Stratonovich effects is examined. Positivity of solutions is discussed and demonstrated numerically.

1 Introduction

The Navier-Stokes equations for incompressible fluid flow are given by the system

$$\begin{aligned} &\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \nu \Delta u + f, \\ &\nabla \cdot u = 0, \end{aligned} \tag{1.1}$$

where p is the pressure, ν is the viscosity coefficient, u is the velocity vector field and f is some external force. One important property of the Navier-Stokes equations is that they model the energy transfer from large scales to small scales, according to Kolmogorov's law; see [13], [18] and [14] and the references therein. This is achieved

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via nonlinear interaction between the modes in the Fourier space. There is extensive experimental, numerical and analytical literature about this energy transfer; see [13], [11] and the references therein.

Shell models are some of the most interesting and most popular artificial phenomenological models of turbulence that capture some properties of structure function in some range of wave numbers. We refer the reader to Biferale [4] for several descriptions and results. Shell models are simplified models of the Fourier system of Navier-Stokes equations that consider interactions only between the nearest neighbors. The transfer of energy from large scales to small scales achieved through the nonlinear term in the Navier-Stokes system is preserved in the shell models. We are interested in the following particular shell model written as an infinite system of coupled equations:

$$\begin{cases} X_0(t) = 0 & t \geq 0, \\ \frac{d}{dt}X_n(t) + \nu k_n^2 X_n(t) = k_{n-1}X_{n-1}^2(t) - k_n X_n(t)X_{n-1}(t) + f_n & t \geq 0,\, n \geq 1. \end{cases} \tag{1.2}$$

Here $k_n = \lambda^n$, for $\lambda > 1$, and f_n is a deterministic forcing term. There is extensive literature for the study of this model and its variants (see for example, [16], [5], [2], [6], [7], [8], [12], [9] and [10]). In the stochastic case, we refer to [1].

The viscosity term, $\nu k_n^2 X_n(t)$, causes the energy of (1.2) to dissipate quickly. Recent work has shown, however, that the energy dissipates even in the absence of viscosity [8], [2]. This result is surprising, because when $\nu = 0$, (1.2) is formally conservative. Here we consider system (1.2) with a stochastic forcing term. We examine the long-term behavior of the energy of the model with stochastic forcing. Previous work on the deterministic system (1.2) relied heavily on the positivity of solutions [2]. With the addition of the stochastic forcing term, positivity of solutions is not guaranteed. Because of this, our analytical bounds are general. However, our numerical results demonstrate that energy dissipation is likely occurring in the stochastic case in a manner similar to the dissipation in the deterministic case.

Our paper is organized as follows. In Section 2 we introduce two versions of the stochastic model, the Itô and Stratonovich systems. In Section 3 we present our analytical results for the energy of these systems. This is followed in Section 4 by the numerical approximations for the energy and solution paths. In Section 5, we compare the results for these two systems and account for the differences. In Section 6 we discuss the necessity of positivity for a bound for $E(X(t))$. In the Appendix we provide a review of elements of stochastic analysis.

2 The Stochastic Model

Let $(\Omega, \mathcal{F}, P)$ be a probability space and E be the expected value. For $n \geq 1$, let $B_n : \Omega \times [0,T] \mapsto \mathbb{R}$ be a sequence of real-valued Brownian motion. We consider the stochastic forcing term $f_n = g(X_n)dB_n$ in both the Itô and Stratonovich sense. The difference between these two stochastic integrals is provided in Appendix A. We will denote the difference between the two stochastic integrals by writing the Itô as $\int_0^t g(X_n(s))dB_n$ and the Stratonovich as $\int_0^t g(X_n(s)) \circ dB_n$.

Replacing f_n in (1.2) with the multiplicative stochastic forcing term and taking $\nu = 0$ gives the Itô system

$$\begin{cases} X_0(t) = 0 & t \geq 0, \\ dX_n(t) = \big(k_{n-1}X_{n-1}^2(t) - k_n X_n(t) X_{n+1}(t)\big)dt + g(X_n(t))dB_n & t \geq 0,\ n \geq 1, \\ X_n(0) = X_n^0, & \end{cases} \tag{2.1}$$

and the Stratonovich system

$$\begin{cases} X_0(t) = 0 & t \geq 0, \\ dX_n(t) = \big(k_{n-1}X_{n-1}^2(t) - k_n X_n(t) X_{n+1}(t)\big)dt + g(X_n(t)) \circ dB_n & t \geq 0,\ n \geq 1, \\ X_n(0) = X_n^0. & \end{cases} \tag{2.2}$$

Let us denote by H the space of all square summable sequences of real numbers, that is,

$$H = \left\{ u_n \in \mathbb{R};\ \sum_{n=1}^{\infty} u_n^2 < \infty \right\}.$$

Thus, H is a Hilbert space with the inner product $< u, v >_H = \sum_{n=1}^{\infty} u_n v_n$, where $u = (u_n)_{n\in\mathbb{N}}$, $v = (v_n)_{n\in\mathbb{N}}$, $u, v \in H$. The corresponding norm in H will be denoted by $|\cdot|_H$. We denote the space of all sequences of real numbers by $\mathbb{R}^N$, and its subset of all non-negative real numbers by $\mathbb{R}_+^N$. We will use the term "energy" for the quantity $|X|_H^2$, for an element $X \in H$.

