Virtual Mosaic Knots

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Dedicated to my mother, Maria E. Luna, and my father, Enrique Garduño.

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Abstract

We extend mosaic knot theory to virtual knots and define a new type of knot: virtual mosaic knot. As in classical knots, Reidemeister moves are applied to a virtual mosaic knot to transform one knot diagram into another. Additionally, given the mosaic number of a virtual knot, we find an upper bound on the sum of the classical and virtual crossing numbers. Furthermore, given the classical and virtual crossing numbers of a knot, we find a lower bound on the virtual mosaic number of a knot.

1 Introduction

In the nineteenth century, Johann Carl Friedrich Gauss (1777-1855) began to mathematically study knots [4]. Gauss showed in 1833 that the linking number of two knots can be computed by an integral. In 1867, William Thomson (Lord Kelvin) (1824-1907) [4] presented his *vortex-atom theory*, which states that each chemical element has its own knot form and can be differentiated by their given knot form [4]. Although Thomson's theory was misguided, Peter Guthrie Tait (1831-1901) created the first knot tables in 1876. Since then, knot theory has expanded and has become a major part of the mathematics field called topology.

Vaughan Jones discovered the Jones polynomial in 1983 and in 1990 he won the Fields Medal for his contribution in the field [6], inspiring substantial activity in knot theory. In 1999, Louis H. Kauffman defined virtual knots [1]. Almost ten years later mosaic knots were defined by Samuel J. Lomonaco and Louis H. Kauffman [2].

In Section 2, we provide background information and definitions. We begin with classical knots and Reidemeisters theorem. Next, we recall virtual knot theory and mosaic knots. In Section 3, we introduce and describe virtual mosaic knots. In Section 4, we discuss the lower bound of a mosaic number. We also show that these bounds can be applied to virtual mosaic knots.

2 Background

2.1 Classical Knots

A *knot* is an embedding (an injective map) of a circle, S^1, into $\mathbb{R}^3$. A *knot diagram* is a projection of a knot onto the plane in which crossing information is preferred. Figure 1 is an example of a knot diagram.

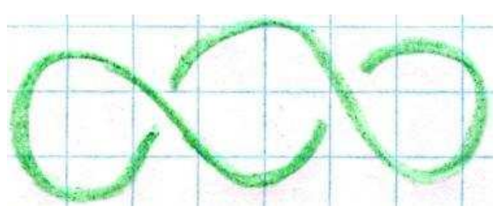


Figure 1: A Knot Diagram

We can deform one knot diagram to another by way of stretching and bending. This process is called *planar isotopy*, as shown in Figure 2.

Reidemeister moves involve three different types of moves applied to knot diagrams. We can apply these moves to reduce the number of crossings and

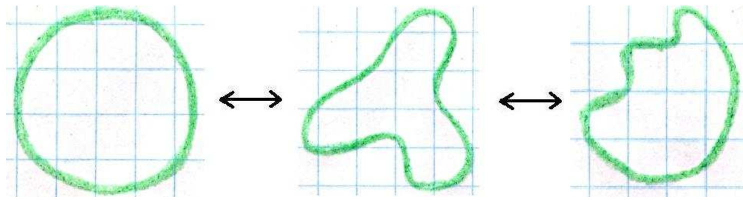


Figure 2: Example of Planar Isotopy

to transform a given knot diagram into a simpler yet equivalent knot diagram. The Reidemeister moves are shown in Figure 3.

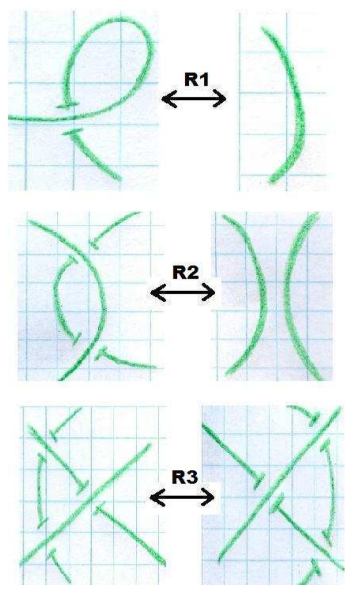


Figure 3: Reidemeister Moves

Note that these three moves are actually eight moves after accounting for over and under crossings.

Figure 4 is an example of a knot diagram transformed into another knot diagram using Reidemeister moves. Reidemeister proved these moves describe all possible three-dimensional moves, as stated in Theorem 2.1.

