# A General Model for Variations of the Even Cycle Problem

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**Abstract**

We consider three related problems in graph theory: determining if a directed graph has a directed even cycle, determining if a two edged-colored graph has an alternating colored even cycle, and determining if a directed graph has an anti-directed even cycle. We show that each of these, and two other variations, are special cases of a more general graph problem. We also show that one of these variations is NP-complete.

**Section 1: Introduction**
We present a general model to represent 5 different variations of the even cycle problem:

1. Even cycle problem (EC). Given a directed graph *D*, determine if *D* has a directed cycle of even length.
2. Alternating cycle problem (AC). Given a 2-colored graph *D*, determine if *D* has a cycle with alternating colors.
3. Alternating directed cycle problem (ADC). Given a directed 2-colored graph *D*, determine if *D* has an alternating directed cycle.
4. Anti-directed cycle problem (AD). Given a directed graph *D*, determine if *D* has an anti-directed cycle.
5. Alternating anti-directed cycle problem (AAC). Given a directed 2-colored graph *D*, determine if *D* has an alternating anti-directed cycle.

There is a cycle of the desired type in the original graph if and only if there is a corresponding cycle in the general model.

The classic even cycle problem (EC), simply to determine if a directed graph has a directed even cycle, was originally put forth by Polya in 1913 in an effort to evaluate matrix permanents. Partial results were found by Kasteleyn [5], and the problem of finding an efficient algorithm was solved by McCuaig [6].

Another even cycle variation involves cycles of alternating colors (AC). In this problem we are given a graph with edges that are colored one of two different colors. We ask if the graph has an alternating cycle, which is a cycle where edges alternate between the two colors (i.e. color 1, color 2, color 1, color 2,…..). Clearly such a cycle has an even number of edges. A method for determining if such a cycle exists was given by Grossman and Haggkvist [3].

The third basic problem involves determining if there is an anti-directed cycle (AD) in a directed graph. In an anti-directed cycle the edge orientation alternates, i.e. no two consecutive edges form a directed path. Definition: An anti-directed cycle is a cycle that alternates between following the given direction of a digraph and going against the given direction; it is even by nature. The manuscript put forth by Gannon [1] gives conditions that guarantee an anti-directed cycle, but does not give an efficient general algorithm for determining if there is one.

We combine these three types to make two more problem variations: finding an alternating directed cycle (ADC) and an alternating anti-directed cycle (AAC). Finding a directed cycle with alternating colors has been shown to be NP-complete by Gutin, Sudakov, and Yeo.[4]

In complexity theory, NP-complete problems (standing for Nondeterministic Polynomial time) have two properties; any given solution to the problem can be verified quickly and if the problem can be solved quickly, then so can every problem in NP. The time required to solve the problem using any currently known algorithm increases very quickly as the size of the problem grows. Thus we do not expect to be able to find an efficient general algorithm to solve such a problem.

We prove here that the fifth problem, that of finding an anti-directed cycle with alternating colors, is also NP-complete. Thus of our five variations, two (EC and AC) have known efficient solutions, two (ADC and AAC) are NP-complete, and one (AD) is unknown status.

**Section 2: General Model**

We give a general model that can be used to express each of the five cases of the even cycle problem. In particular, a specially constructed bipartite graph can be used as the general model. Note that experimenting with a bipartite graph as a general model was natural considering that all cycles in a bipartite graph are forced to be even. For example, the classic result put forth by McCuaig [6] for the directed even cycle problem began by translating the problem into one on bipartite graphs. Let $B$ be a bipartite graph (in some cases $B$ may instead be a directed bipartite graph) with vertex parts $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_m\}$. We call a cycle in $B$ a "choice cycle" if for each *i,* the cycle includes at most one vertex of $\{x_i, y_i\}$. There is a choice cycle in the bipartite graph if and only if there is an even cycle of the desired type in the original graph for each of the five cases.

1. First consider the even directed cycle problem on a digraph D with vertex set {1,2,…,m}. We set up a bipartite graph with vertex parts $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_m\}$. For each arc $i \to j$ in $D$, we include directed edges $x_i \to y_j$ and $y_i \to x_j$ in $B$. An example is given in Figure 1.