Let us assume the following assumptions on the function g and the Brownian motion $B = (B_n)_{n\in\mathbb{N}}$:

Assumptions (A):

1. $g : \mathbb{R} \longmapsto \mathbb{R}$.
2. $|g(x)|^2 \leq C_1 |x|^2$.
3. B is a Brownian Motion with values in H.

3 Analytical Energy Bounds

Here we present formal energy bounds for the Itô and Stratonovich systems.

3.1 The Itô System

For a more general shell model with an Itô multiplicative noise, it has been proven in [3] that under the assumptions **(A)**, one has the existence of solutions. The uniqueness is an open problem.

Theorem 1 *Under the assumptions* **(A)***, if the initial condition* $X^0 \in H$*, then there exist at least one solution to the problem (2.1) such that* X *is a continuous process with values in* H*. Moreover, we have the following estimate*

$$E(\Phi(t)) \leq E(\Phi(0))e^{C_1 t}, \tag{3.1}$$

where $\Phi(t) = |X(t)|_H^2$*.*

Proof. First, we apply Itô's formula to $X_n^2(t)$:

$$\begin{aligned} dX_n^2(t) &= 2X_n(t)dX_n(t) + [g(X_n(t))]^2 dt \\ &= 2\left(k_{n-1}X_{n-1}^2(t)X_n(t) - k_n X_n^2(t)X_{n+1}(t)\right) dt \\ &\qquad + 2g(X_n(t))X_n(t)dB_n + [g(X_n(t))]^2 dt. \end{aligned}$$

We take the infinite sum of the terms, under the assumption that each summation converges, and we get

$$\begin{aligned} d\sum_{n=1}^{\infty} X_n^2(t) &= 2\sum_{n=1}^{\infty} \left(k_{n-1}X_{n-1}^2(t)X_n(t) - k_n X_n^2(t)X_{n+1}(t)\right) dt \\ &\qquad + 2\sum_{n=1}^{\infty} g(X_n(t))X_n(t)dB_n + \sum_{n=1}^{\infty}[g(X_n(t))]^2 dt \\ &= 2\sum_{n=1}^{\infty} g(X_n(t))X_n(t)dB_n + \sum_{n=1}^{\infty}[g(X_n(t))]^2 dt. \end{aligned}$$

Then, we integrate each side to obtain:

$$\sum_{n=1}^{\infty} X_n^2(t) = \sum_{n=1}^{\infty}(X_n^0)^2 + 2\int_0^t \sum_{n=1}^{\infty} g(X_n(s))X_n(s)dB_n + \int_0^t \sum_{n=1}^{\infty}[g(X_n(s))]^2 ds. \tag{3.2}$$

Recalling that the expected value of an Itô integral is zero, we take the expected value of each side of the equation,

$$E\left(\sum_{n=1}^{\infty} X_n^2(t)\right) = E\left(\sum_{n=1}^{\infty} (X_n^0)^2\right) + E\left(\int_0^t \sum_{n=1}^{\infty} [g(X_n(s))]^2 ds\right).$$

Using assumption $\mathbf{(A)}_2$ and substituting in $\Phi(t)$ gives

$$E(\Phi(t)) \leq E(\Phi(0)) + C_1 \int_0^t E(\Phi(s))ds.$$

Finally, by Gronwall's lemma, we obtain the estimate (3.1) and this completes the proof. ■

Remark 2 *The bound* (3.1) *given in Theorem 1 is a formal calculation that assumes the infinite summations converge. Consider the partial summation*

$$\begin{aligned} d\sum_{n=1}^{N} X_n^2(t) &= \sum_{n=1}^{N} \left(2k_{n-1}X_n(t)X_{n-1}^2(t) - 2k_n X_n^2(t)X_{n+1}(t)\right) dt \\ &\qquad + 2\sum_{n=1}^{N} X_n(t)g(X_n(t))dB_n + \sum_{n=1}^{N}[g(X_n(t))]^2 dt \\ &= -2k_N X_N^2(t)X_{N+1}(t)dt + 2\sum_{n=1}^{N} X_n(t)g(X_n(t))dB_n + \sum_{n=1}^{N}[g(X_n(t))]^2 dt. \end{aligned}$$