Theorem 2.1 (Reidemeister)**.** *Two knot diagrams represent the same knot if and only if one can be transformed into the other by a finite sequence of Reidemeister moves and planar isotopy moves [3].*

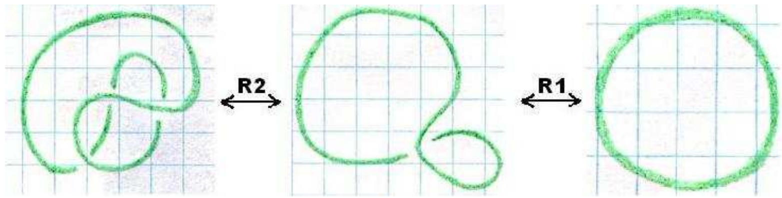


Figure 4: Example of Reidemeister Moves

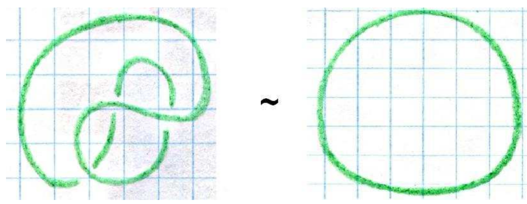


Figure 5: Equivalence Relation of Two Knots Using Reidemeister Moves

Thus, two knot diagrams are related if their diagrams are associated by a finite sequence of Reidemeister moves and/or planar isotopy moves. Reidemeister moves allow us to describe the three-dimensional motions of knots as diagrams. Furthermore, Reidemeister moves allow us to place knots in equivalence classes.

Recall that an *equivalence relation*, $\equiv$, on a set X must satisfy the following for all x, y, and z in X:

- *Reflexive*: $x \equiv x$
- *Symmetry*: if $x \equiv y$, then $y \equiv x$
- *Transitive*: if $x \equiv y$ and $y \equiv z$, then $x \equiv z$

This idea can be applied to knots. For example, any two diagrams of the unknot related by the Reidemeister moves are members of the equivalent class of the unknot. An equivalence class of the element $y \in X$ is the set $\{x \in X : x \equiv y\}$.

In Figure 6 we see several elements of the equivalence class containing the unknot. The knot diagrams are equivalent because they are related by a sequence of Reidemeister moves. That is, Reidemeister moves form an equivalence relation and the knot diagrams are elements of the same equivalence class. In this sense we can define a knot to be an equivalence class of knot diagrams.

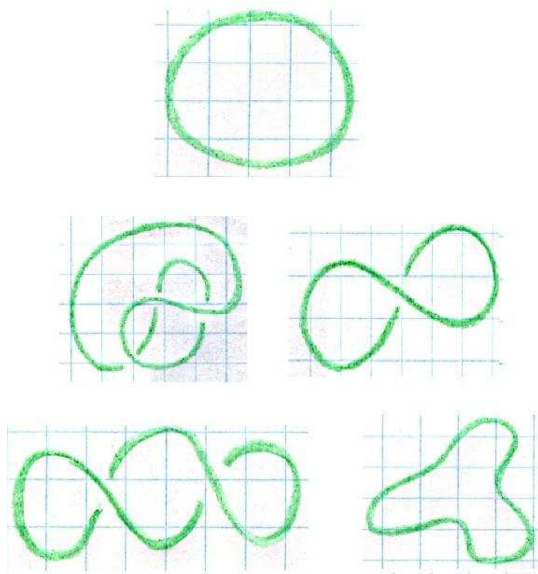


Figure 6: Equivalence Class of Knot Diagrams

2.2 Virtual Knots

Virtual knot diagrams are an extension of classical knot diagrams [1]. Classical and virtual knots are represented in diagram form, and there are an extended set of Reidemeister moves and planar isotopy moves. The major difference between virtual and classical knot diagrams is the type of crossings. A *virtual knot diagram* is a mapping of S^1 into $\mathbb{R}^2$ with classical crossings and an additional type of crossing called a *virtual crossing*.

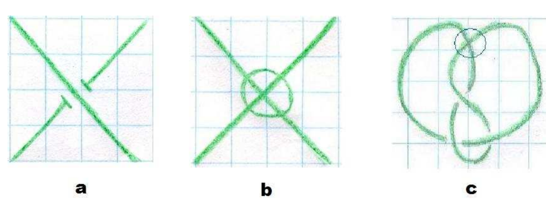


Figure 7: A Classical Crossing, a Virtual Crossing, and a Virtual Knot Diagram.

Notice in Figure 7 that the virtual crossing of a knot diagram does not include over/under markings.

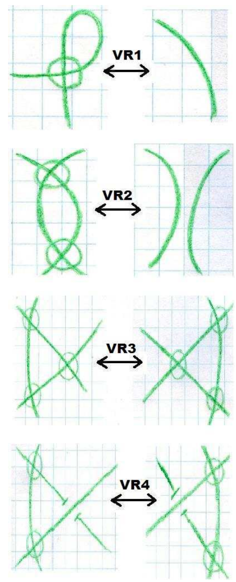


Figure 8: Virtual Reidemeister Moves

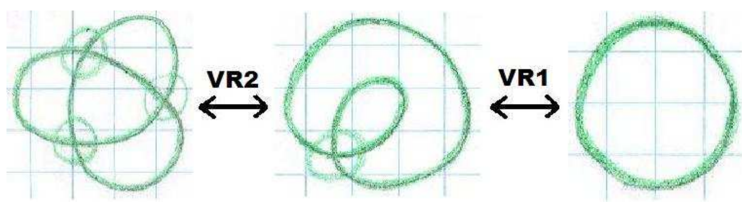


Figure 9: Example of Virtual Reidemeister Moves

In virtual knot theory there are four additional Reidemeister moves. The first three virtual Reidemeister moves are analogous to the classical Reidemeister moves. The fourth move involves both types of crossings as shown in Figure 8. Two virtual knot diagrams are *equivalent* if their diagrams are related by a finite sequence of virtual and classical Reidemeister moves and planar isotopy.