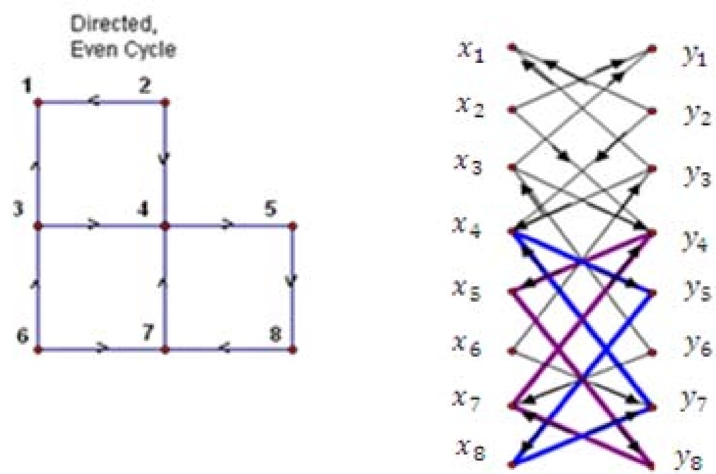


Figure 1. Construction for the EC problem

It is straightforward to check that there is a directed even cycle, $i_1 \to i_2 \to \cdots \to i_k \to i_1$ , in $D$ if and only if the corresponding bipartite graph has directed choice cycles

$$x_{i_1} \to y_{i_2} \to x_{i_3} \to \cdots \to y_{i_k} \to x_{i_1}$$

and

$$y_{i_1} \to x_{i_2} \to y_{i_3} \to \cdots \to x_{i_k} \to y_{i_1}.$$

These are choice cycles because no subscript is repeated. Therefore, a directed even cycle in the original graph exists if and only if two directed cycles exist in the bipartite graph.

2. When translating a graph with alternating colors, again each edge ij corresponds to two arcs. Now the columns of the bipartite graph are labeled red outgoing and green outgoing with red edges directed to the right and green edges directed to the left; so red edge ij corresponds to arcs $x_i \rightarrow\ y_j$ and $x_j \rightarrow y_i$, while green edge ij corresponds to arcs $y_i \rightarrow x_j$ and $y_j \rightarrow x_i$. The number of edges in the bipartite graph is double the number of edges in the original once again, but this time because the colored edges in the original graph do not have direction. An example is given in Figure 2.

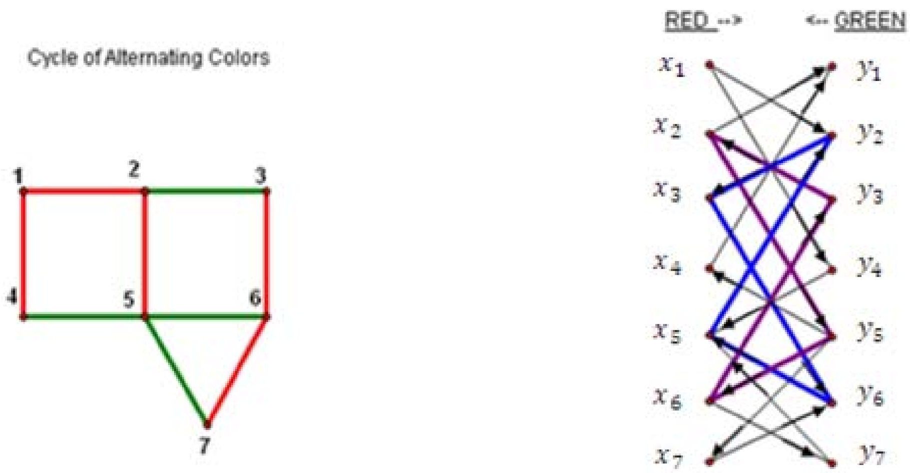

Figure 2. Construction for the AC problem

Once again, an alternating cycle in the original corresponds to two directed choice cycles in the bipartite digraph.

3. When translating a directed graph with alternating colors into the bipartite graph, combine the methods used in the first two variations. We label the first column in the bipartite graph Red out /Green in and label the other column Green out / Red in. A directed cycle in the bipartite graph means that a directed cycle with alternating colors is present in the original graph. Thus, an arc ij in the original graph will generate in the bipartite graph arc $x_i \rightarrow\ y_j$ if the original is green, and arc $y_i \rightarrow x_j$ if the original is red. An example is given in Figure 3.