For the result of Theorem 1 to hold, then

$$\lim_{N\to\infty} -2k_N X_N^2(t)X_{N+1}(t) = 0.$$

This limit is of interest and will be left for an upcoming project; for more details, see Barbato [2]. However, in light of Remark 2, we performed a computation similar to that in the proof of Theorem 1 using the finite summation of the terms instead of the infinite summation. Define the energy for the first N modes of a solution $X(t) \in H$ of (2.1) or (2.2) by

$$\Phi_N(t) := \sum_{k=1}^{N} X_k^2(t).$$

This gives,

$$E\left(\Phi_N(t)\right) = E\left(\Phi_N(0)\right) - E\left(\int_0^t 2k_N X_N^2(s)X_{N+1}(s)ds\right) + E\left(\int_0^t \sum_{n=1}^{N}[g(X_n(s))]^2 ds\right).$$

Hence, if

$$E\left(\int_0^t \sum_{n=1}^{N}[g(X_n(s))]^2 ds\right) - E\left(\int_0^t 2k_N X_N^2(s)X_{N+1}(s)ds\right) \leq 0, \tag{3.3}$$

then $E(\Phi_N(t))$ is non-increasing. While we have yet to prove this inequality analytically, we demonstrated that it holds numerically.

3.2 The Stratonovich System

A similar approach to that used in the proof of Theorem 1 was applied to the Stratonovich system (2.2). First, we apply the chain rule to (2.2) to find $dX_n^2(t)$:

$$\begin{aligned} dX_n^2(t) &= 2X_n(t)dX_n(t) \\ &= 2\left(k_{n-1}X_{n-1}^2(t)X_n(t) - k_n X_n^2(t)X_{n+1}(t)\right)dt + 2g(X_n(t))X_n(t)\circ dB_n. \end{aligned}$$

Here we let $g(x) = x$ and take the infinite sum of the terms, assuming that each summation converges. Thus,

$$\begin{aligned} d\sum_{n=1}^{\infty} X_n^2(t) &= 2\sum_{n=1}^{\infty}\left(k_{n-1}X_{n-1}^2(t)X_n(t) - k_n X_n^2(t)X_{n+1}(t)\right)dt + 2\sum_{n=1}^{\infty} X_n^2(t)\circ dB_n \\ &= 2\sum_{n=1}^{\infty} X_n^2(t)\circ dB_n. \end{aligned}$$

Then we integrate each side to obtain:

$$\sum_{n=1}^{\infty} X_n^2(t) = \sum_{n=1}^{\infty}(X_n^0)^2 + 2\int_0^t \sum_{n=1}^{\infty} X_n^2(s)\circ dB_n. \tag{3.4}$$

The Stratonovich integral in (3.4) can be rewritten as an Itô integral (see the Appendix). This gives

$$\begin{aligned} \sum_{n=1}^{\infty} X_n^2(t) &= \sum_{n=1}^{\infty}(X_n^0)^2 + 2\int_0^t \sum_{n=1}^{\infty} X_n^2(s)dB_n + 2\int_0^t \sum_{n=1}^{\infty} X_n^3(s)ds \\ &\leq \sum_{n=1}^{\infty}(X_n^0)^2 + 2\int_0^t \sum_{n=1}^{\infty} X_n^2(s)dB_n + 2\sum_{n=1}^{\infty}\left(\int_0^t (X_n^2)(s)ds\right)^{1/2}\left(\int_0^t (X_n^4)(s)ds\right)^{1/2} \\ &\leq \sum_{n=1}^{\infty}(X_n^0)^2 + 2\int_0^t \sum_{n=1}^{\infty} X_n^2(s)dB_n + \sum_{n=1}^{\infty}\left(\int_0^t X_n^2(s)ds + \int_0^t X_n^4(s)ds\right). \end{aligned}$$

We have used the Holder inequality in the inequality above. We then take the expected value and obtain

$$E\left(\sum_{n=1}^{\infty} X_n^2(t)\right) \leq E\left(\sum_{n=1}^{\infty} (X_n^0)^2\right) + E\left(\sum_{n=1}^{\infty} \int_0^t X_n^2(s)ds\right) + \left(\sum_{n=1}^{\infty} \int_0^t X_n^4(s)ds\right). \tag{3.5}$$

It is clear that because the final term in (3.5) is positive, this estimate for the energy is worse than the estimate for the Itô system provided in Theorem 1.