We see two equivalent virtual knot diagrams in Figure 9 and can now define a virtual knot [1]. A *virtual knot* is an equivalence class of virtual knot diagrams determined by all the Reidemeister moves.

2.3 Knot Invariant

We now introduce the idea of an invariant. A *knot invariant* is a numerical quantity or a property associated with knot diagrams where the Reidemeister moves do not change the numerical quantity or the property. The *crossing number* of a knot is the minimum number of classical crossings in any equivalent knot diagram.

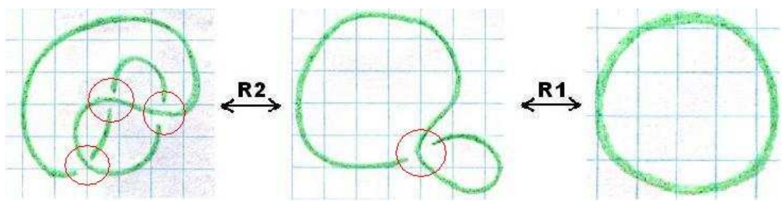


Figure 10: Classical Crossing Number

Figure 10 shows two different yet equivalent knot diagrams related by Reidemeister moves. Notice that the number of crossings decreases with each Reidemeiste move. Therefore, the crossing number is zero because we want the least number of crossings in any of its equivalent knot diagrams.

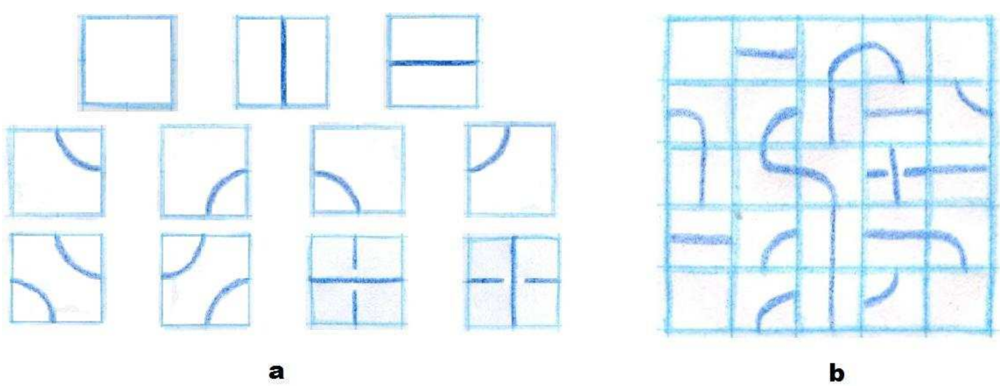


Figure 11: Mosaic Tiles, and a 5-mosaic

In the same manner we can define the virtual crossing number. A *virtual crossing number* of a knot is the minimum number of virtual crossings in any equivalent virtual knot diagram. We can also define the *complete crossing number* as the sum of the virtual and classical crossing numbers.

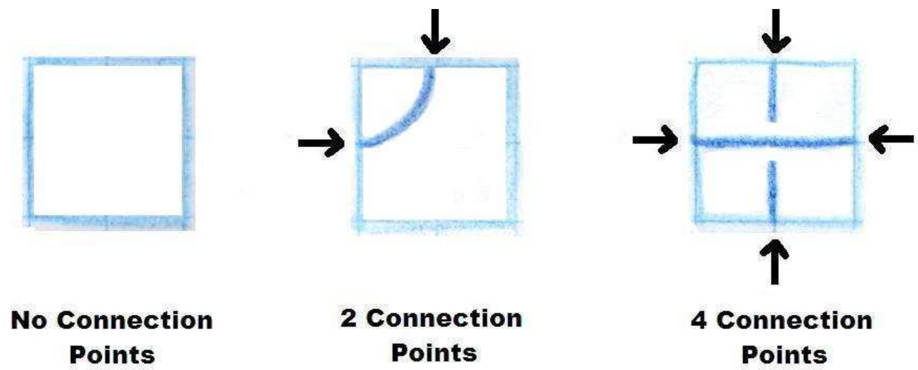


Figure 12: Connection Points

2.4 Mosaic Knots

We now introduce the concept of mosaic knots [2].