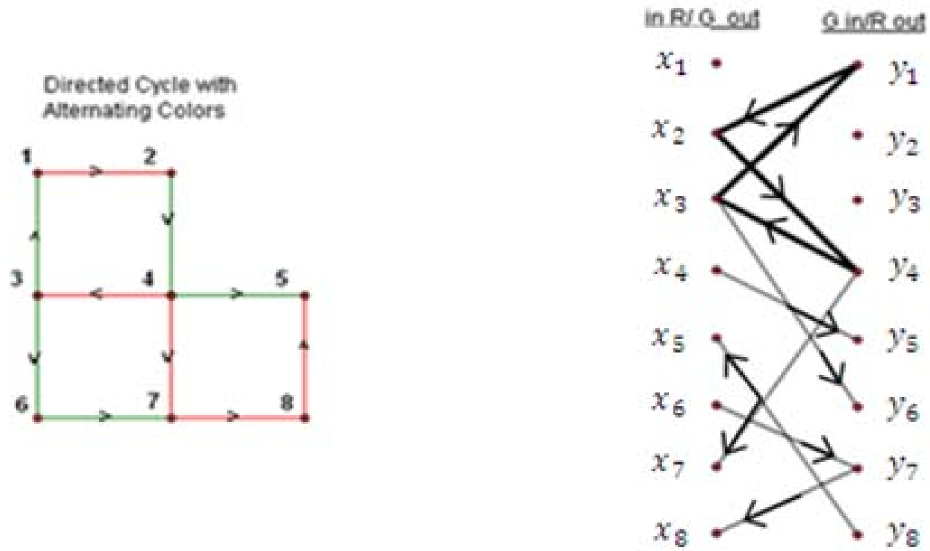


Figure 3. Construction for the ADC problem

By using this procedure, the cycle $y_1 \rightarrow x_2 \rightarrow y_4 \rightarrow x_3 \rightarrow y_1$ exists in the bipartite graph because a directed cycle of alternating color exists in the original graph. It is easy to check that the original has an alternating directed cycle if and only if the constructed graph has a directed choice cycle.

4. Translating a directed graph into the bipartite graph model when seeking an anti-directed cycle is very similar to translating the directed graph when seeking a directed cycle in Variation #1. The columns of the bipartite graph are labeled the same and the edges are translated using the same procedure used in Variation #1. In this instance though, the edges do not need to be reflected/doubled and need not be directed in the bipartite graph. So arc ij generates edge $x_i \rightarrow y_j$ in the bipartite construction, which has a directed choice cycle if and only if the original digraph has an anti-directed cycle. An example is given in Figure 4.

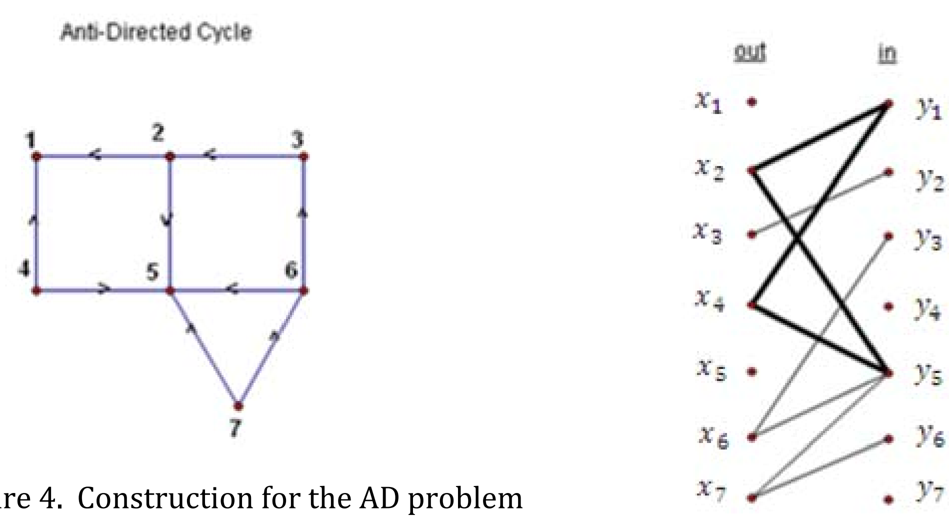


Figure 4. Construction for the AD problem

5. Translating a directed graph with alternating colors into the bipartite graph when seeking an anti-directed cycle is very similar to translating a directed graph with alternating colors when seeking a directed cycle in Variation #3.

**Theorem 1**: The problem of finding an alternating, directed cycle is equivalent to the problem of finding an alternating, anti-directed cycle.
**Proof**: By its definition, an anti-directed cycle alternates between following the given direction and going against the given direction. A cycle of alternating colors is a cycle with edges that alternate between two colors. Trying to find an anti-directed cycle with alternating colors combines these two types; therefore, since both are alternating in some way, edges of color 1 will follow the given direction and edges of color 2 will go against the given direction. Reversing the direction of color 1 or color 2 will then make the edges of the cycle all follow the new given directions. Therefore, by reversing the direction of all arcs of one color in an anti-directed cycle of alternating color, the cycle becomes directed. This implies that finding an anti-directed cycle of alternating color in a given graph is equivalent to finding a directed cycle of alternating color.