4 Numerical Results

In order to approximate systems (2.1) and (2.2) using numerical methods, we first made the systems finite by redefining them as

$$\begin{cases} X_0(t) = 0 & t \geq 0, \\ dX_n(t) = \left(k_{n-1}X_{n-1}^2(t) - k_n X_n(t) X_{n-1}(t)\right) dt + g(X_n(t))dB_n & t \geq 0,\ n \in \{1, 2, ..., N\}, \\ X_{N+1}(t) = 0 & t \geq 0, \end{cases} \tag{4.1}$$

and

$$\begin{cases} X_0(t) = 0 & t \geq 0, \\ dX_n(t) = \left(k_{n-1}X_{n-1}^2(t) - k_n X_n(t) X_{n-1}(t)\right) dt + g(X_n(t)) \circ dB_n & t \geq 0,\ n \in \{1, 2, ..., N\}, \\ X_{N+1}(t) = 0 & t \geq 0, \end{cases} \tag{4.2}$$

respectively.

4.1 Energy Approximations

Inspection of (4.1) reveals that the final mode, $X_N(t)$, ‘collects’ the energy of the entire system, while the infinite system has no final term and therefore the energy is passed on to smaller scales indefinitely. In order to better simulate the infinite system we drop the final mode in our energy approximation. This is comparable to defining some smallest observable scale. We will proceed by defining the adjusted energy as

$$\Phi_{N-1}(t) := \sum_{k=1}^{N-1} X_k^2(t).$$

In order to simulate the expected value of the energy, we computed multiple paths and then averaged them. For a large number of paths, this gives an approximation of the expected value.

For the numerical approximation of the Itô system (2.1) we used the Milstein method. This method converges to the Itô integral and is of higher order than the simple Euler method [17]. For the Stratonovich system, we used the modified Euler method, also known as the Heun-Trapezoidal method, which approximates the Stratonovich integral [15]. These approximations were computed in Matlab.

Barbato *et al.* [2] demonstrated that the deterministic system (1.2) decays like $1/t^2$ when $\nu = 0$. Here we compare the average adjusted energy of the stochastic system, $\Phi_{N-1}(t)$, with the function $\Phi_{N-1}(0)/t^2$. Figure 1 shows the values of the adjusted energy for the Itô system when $g(x) = x$.

Figures 2-3 show the value of the adjusted energy for the Stratonovich system with $g(x) = x$ and $g(x) = |x|^{1/2}$. In each case, the energy is clearly dissipating.

Remark 3 *As p is decreased, it appears that the energy dissipates at a greater rate. This can be seen in Figure 4, which compares the data from Figure 2 with the data from Figure 3.*

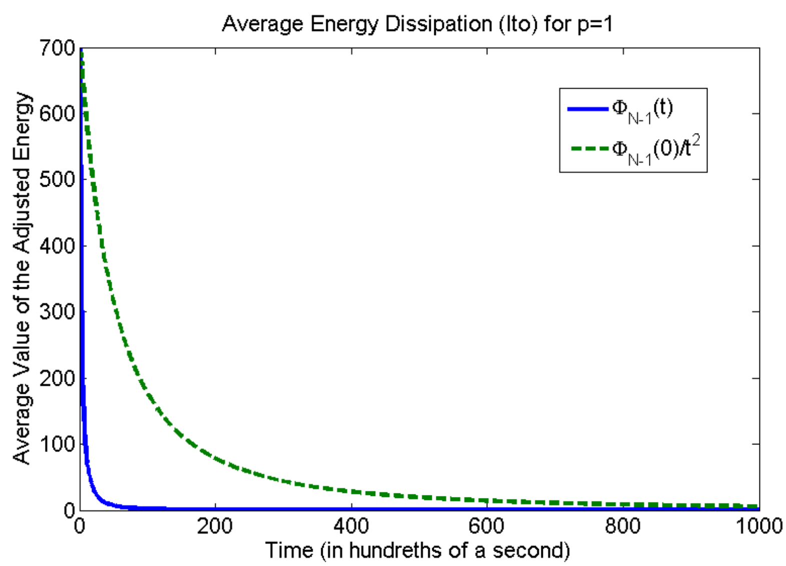

Figure 1: Average adjusted energy of (4.1) for $N = 8$, $X_n^0 = 10$ and 50 paths when $g(x) = x$.

