A special case of the Yelton-Gaines Conjecture on Isomorphic Dessins

by
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Abstract

Let (ρ_0, ρ_1) and (ρ_0', ρ_1') be two ordered pairs of permutations in S_n and let t be a divisor of n. The Yelton-Gaines conjecture states that if at least one of these four permutations is a product of n/t disjoint t-cycles, and if there is a *strong* isomorphism (definition below) $\phi : \langle\rho_0, \rho_1\rangle \to \langle\rho_0', \rho_1'\rangle$ between the two subgroups of S_n generated by the elements in each ordered pair, then there is a fixed permutation τ in S_n that simultaneously conjugates ρ_i to ρ_i' for $i = 0, 1$. The conclusion of this conjecture can be restated to say that the two *dessins d'enfants* corresponding to the two ordered pairs are isomorphic.

In this paper a proof of this conjecture is given in the case in which all of the initial four permutations are fixed-point-free involutions.

1 Introduction

The term *dessin d'enfant* was coined by Grothedieck to refer to a bipartite graph that is embedded in a compact, oriented Riemann surface, that is, into a torus with $g \geq 0$ holes. Every dessin is determined up to isomorphism by an ordered pair of permutations from a symmetric group. We emphasize that this pair of permutations is ordered and that the dessin $\mathbb{D}(\rho_0, \rho_1)$ determined by the pair (ρ_0, ρ_1) is not usually isomorphic to the dessin $\mathbb{D}(\rho_1, \rho_0)$ obtained by reversing the order. Two ordered pairs of permutations are considered to be the same when there is a $\tau \in S_n$ that simultanesouly conjugates each component of the first ordered pair into the corresponding component of the second ordered pair. Every dessin gives rise to a graph obtained by ignoring the embedding into the surface of a torus. It frequently happens that two non-isomorphic dessins have isomorphic underlying graphs.

A *Gassmann triple* consists of a group G and two *locally conjugate* subgroups H and H', meaning that there exists a bijection $\psi : H \to H'$ such that $\psi(h)$ is conjugate to h in G for every $h \in H$. Let (G, H, H') be a Gassmann triple and note that H and H' must have the same index n in G. Choose an ordered pair (g_0, g_1) of elements of G. These elements act on the set of cosets G/H by left multiplication, giving an ordered pair of permutations (ρ_0, ρ_1) in S_n. The same elements g_0, g_1 act on the cosets G/H' giving a second ordered pair of permutations (ρ_0', ρ_1'). In turn, these two ordered pairs give rise to two dessins $\mathbb{D}(\rho_0, \rho_1)$ and $\mathbb{D}(\rho_0', \rho_1')$. These two dessins are called Gassmann equivalent.

We now describe the construction by which a pair of permutations yields a dessin. Let $\rho_0, \rho_1 \in S_n$. Then each cycle in the permutations corresponds to a vertex in our dessin. The cycles of ρ_0 correspond to black vertices and the cycles of ρ_1 correspond white vertices. If a vertex is induced by an n-cycle it is then endowed with n branches. The branches of a vertex are labeled in counter-clockwise order with the elements permuted by the corresponding cycle. After this process is complete, for all $n \in \{1, ..., n\}$ there is a branch, labeled n, attached to a white vertex and another attached to a black vertex. These two branches are then connected. This is done for each $n \in \{1, ..., n\}$ and we, thereby, produce the underlying graph of our dessin.

Example 1: Let $\rho_0 = (12)(34)(56)$ and $\rho_1 = (13)(456)(2)$. These permutations correspond to the following graph:

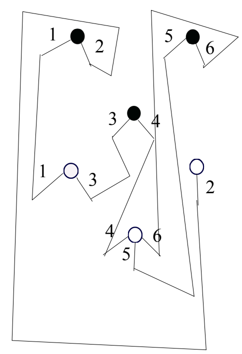

Figure 1: This is the dessin corresponding to ρ_0 and ρ_1.

The monodromy group of a dessin is the subgroup of S_n generated by the defining permutations. When two dessins are Gassmann equivalent, there exists an isomorphism ϕ between the monodromy groups that is a local conjugation in S_n and that maps ρ_0 to ρ_0' and maps ρ_1 to ρ_1'; see [M-P] for details. We refer to such an isomorphism between subgroups of a symmetric group as a *strong* isomorphism.

In 2007 Jeff Yelton examined many examples of Gassmann equivalent dessins. (See [Y]). He was interested in knowing when the underlying graphs were isomorphic. He conjectured that if $\mathbb{D}(\rho_0, \rho_1)$ and $\mathbb{D}(\rho_0', \rho_1')$ are Gassmann equivalent dessins and if at least one of $\rho_0, \rho_1, \rho_0', \rho_1'$ is composed entirely t-cycles for some fixed t, then the underlying graphs are isomorphic. In 2008, Ben Gaines expressed the opinion that Yelton's hypotheses should imply the stronger conclusion that the dessins are isomorphic. In fact, Gaines not only strengthened the conclusion, but also slightly weakened the hypothesis. He did so by removing the reference to Gassmann equivalent dessins, but still requiring there to be a strong isomorphism between monodromy groups. His work is found in [G]. We refer to Gaines' formulation as the *Yelton-Gaines* conjecture. Here is the precise statement.