Since we have already shown that a directed graph with alternating colors can be translated into the general model of the directed bipartite graph, the anti-directed graph with alternating colors can also be translated by the above statement. An example is shown in Figure 5.

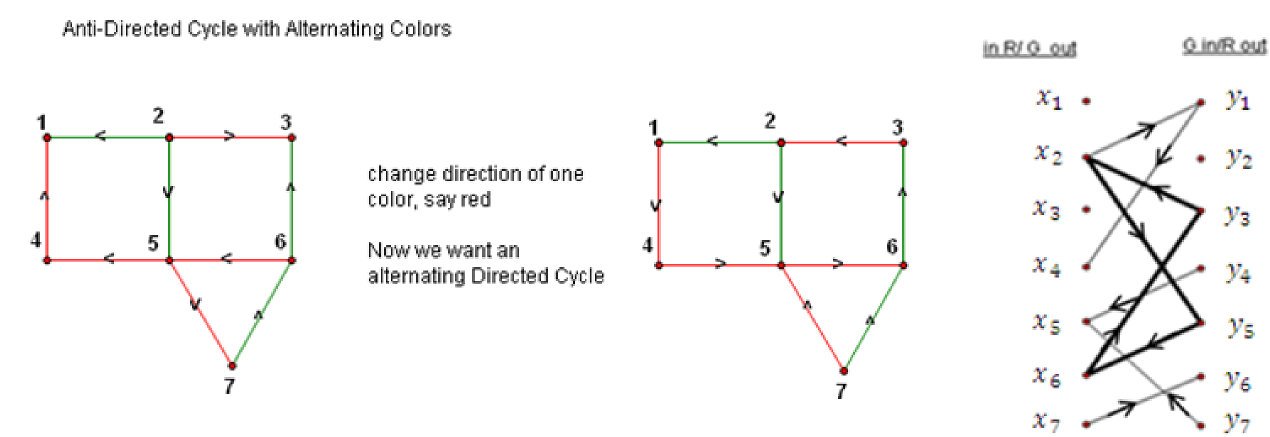


Figure 5. Construction for the AAC problem

Therefore, each of these five variations of the even cycle problem can be translated into the problem of finding a choice cycle or choice directed cycle in a bipartite graph or directed graph.

**Section 3: NP-completeness of the AAC problem**

Using **Theorem 1** from the 5th variation of the even cycle problem, we will show that finding an anti-directed cycle with alternating colors is NP-Complete.

**Theorem 2. Finding an alternating, anti-directed cycle (AAC) is NP-Complete**

Gutin, Sudakov, and Yeo proved the NP-Completeness of the alternating, directed cycle problem (ADC) [4]. By our **Theorem 1**, the alternating, anti-directed cycle problem (AAC) is polynomially equivalent to the ADC problem, so both are NP-complete. We provide a full proof by reduction of the 3-SAT problem to an instance of an AAC problem, based closely on the proof in Gutin.

3-SAT problems can be defined as follows: We are given a finite set $U$ of variables, and a collection of m clauses, $C = \{c_1, c_2, \ldots, c_m\}$. Each clause consists of 3 literals (which are each a variable "x" from $U$, or its negation "not x"). The 3-SAT problem is to determine if there is a truth assignment (T/F) to the variables of $U$ so that each clause in $C$ contains at least one literal whose value is "true". Note that if variable "x" is assigned a "true" value, then "not x" is "false".

(see reference [2] for more background on 3-SAT problems)

**<u>Proof</u>**:

Begin with any instance of the 3-SAT problem. Let us denote $U = \{u_1, u_2, \ldots, u_k\}$ as the set of k possible variables. Let us denote $C = \{c_1, c_2, \ldots, c_m\}$ as the m clauses in the 3-SAT problem such that every clause has 3 literals. Let us also denote each of these literals as $v_{(i,l)}$, where *i* = {the clause that the literal is in} and *l* = {the position of the literal in clause *i*, *l* ≤ 3}.

We then construct a 2-colored digraph *D* which has an alternating, anti-directed cycle if and only if *C* (the set of clauses) is satisfiable. The vertices in digraph *D* consist of two disjoint sets:

$X = \{x_i : 1 \leq i \leq m+2\}$ and coordinates

$Y = \{(j,0), (j,1), (\overline{j,1}), (j,2), (\overline{j,2}), \ldots, (j,t), (\overline{j,t}), (j,t+1)\}$, where j = 1, 2, ..., k and t = 6m.