Remark 2.1. *The only difference between classical knots and mosaic knots is that mosaic knot diagrams are laid on tiles.*

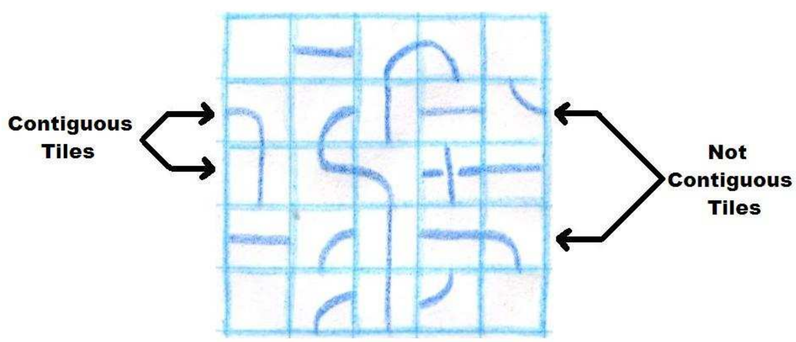


Figure 13: Contiguous Tiles

An *n-mosaic* is an $n \times n$ matrix composed of mosaic tiles, as shown in Figure 11.

A *connection point* is the end point of any curve or line within a tile and located at the midpoint of a tile's edge. These connection points allow us to create n-mosaic knots. A *knot n-mosaic* is a mosaic with all of its tiles joined at the connection points.

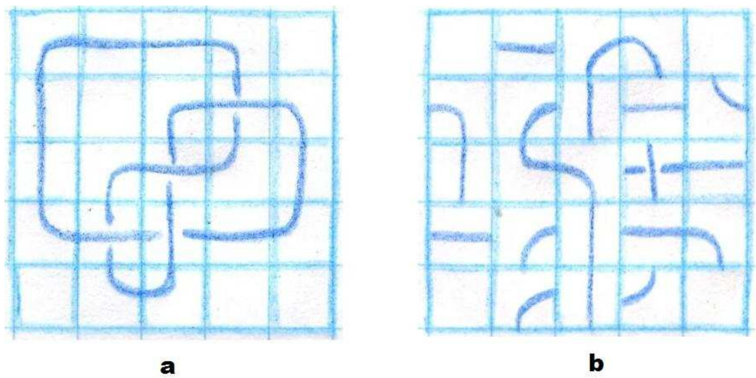


Figure 14: Suitably Connected Tiles and Not Suitably Connected Tiles

Contiguous tiles, as shown in Figure 13, are two tiles in a mosaic diagram that are placed immediately next to each other.

Furthermore, two tiles are *suitably connected tiles* when all of the connection points of a tile touch the connection points of a contiguous tile in a mosaic, as shown in Figure 14.

A *mosaic number* is the smallest integer for which a knot is representable as a knot n-mosaic. Figure 15 exhibits the 11 mosaic planar isotopy moves.

Remark 2.2. *We can use these moves to obtain a diagram with the mosaic number. Furthermore, we reduce a column and a row using mosaic planar isotopy moves if there are no crossings or corners involved in the row or column (prove this in section 4).*

Notice that in Figure 16 the left hand knot is a 6×6 mosaic knot diagram that is reduced to a 5×5 mosaic knot diagram.

In Figure 18, the left hand knot is a 7×7 mosaic knot diagram that is reduced to a 6×6 mosaic knot diagram and finally to a 5×5 mosaic knot diagram. Moreover, we see that the left hand knot is equivalent to the right hand knot. Hence, two mosaic knot diagrams are *equivalent* if one can be transformed into the other by a finite sequence of mosaic Reidemeister moves and/or mosaic planar isotopy.

Notice that in both the mosaic planar isotopy and mosaic Reidemeister moves we have tiles with dotted lines. These tiles are called *non-deterministic tiles.* A *non-deterministic tile* is a tile that denotes two possible outcomes. For example, in Figure 17 we see that the mosaic Reidemeister moves have