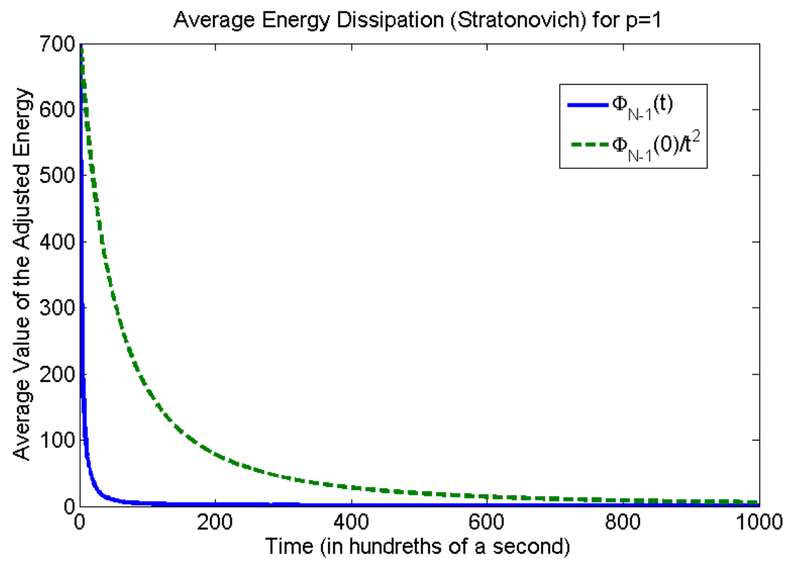

Figure 2: Average adjusted energy of (4.2) for $N = 8$, $X_n^0 = 10$ and 50 paths when $g(x) = x$.

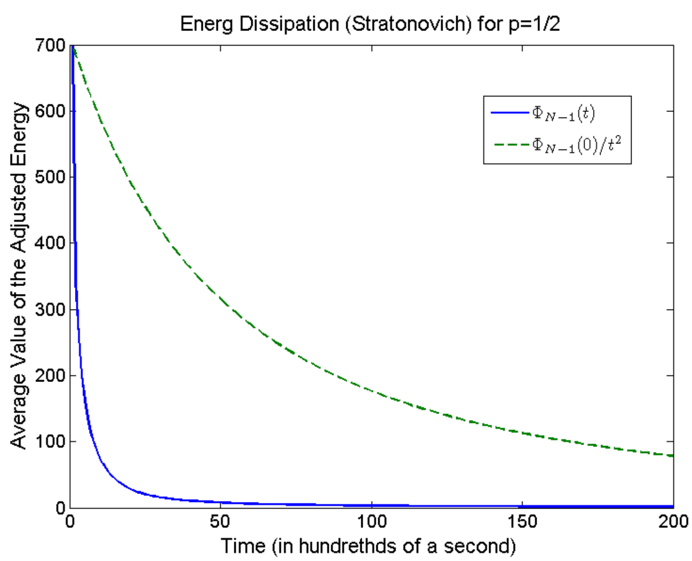

Figure 3: Average adjusted energy of (4.2) for $N = 8$, $X_n^0 = 10$ and 100 paths when $g(x) = |x|^{1/2}$.

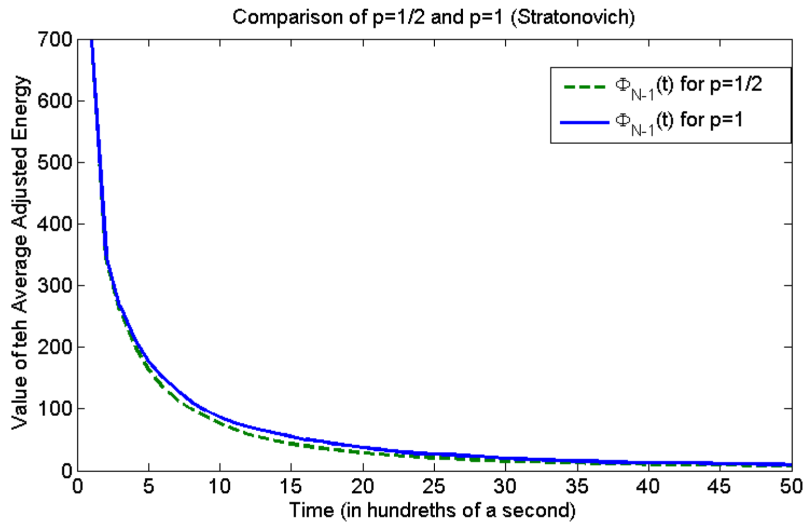

Figure 4: Comparison of the average energy for p=1 and p=1/2 (Stratonovich).

4.2 Solution Paths

Using the same methods as for the energy simulations, we approximated the expected values of the paths of solutions to (2.2). This is provided in Figure 5. Here it is clear that the average of each path is positive.