Yelton-Gaines Conjecture Let $\rho_0, \rho_1, \rho_0', \rho_1'$ be permutations in S_n for which there is a group isomorphism $\phi : \langle \rho_0, \rho_1 \rangle \longrightarrow \langle \rho_0', \rho_1' \rangle$ which is a local conjugation in S_n and which maps ρ_i to ρ_i' for $i = 0, 1$. If at least one of the four permutations is composed entirely of t-cycles for some fixed divisor t of n, then there is a permutation $\tau \in S_n$ that conjugates ρ_i into ρ_i' for $i = 0, 1$.

2 Proof in the Special Case of Fixed-Point-Free Involutions

In this paper we prove the conjecture in the case when each of the four permutations $\rho_0, \rho_1, \rho_0', \rho_1'$ is a product of disjoint 2-cycles (and contains no 1-cycles). This forces n to be even. Write $n = 2m$.

We begin with a lemma.

Zigzag Lemma For $i = 0, 1$ let ρ_i be a product of m disjoint transpositions in S_{2m}. Let ρ_2 denote the product $\rho_1\rho_0$, with ρ_0 acting first. Fix an element a_1 in $\{1, \ldots, 2m\}$, let c denote the ρ_2-orbit of a_1, and let $s = |c|$ be the length of the cycle c. Then there exist a set of pairwise distinct elements $\{a_2, \ldots, a_s, b_1, \ldots, b_s\}$ in $\{1, \ldots, 2m\}$ each different from a_1, for which

$\rho_0 = (a_1, b_1)(a_2, b_2)...(a_{s-1}, b_{s-1})(a_s, b_s) * **$
$\rho_1 = (b_1, a_2)(b_2, a_3)...(b_{s-1}, a_s)(b_s, a_1) * **$

are the cycle decompositions, with $***$ denoting products of 2-cycles disjoint from the displayed cycles. Moreover, the ρ_2-orbit c is $c = (a_1, \ldots, a_s)$ and another orbit in ρ_2 is $d = (b_s, \ldots, b_1)$, disjoint from c.

Proof. Define $b_1 = \rho_0(a_1)$. Then $b_1 \neq a_1$ since ρ_0 has no fixed-points. Let $x_1 = \rho_1(b_1)$. Then $x_1 \neq b_1$ since ρ_1 has no fixed points. We distinguish two cases: either $x_1 = a_1$ or $x_1 \neq a_1$. If $x_1 = a_1$, then

$\rho_0 = (a_1, b_1) * **$
$\rho_1 = (b_1, a_1) * **$

In this case, $\rho_2(a_1) = a_1$ and $c = (a_1)$ is the ρ_2-orbit of a_1 so the length of c is $s = 1$. Moreover, $d = (b_1)$ is also cycle of length $s = 1$ in ρ_2 disjoint from c, as claimed by the lemma. Now consider the case $x_1 \neq a_1$, and rename x_1 to a_2. Then a_1, b_1, a_2 are pairwise distinct. Define $b_2 = \rho_0(a_2)$. Then b_2 is distinct from each of a_1, b_1, a_2. Let $x_2 = \rho_1(b_2)$. Then x_2 is different from b_1, a_2, b_2. We again distinguish two cases: Either $x_2 = a_1$ or $x_2 \neq a_1$. If $x_2 = a_1$, then

$\rho_0 = (a_1, b_1)(a_2, b_2) * **$
$\rho_1 = (b_1, a_2)(b_2, a_1) * **$

In this case, the ρ_2-orbit of a_1 is $c = (a_1, a_2)$ of length $s = 2$, and $d = (b_2, b_1)$ is another length 2 cycle in ρ_2 that is disjoint from c, as the lemma claims. In the other case when $x_2 \neq a_1$ then rename x_2 to a_3. Continue this process. If after $m - 1$ iterations we have not yet produced an element $x_i = a_1$ then necessarily in the mth iteration we will have $x_m = a_1$ since a_1 must occur somewhere in the cycle decomposition of ρ_1. Let $s \leq m$ be the first (and in fact only) integer i for which this process produces $x_i = a_1$. Then

$\rho_0 = (a_1, b_1)(a_2, b_2) \ldots (a_s, b_s) * **$
$\rho_1 = (b_1, a_2)(b_2, a_3) \ldots (b_s, a_1) * **.$

Then $c = (a_1, a_2, \ldots, a_s)$ and $d = (b_s, b_{s-1}, \ldots, b_1)$ are disjoint cycles of ρ_2 of length s. This proves the lemma. $\square$

We note a corollary.