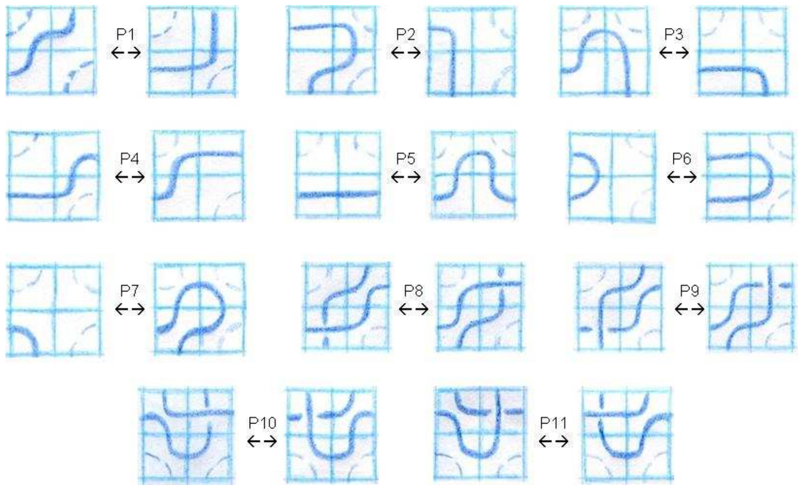


Figure 15: Mosaic Planar Isotopy

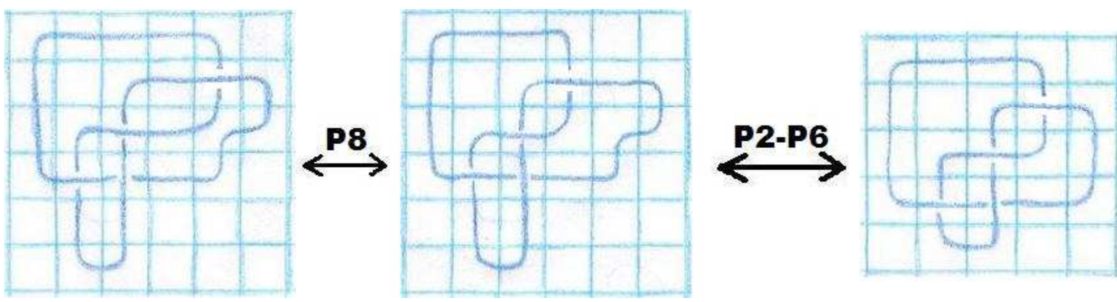


Figure 16: Example of Mosaic Planar Isotopy

dotted lines that go in two directions. In these cases, the knot continues in one of the two directions. The same holds for a tile that has dotted lines in only one direction, as shown in Figure 19.

3 Virtual Mosaic Knots

We extend mosaic knot theory and introduce *virtual mosaic knot theory*. Recall that n-mosaics are composed of mosaic tiles. *Virtual mosaic tiles* are a collection of mosaic tiles that include the virtual mosaic tile.

A *virtual n-mosaic* is an $n \times n$ matrix of mosaic tiles that potentially includes the virtual tile, as shown in Figure 20. We can now define a knot

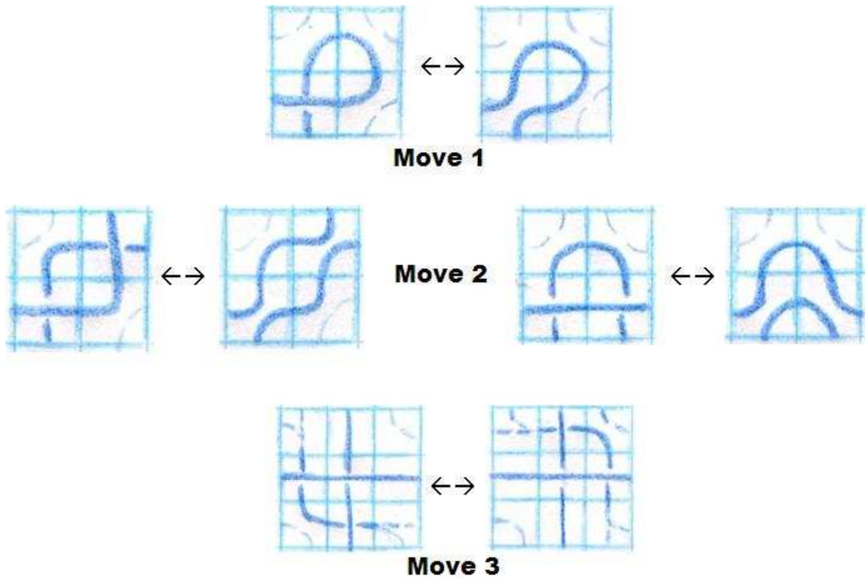


Figure 17: Mosaic Reidemeister Moves

virtual mosaic. A *virtual n-mosaic knot* is a virtual n-mosaic where all of its tiles are suitably connected [Figure21b].

Remark 3.1. *Virtual mosaic knot diagrams do not have specification on the matrix size.*

A *virtual mosaic number* is the smallest integer m for which a virtual mosaic knot is representable as a virtual n-mosaic knot.

We extend mosaic planar isotopy to virtual mosaic planar isotopy by adding two virtual mosaic planar isotopy moves, P12 and P13 shown in Figure 22, and four virtual mosaic Reidemeister moves shown in Figure 23.