The edges in digraph *D* fall into three disjoint sets. Subscript *R/B* denotes the arc's color as Red/Blue. Two of the sets (*B* and *P* below) remain the same, regardless of the variables that appear in the clauses.

$$B = \{(x_{m+2}, x_{m+1})_R, (x_{m+2}, (1,0))_B, ((k,t+1), x_1)_B\} \cup \{((p,t+1), (p+1,0))_B : 1 \leq p \leq k-1\}$$

$$P = \left\{ \big((j,1),(j,0)\big)_R, \big((j,1),(j,2)\big)_B, \big((j,3),(j,2)\big)_R, \ldots, \big((j,t-1),(j,t)\big)_B, \big((j,t),(j,t+1)\big)_R \right\} \cup$$
$$\{ \big((j,0),(\overline{j,1})\big)_R, \big((\overline{j,1}),(\overline{j,2})\big)_B, \big((\overline{j,3}),(\overline{j,2})\big)_R, \ldots, \big((\overline{j,t-1}),(\overline{j,t})\big)_B, \big((\overline{j,t}),(j,t+1)\big)_R \}$$

Figure 6 shows an example digraph with all needed vertices and only arc sets $B$ and $P$ for $U = \{u_1, u_2, u_3\}$ and $C = \{c_1, c_2\}$. Here k=3, m=2, t=12.