5 Itô and Stratonovich Comparisons

Recall that the Stratonovich integral can be written in terms of the Itô formula:

$$\begin{aligned} X_n(t) &= X_n(0) + \int_0^t f(X_n(s))ds + \int_0^t g(X_n) \circ dB_n \\ &= X_n(0) + \int_0^t f(X_n(s))ds + \int_0^t g(X_n)dB_n + \int_0^t \frac{1}{2} g(X_n)g'(X_n)ds. \end{aligned}$$

Now comparing this to the Itô representation,

$$dX_n(t) = f(X_n(t))dt + g(X_n)dB_n,$$

we can see that for the Itô SDE, $dX_n(t)$ is less than in the Stratonovich SDE if $\frac{1}{2}g(X_n)g'(X_n)dt > 0$. The approximation in Figure 5 suggests that this inequality

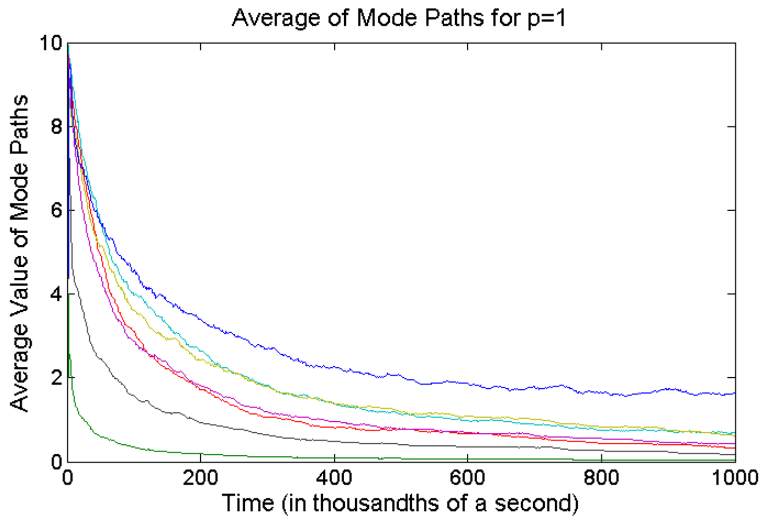

Figure 5: Average values of each mode for (4.2) for $N = 7$

holds. This suggests that the rate of dissipation using the Itô method will be greater than when using the Stratonovich method. This can be seen in Figure 6, which compares the average energy from Figure 1 with the average energy from Figure 3.

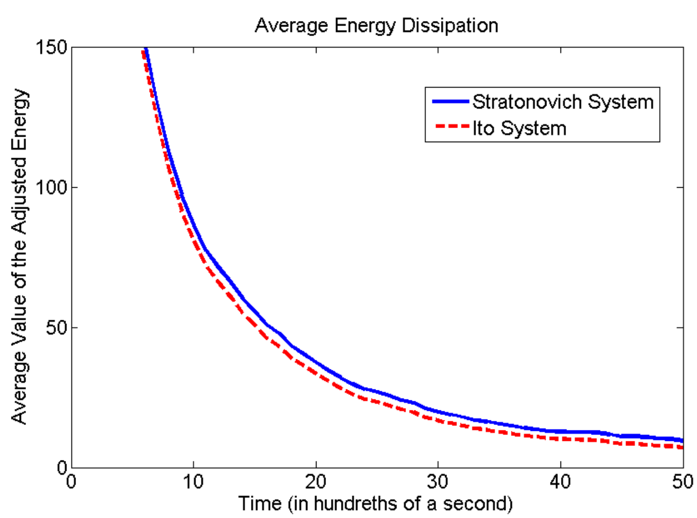

Figure 6: Comparison of the average energy of the Itô and Stratonovich systems.

This difference in dissipation is indicative of the dissipative effect of the Itô integral in the general case. Consider the SDE

$$dY = \alpha X dt + \beta X dW,\ Y(0) = Y_0 \tag{5.1}$$

If this is understood to be Itô multiplicative noise, then the solution is

$$Y = Y_0 e^{(\alpha - \frac{1}{2}\beta^2)t} e^{\beta W(t)}.$$

But if the noise is Stratonovich, then

$$Y = Y_0 e^{\alpha t} e^{\beta W(t)}$$

is the solution. However, if there is no stochastic forcing present, (5.1) becomes

$$dY = \alpha X dt,$$

and the solution is

$$Y = Y_0 e^{\alpha t}. \tag{5.2}$$

For β large, the solution with Itô forcing will dissipate regardless of the sign of α. However, the solution to the Stratonovich equation will only dissipate if $\alpha < 0$.

Thus, the Stratonovich solution will dissipate only if the deterministic solution (5.2) dissipates, when $\alpha < 0$, while the Itô solution will dissipate regardless of whether or not (5.2) is dissipative.

6 Positivity of Solutions: Analytical Implications

The positivity of solutions to (2.1) and (2.2) is desirable for further analytical results. In Section 4.2 it was shown numerically that $E(X(t)) > 0$. However, this has not been proven analytically. Numerically, the second term of (3.3) causes that inequality to hold, indicating dissipation of energy. This is shown in Figure 7. If solutions are shown to be positive, then this result could be explored further analytically.