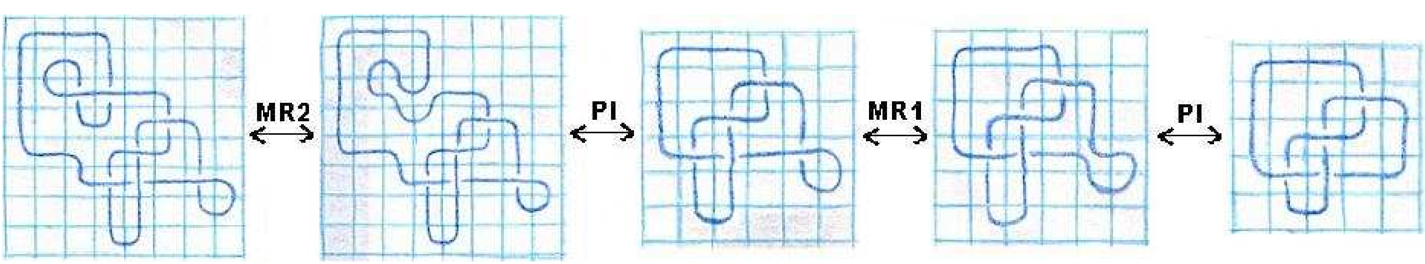
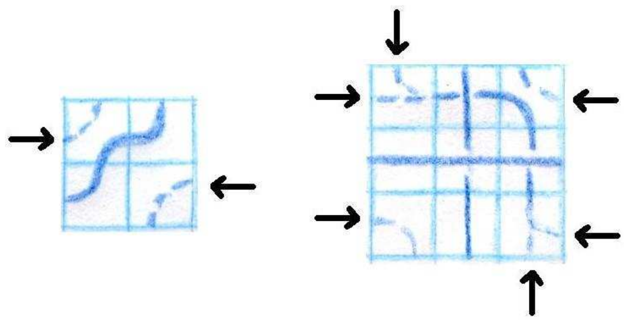


Figure 18: Example of Mosaic Reidemeister Moves



a b

Figure 19: Non-Deterministic Tiles

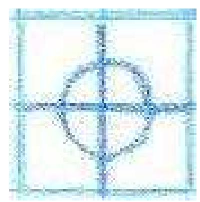


Figure 20: Virtual Mosaic Tile

In the following example, Figure 24, we apply a virtual mosaic Reidemeister move 2 so that the left hand knot is equivalent to the right hand knot.

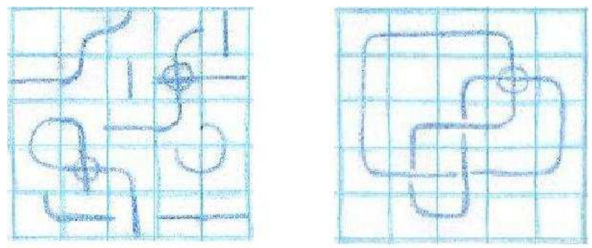
Two virtual n-mosaic knot diagrams are *equivalent* if one can be transformed into the other by a finite sequence of classical and virtual mosaic Reidemeister moves and/or classical and virtual mosaic planar isotopy moves.

4 Lower and Upper Bounds of a Mosaic

4.1 Upper Bound on The Crossing Number

Proposition 4.1. *The edges of a virtual n-mosaic knot cannot contain classical crossings or virtual crossings.*

Proof Suppose we have a virtual n-mosaic knot with virtual mosaic number m. By definition a virtual n-mosaic knot is a virtual mosaic with all tiles suitably connected. A virtual n-mosaic knot holding a crossing in an



a b

Figure 21: A Virtual 5-Mosaic and a Knot Virtual 5-Mosaic

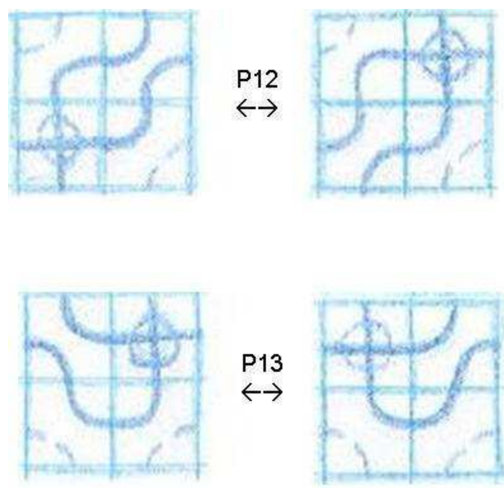


Figure 22: Virtual Mosaic Planar Isotopy Moves

edge tile is not suitably connected. So, the edges of a virtual n-mosaic knot cannot hold crossings. ∎

For example,in Figure 25a we have a suitably connected virtual 5-mosaic. Notice that none of the crossings are at the edges. The not suitably connected virtual 5-mosaic in Figure 25b has crossings at the edges that do not 'lead' to any connection points. This is an example where all the crossings must not be at the edges. Additionally, all crossings must be inside the 5-mosaic because if there were any crossings at the edges of the 5-mosaic, then the 5-mosaic will not be suitably connected.