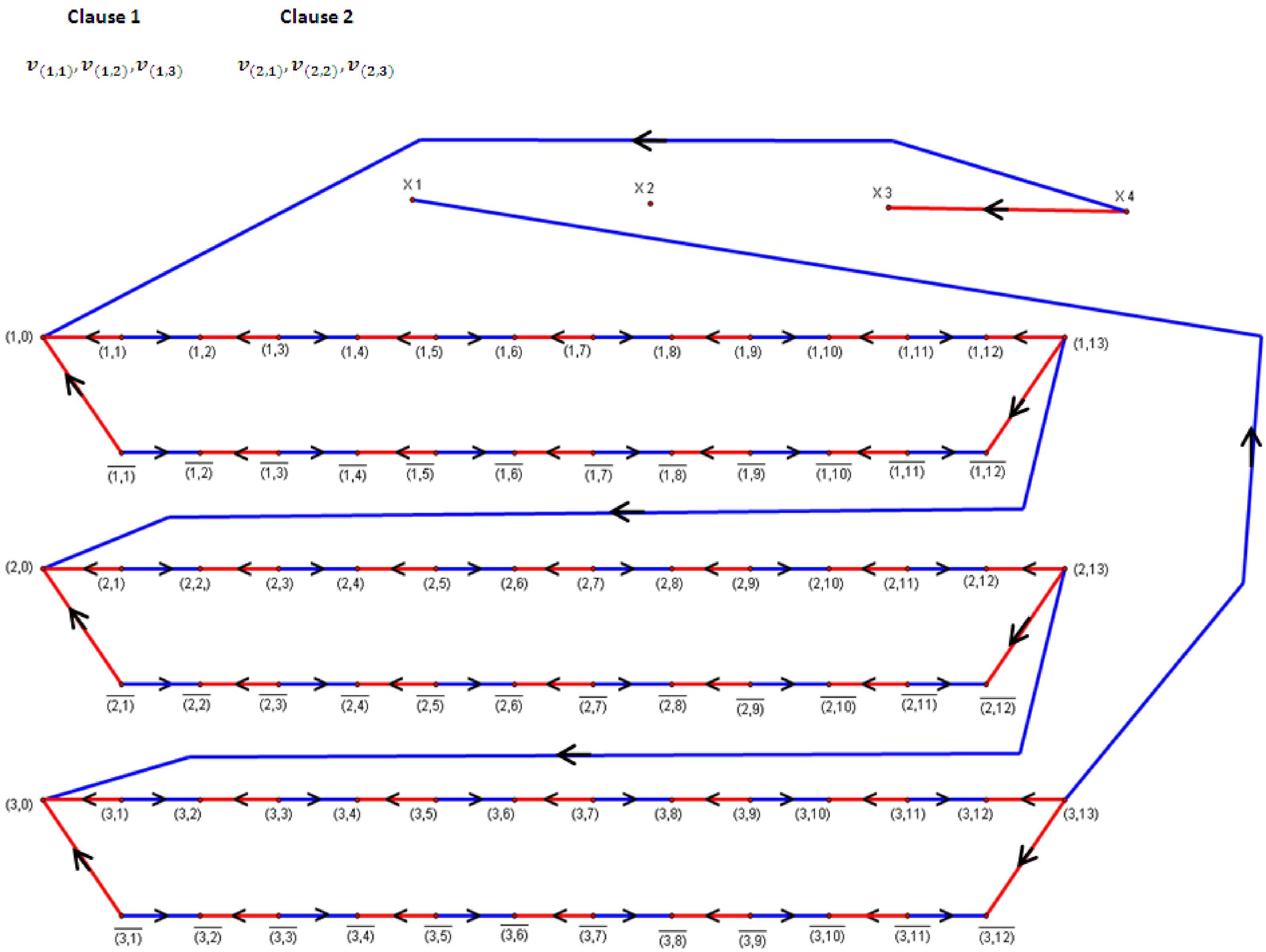


Figure 6. Edge sets *B* and *P*.

The third set of edges will depend on the clauses. Each variable in a clause will correspond to one of the coordinates of Y. This mapping gives us our third set of edges,

$$Q = \left\{ \left(coord.v_{(i,l)}, x_i\right)_R, \left(coord.v_{(i,l)}, x_{i+1}\right)_B \right\};\ i = 1,2,\dots,m\ ;\ l = 1,2,3$$

where if the variable $v_{(i,l)}$ corresponds with literal $u_j$, then

$$coord.v_{(i,l)} = \begin{cases} \overline{(j\,, 6(i-1)+2l)} & if\ v_{(i,l)} = \bar{u}_j \\ (j\,, 6(i-1)+2l) & if\ v_{(i,l)} = u_j \end{cases}.$$

Figure 7 combines Figure 6 with edge set Q for $c_1 = (a, b, \bar{b})$ and $c_2 = (a, \bar{b}, \bar{c})$. So, for example, from clause 1 we have red and blue edges going from (1,2) to $x_1$ and $x_2$ , from (2,4) to $x_1$ and $x_2$ , and from $(\overline{2,1})$ to $x_1$ and $x_2$ .