Here we present a bound for $E(X(t))$ that also relies upon the positivity of solutions.

Let $g(x) = |x|^{\frac{3}{2}}$ in (2.1). The system becomes

$$dX_n(t) = k_{n-1}X_{n-1}^2 dt - k_n X_n X_{n+1} dt + |X_n|^{\frac{3}{2}} dB_n(t) \tag{6.1}$$

Consider self-similar solutions, that is, solutions of the form $X_n(t) = a_n \varphi(t)$ where $a = (a_n)_{n\in\mathbb{N}} \in \mathbb{R}^N$. Suppose that $\varphi(t) > 0$; then

$$dX_n(t) = d(a_n \varphi(t)) = a_n d\varphi(t).$$

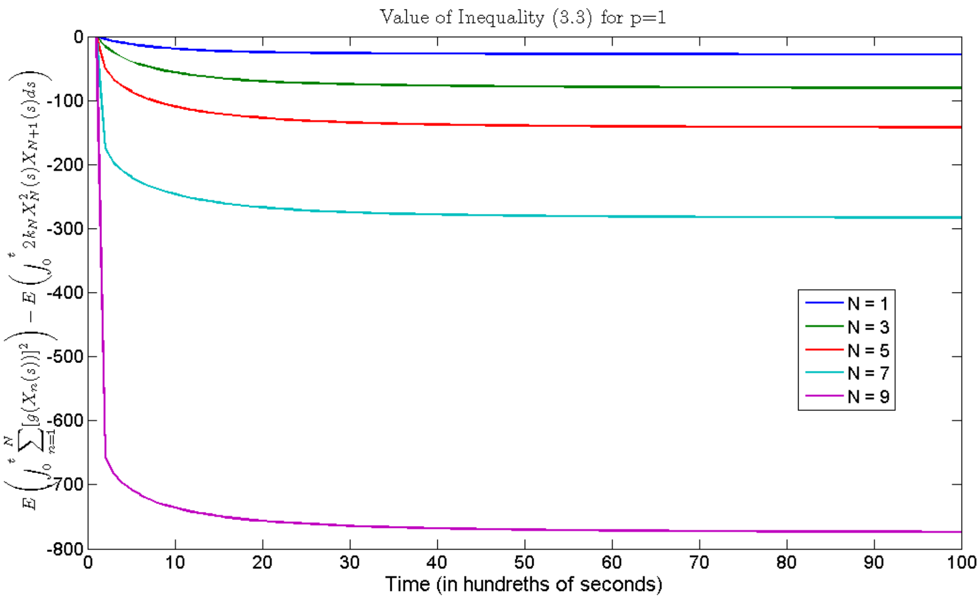

Figure 7: Average values of (3.3) for different values of N

Thus,

$$a_n d\varphi(t) = k_{n-1}a_{n-1}^2\varphi(t)^2 dt - k_n a_n a_{n+1}\varphi(t)^2 dt + |a_n|^{\frac{3}{2}}|\varphi(t)|^{\frac{3}{2}} dB_n(t)$$

Now we define $\Theta(t) = \frac{1}{\varphi(t)}$ and use the Itô formula to find the differential

$$d\Theta(t) = \frac{-1}{\varphi^2(t)} d\varphi(t) + \frac{1}{\varphi^3(t)} \frac{|a_n|^3|\varphi(t)|^3}{a_n^2} dt.$$

Substituting $\varphi(t)$ and integrating gives

$$\Theta(t) = \Theta(t_0) - \left(k_{n-1}\frac{a_{n-1}^2}{a_n} - k_n a_{n+1} + sign(a_n)a_n^2\right)(t - t_0) + \int_0^t \frac{|a_n|^{\frac{3}{2}}}{a_n} \frac{1}{\sqrt{|\varphi(t)|}} dB_n(t).$$

We then take the expected value to obtain

$$E\Theta(t) = E\Theta(t_0) - E\left(k_{n-1}\frac{a_{n-1}^2}{a_n} - k_n a_{n+1} + sign(a_n)a_n^2\right)(t - t_0).$$

Because $\varphi(t) > 0$, then by Jensen's inequality,

$$E(\varphi(t)) = E(\frac{1}{\Theta(t)}) \geq \frac{1}{E(\Theta(t))}.$$

Therefore we have two different cases for $E(X_n(t))$: if $a_n > 0$, then

$$E(X_n(t)) \geq \frac{1}{X_n(0) + \left(1 - k_{n-1}\frac{a_{n-1}^2}{a_n^2} - k_n\frac{a_{n+1}}{a_n}\right)t}$$

and if $a_n < 0$, then

$$E(X_n(t)) \leq \frac{1}{X_n(0) - \left(1 + k_{n-1}\frac{a_{n-1}^2}{a_n^2} + k_n\frac{a_{n+1}}{a_n}\right)t}.$$