Proposition 4.2. *Given a virtual n-mosaic knot, we can reduce a column and a row using virtual mosaic planar isotopy moves and mosaic Reidemeis-*

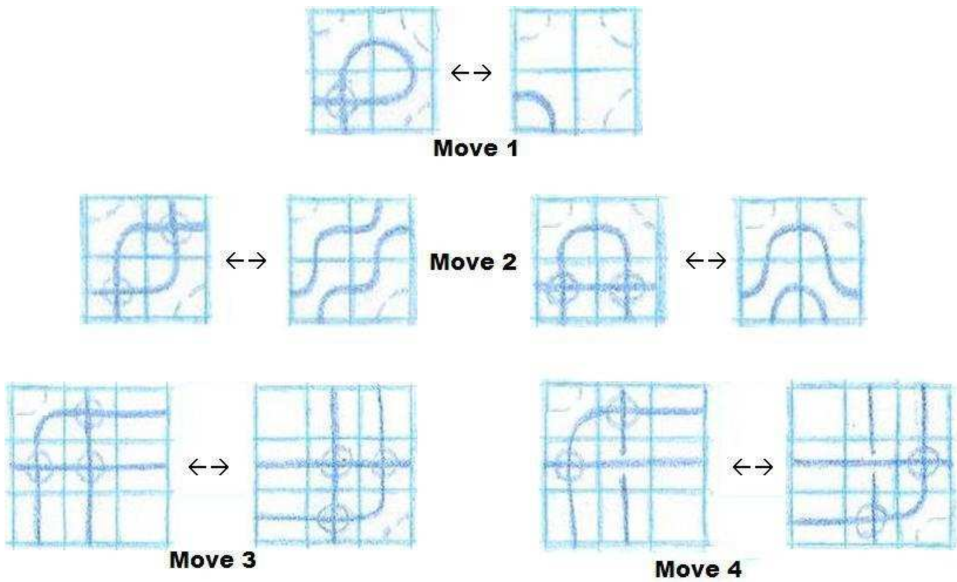


Figure 23: Virtual Mosaic Reidemeister Moves

ter moves if there are no crossings or corners involved in the column or row.

Proof Suppose we have a virtual n-mosaic knot diagram. By definition two virtual mosaic knot diagrams are equivalent if one can be transformed into the other by a finite sequence of classical and virtual mosaic Reidemeister moves and/or classical and virtual mosaic planar isotopy moves. Since all the boundary rows or columns have no crossings or corners which could be reduced via a sequence of mosaic Reidemeister moves and/or virtual mosaic planar isotopy moves, they represent a different yet equivalent virtual n-

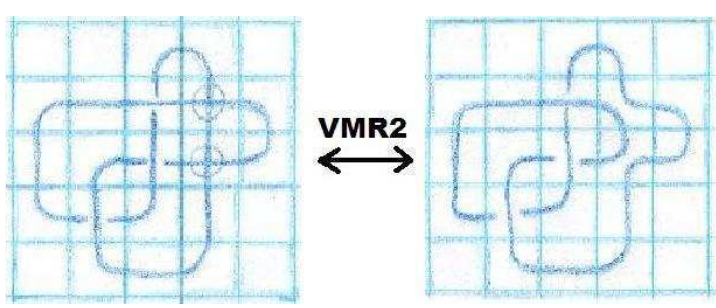


Figure 24: Example of Virtual Mosaic Reidemeister Moves

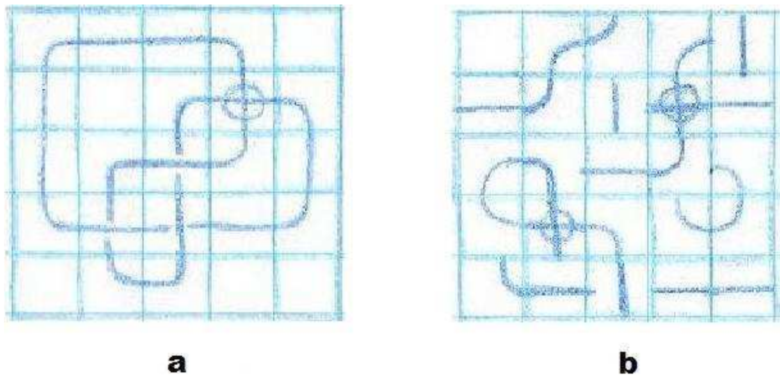


Figure 25: Suitably Connected Virtual 5-Mosaic and a Not Suitably Connected Virtual 5-Mosaic

mosaic knot diagram. ∎

Theorem 4.3. *Given a virtual knot, let m denote the mosaic number, n denote the classical crossing number, and r denote the virtual crossing number. Then $l \leq (m-2)^2$, where $l = n + r$ and $m \geq 2$.*

Proof Suppose we have a virtual mosaic knot with l crossings and mosaic number m. Now, a virtual n-mosaic knot can be reduced to a n-mosaic knot which cannot contain crossings at the tile's edge. Thus all classical and virtual mosaic crossings are within the $m-2$ submatrix. Hence $l \leq (m-2)^2$ is an upper bound on l. ∎