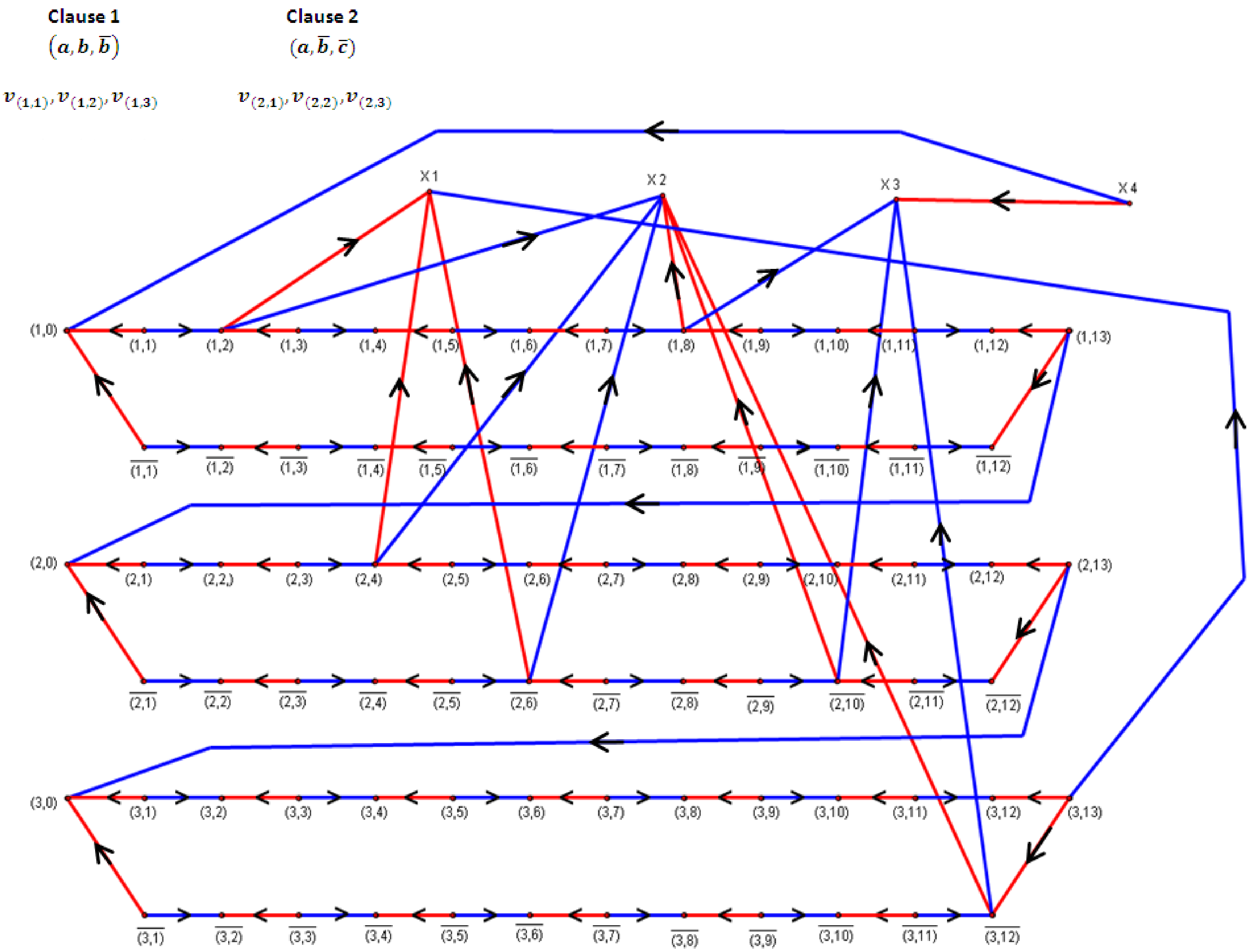


Figure 7. The constructed digraph

We claim that there is an alternating, anti-directed cycle in this graph if and only if these 2 clauses are satisfiable. We see that the truth assignment

a=true, b=false, c=false

holds. Therefore, there must be at least one alternating, anti-directed cycle in this graph. Figure 8 shows one from Figure 7. (In the proof, this alternating, anti-directed cycle will correspond directly to the given truth assignment).