Appendix A Review: Stochastic Analysis

Stochastic processes take place within the context of probability spaces. We refer the reader to Øksendal [19] for a formal definition of probability spaces. To enable practical use of the elements $\omega \in \Omega$ in the probability space, we use random variables, which provide numerical values to associate with each element in the probability space. Again, see Øksendal [19] for a formal definition. Here we present three additional important stochastic analysis definitions from Øksendal [19] : the distribution function, expected value, and Brownian motion.

Definition 1 *The* distribution *of a random variable X is a function $\mu_X : \mathbb{R} \mapsto \mathbb{R}$ defined by*

$$\mu_X(x) = P(\{\omega \in \Omega : X(\omega) < x\}).$$

The derivative of the distribution, if it exists, is called the probability density function.

An important quantity in stochastic analysis is the expected value of a random variable.

Definition 2 *The expected value of a random variable X is given by*

$$E(X) := \int_{\mathbb{R}} x d\mu_X(x).$$

Definition 3 *Brownian motion is stochastic process $B_t : \Omega \mapsto \mathbb{R}$ whose probability density function is a normal distribution with mean $\mu = 0$ and variance $\sigma^2 = t$ denoted as:*

$$\mathrm{f}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)}{2\sigma^2}} = \frac{1}{\sqrt{2\pi t}} e^{\frac{-x}{2t}}$$

Importantly, B_t is not a differentiable function, and so $\int f(t,\omega)dB_t(\omega)$ cannot be understood as a traditional Riemann integral. Rather this integral can be interpreted as either a Stratonovich or Itô stochastic integral. A heuristic explanation of both follows; see [19] for formal definitions.

We consider the Itô integral $\int f(t,\omega)dB_t(\omega)$ to be

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(t_j,\omega)[B_{t_{j+1}} - B_{t_j}]$$

where t_j is taken as the left point of the interval. This leads naturally to the consideration of a stochastic process of the form

$$Y_t = Y_0 + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_s,$$

which can also be written in the differential form

$$dY_t = udt + vdB_t.$$

In order to solve Itô systems, we rely heavily on the Itô formula.

Theorem 4 (Itô's formula) *[19] Let Y_t be a process such that $dY_t = udt + vdB_t$ and $g(t,x) \in C_t^1 \cap C_x^2\,[(0,\infty)\times\mathbb{R}]$. Then $Z_t = g(t,Y_t)$ and*

$$dZ_t = \frac{\partial g}{\partial t}(t,Y_t)dt + \frac{\partial g}{\partial x}(t,Y_t)dY_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,Y_t)v^2dt.$$

In contrast, the Stratonovich stochastic integral $\int f(t,\omega)\circ dB_t(\omega)$ is given by

$$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{f(t_{j+1},\omega) + f(t_j,\omega)}{2}[B_{t_{j+1}} - B_{t_j}]$$

Rather than approximating the integral at the left side, the Stratonovich integral approximates the integral at the midpoint of the interval.

Theorem 5 (Chain rule for Stratonovich Integrals) *Let Y_t be a process such that $dY_t = uY_t dt + vY_t \circ dB_t$ and $h(t,x) \in C_t^1 \cap C_x^2\,[(0,\infty)\times\mathbb{R}]$. Let $M_t = h(t,Y_t)$. Then*

$$dM_t = \frac{\partial h}{\partial t}(t,Y_t)dt + \frac{\partial h}{\partial x}(t,Y_t)dY_t.$$

The relation between an Itô integral and a Stratonovich integral is given by formula [15]

$$\int_a^b f(t,x)\circ dB = \int_a^b f(t,x)dB + \frac{1}{2}\int_a^b \frac{\partial f}{\partial x} f(t,x)dt. \tag{A.1}$$

Because Itô's formula approximates solutions using the left endpoints, it is an underestimate and therefore is less accurate, however, there are many circumstances in which we cannot know what future modes (Z_{i+1}) will yield and therefore it is impossible to use Stratonovich approximations. For the most part, Itô's formula is used to approximate discrete pulses of stochastic noise, whereas Stratonovich's formula is used for continuous fluctuating noise.

An important property of Itô integrals is that

$$E\left(\int_a^b f dB_t(\omega)\right) = 0.$$

Thus, because of the relation (A.1),

$$E\left(\int_a^b f\circ dB_t(\omega)\right) = E\left(\frac{1}{2}\int_a^b ff' dt\right).$$

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