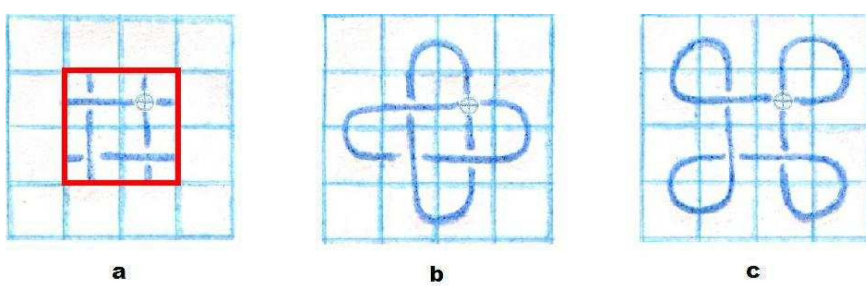


Figure 26: A Virtual 4-Mosaic With Alternating Crossings

For example, Figure 27a shows a virtual 5-mosaic with alternating crossings. As noted, all of the crossings are within the walls of the edges. If we

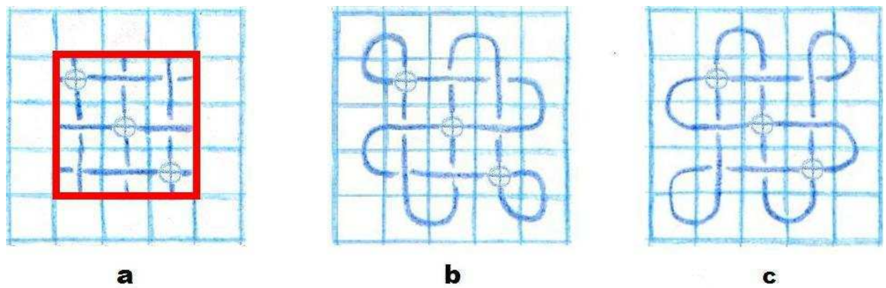


Figure 27: A Virtual 5-Mosaic with Alternating Crossings

connect the crossings in such a way as to create a knot, we get the following two knots as shown in Figure 27b and c.

4.2 Lower Bound on The Mosaic Number

Recall Proposition 4.1 and Proposition 4.2.

Theorem 4.4. *Given a virtual knot, let m denote the mosaic number, n denote the classical crossing number, and r denote the virtual crossing number. Then $m \geq \left\lceil \sqrt{l} \right\rceil + 2$, where $l = n + r$ and $l \geq 0$.*

Proof Suppose we have a virtual mosaic knot with l crossings and mosaic number m. Since there is at most one crossing in each tile, the smallest square-matrix that could contain all classical and virtual crossings is an $\left\lceil \sqrt{l} \right\rceil \times \left\lceil \sqrt{l} \right\rceil$ submatrix and since there cannot be any crossings on the edge tiles, there must be two additional rows and columns. Thus, $m \geq \left\lceil \sqrt{l} \right\rceil + 2$. ∎

Remark 4.1. *Theorem 4.3 is a Corollary of Theorem 4.4.*

For example, in Figure 28a we see a 2-submosaic enclosed in a virtual 4-mosaic knot diagram. Notice that since there cannot be any classical or virtual crossings at a tile's edge, all of the crossings are within the the 2×2 submatrix. Although the number of crossings are different in the virtual 5-mosaic knot diagrams shown in Figures 28b and c, both the two virtual mosaic knot diagrams have a 3-submosaic enclosed in a virtual 5-mosaic knot diagram where all of the crossings are within the 3×3 submatrix.

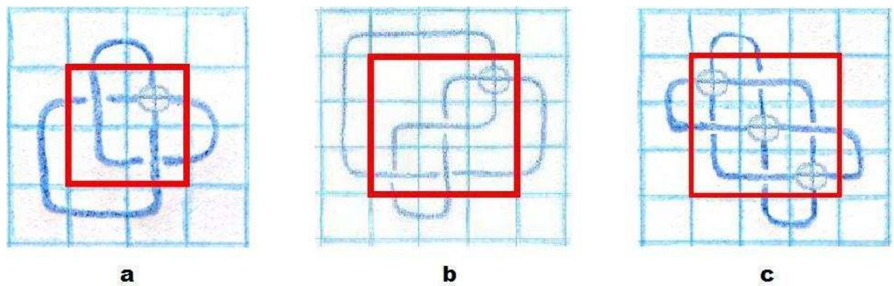


Figure 28: A Virtual 4-Mosaic and Virtual 5-Mosaics

5 Conjecture

We conclude this paper with some conjectures that we plan to investigate in the future. Recall Theorem 4.3. The upper bound on the total crossing number l of a virtual n-mosaic knot given the mosaic number m can be reduced to:

- $(m-2)^2 - 2$ if m is odd
- $(m-2)^2 - 1$ if m is even, where $m \geq 2$

As shown in Figure 27, two corners of the virtual 5-mosaic knot diagram can undergo virtual mosaic Reidemeister move 1 reducing the crossing number by two. This can be applied to any odd virtual n-mosaic knot as we will obtain two corners that will undergo classical or virtual mosaic Reidemeister move 1.

Furthermore, when we pack a virtual 4-mosaic with crossings, as in Figure 26b, we get a *link*, which is non-empty union of a finite number of disjoint knots. Since we are dealing with knots and knot diagrams, we must avoid the use of a link. Thus we must remove at least one crossing to avoid obtaining a link.

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