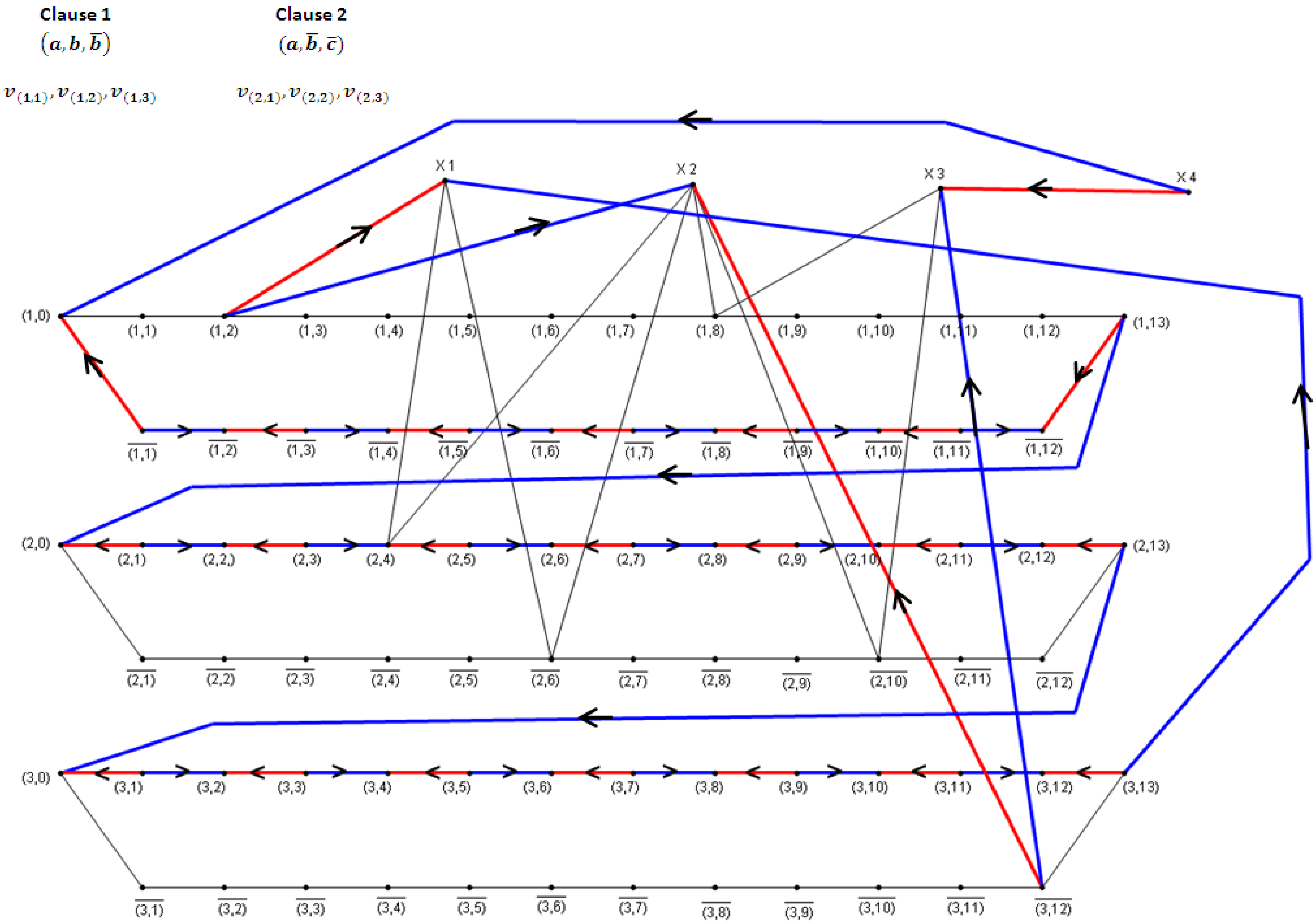


**Figure 8**. An alternating, anti-directed cycle corresponding to valid truth assignment {a=true, b=false, c=false}.

We now prove that $C$ is satisfiable in any 3-SAT problem if an only if $D$ has an alternating cycle.

Looking at the structure of the construction of $D$, it is straightforward to check the structure that any possible alternating, anti-directed cycle must have. There is an alternating, anti-directed cycle if and only if the cycle consists of pieces of the form
( 0 )

$$x_1 \leftarrow coord.v_{(1,l_1)} \rightarrow x_2 \leftarrow coord.v_{(2,l_2)} \rightarrow x_3 \leftarrow, \ldots, \rightarrow x_m \leftarrow coord.v_{(m,l_m)} \rightarrow x_{m+1} \leftarrow x_{m+2}$$

and

( 1 )

$$x_{m+2} \rightarrow (1,0) \leftarrow (1,1) \rightarrow (1,2) \leftarrow \cdots \rightarrow (1,6m) \leftarrow (1,6m+1)$$

or

$$x_{m+2} \rightarrow (1,0) \leftarrow (\overline{1,1}) \rightarrow (\overline{1,2}) \leftarrow \cdots \rightarrow (\overline{1,6m}) \leftarrow (1,6m+1)$$

and

( 2 )

$$(1,6m+1) \rightarrow (2,0) \leftarrow (2,1) \rightarrow (2,2) \leftarrow \cdots \rightarrow (2,6m) \leftarrow (2,6m+1)$$

or

$$(1{,}6m+1) \rightarrow (2{,}0) \leftarrow (\overline{2{,}1}) \rightarrow (\overline{2{,}2}) \leftarrow \cdots \rightarrow (\overline{2,6m}) \leftarrow (2,6m+1)$$

.

.

.

and

( $k$ )

$$(k-1,6m+1) \rightarrow (k,0) \leftarrow (k,1) \rightarrow (k,2) \leftarrow \cdots \rightarrow (k,6m) \leftarrow (k,6m+1) \rightarrow x_1$$

or

$$(k-1,6m+1) \rightarrow (k,0) \leftarrow (\overline{k,1}) \rightarrow (\overline{k,2}) \leftarrow \cdots \rightarrow (\overline{k,6m}) \leftarrow (k,6m+1) \rightarrow x_1$$

Such a cycle will exist if and only if in line ( 0) we can choose "*coord*" vertices that do not appear in the lines chosen from each of the parts ( 1 ), ( 2 ),…, ( $k$ ). The choices of the "*coord"* vertices in (0) in such a cycle determines the truth assignments for the corresponding 3-SAT problem; including arc $x_i \leftarrow coord.v_{(i,l_i)}$ in the alternating, anti-directed cycle means that in the 3-SAT problem clause *i* will be satisfied by literal $u_j$ if $v_{(i,l)} = u_j$ (so $u_j$ is assigned a "true" value), or clause *i* will be satisfied by literal $\overline{u_j}$ if $\overline{v_{(i,l)}} = \overline{u_j}$ (so $u_j$ is assigned a "false" value). But since each "*coord*" vertex corresponds to a literal with a "true" value, this will be possible if and only if :

(i) each clause contains a true literal, because of line ( 0 )

(ii) variable truth assignments are consistent (we do not use $u_j$ and $\overline{u_j}$ both true to satisfy clauses, because then neither string in line ( j ) can be used in the cycle).

Thus the clauses in *C* are only satisfiable if and only if the constructed digraph *D* has an alternating, anti-directed cycle. Since the constructed digraph has ($m$ + 2) + $k$ (6$m$ + 2) vertices, the reduction is clearly polynomial time.

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