Corollary 1: Let ρ_0, ρ_1, and ρ_2 be as in the zigzag lemma, and let $M = \langle \rho_0, \rho_1 \rangle$ be the monodromy group. Then the number of cycles of ρ_2 of a given length is even and the total number of cycles in

ρ_2 is even. The cycles in ρ_2 of a given length can be paired as c_i and d_i so that the M-orbits in $\{1, ..., 2m\}$ are precisely the union of the elements permuted by c_i and d_i.

Proof: Fix an element a_1 and let the ρ_2-orbit of a_1 be $c = (a_1, a_2, \ldots, a_s)$. According to the zigzag lemma, there are elements $b_1, b_2, \ldots, b_s$ for which $d = (b_s, \ldots, b_1)$ is another orbit of ρ_2 of length s and disjoint from c. Moreover, ρ_0 and ρ_1 interchange the set of elements of c and with the set of elements of d. The map ρ_0 preserves the order while ρ_1 doesn't, but the order doesn't matter here. So the union $c \cup d$ (with minor notational liberty) is mapped to itself by the monodromy group M. And this union is the M-orbit of a_1 as can be seen by letting the generators ρ_0, ρ_1 alternately act on a_1. $\square$

We are now ready to prove the main result of this paper.

Theorem: Let each of $\rho_0, \rho_1, \rho_0', \rho_1' \in S_{2m}$ be products of m disjoint 2-cycles. If there exists a strong isomorphism of monodromy groups $\phi : \langle \rho_0, \rho_1 \rangle \to \langle \rho_0', \rho_1' \rangle$ then there is an element $\tau \in S_{2m}$ that simultaneously conjugates ρ_i to ρ_i' for $i = 0, 1$. In otherwords, there is an isomorphism of dessins $\mathbb{D}(\rho_0, \rho_1) \cong \mathbb{D}(\rho_0', \rho_1')$.

Proof. Let $\rho_2 = \rho_1 \rho_0$ and let $\rho_2' = \rho_1' \rho_0'$. The strong isomorphism ϕ maps ρ_2 to ρ_2'. Since ϕ is a local conjugation, then ρ_2 and ρ_2' have the same cycle structure. By Corollary 1, there are an even number, say $2r$, of cycles of ρ_2, and the set of these cycles can be written as $\{c_1, d_1, \ldots, c_r, d_r\}$ where each pair c_i, d_i is as described in the zigzag lemma. That is, we can write $c_i = (a_{i,1}, \ldots, a_{i,s_i})$ and $d_i = (b_{i,s_i}, \ldots, b_{i,1})$ so that ρ_0, ρ_1, and ρ_2 are given by

$$\begin{aligned}
\rho_0 &= \textstyle\prod_{i=1\ldots r} (a_{i,1}, b_{i,1}) \ldots (a_{i,s_i}, b_{i,s_i}) \\
\rho_1 &= \textstyle\prod_{i=1\ldots r} (b_{i,1}, a_{i,2}) \ldots (b_{i,s_i}, a_{i,1}) \\
\rho_2 &= \textstyle\prod_{i=1\ldots r} (a_{i,1}, \ldots, a_{i,s_i})(b_{i,s_i} \ldots, b_{i,1}).
\end{aligned}$$

Similarly, there are $2r$ cycles in ρ_2' and the set of these cycles can be written as $\{c_1', d_1', \ldots, c_r', d_r'\}$ with c_i and c_i' having the same length for $i = 1, \ldots, r$. Moreover, we can write $c_i' = (a_{i,1}', \ldots, a_{i,s_i}')$ and $d_i' = (b_{i,s_i}', \ldots, b_{i,1}')$ so that $\rho_0', \rho_1', \rho_2'$ are given by

$$\begin{aligned}
\rho_0' &= \textstyle\prod_{i=1\ldots r} (a_{i,1}', b_{i,1}') \ldots (a_{i,s_i}', b_{i,s_i}') \\
\rho_1' &= \textstyle\prod_{i=1\ldots r} (b_{i,1}', a_{i,2}') \ldots (b_{i,s_i}', a_{i,1}') \\
\rho_2' &= \textstyle\prod_{i=1\ldots r} (a_{i,1}', \ldots, a_{i,s_i}')(b_{i,s_i}' \ldots, b_{i,1}')
\end{aligned}$$

With the notation just established, define the permutation $\tau \in S_{2m}$ by declaring that for each $i = 1, \ldots, r$ and for each $j = 1, \ldots s_i$ that $\tau(a_{i,j}) = a_{i,j}'$ and $\tau(b_{i,j}) = b_{i,j}'$. This τ gives the desired conjugation. We note in passing that the given strong isomorphism ϕ agrees with conjugation by τ since both maps are group isomorphisms taking the same values on generators of the monodromy group M.
$\square$

With the discussion above, we conclude:

Corollary 2: If the fixed-point-free involutions $\rho_0, \rho_1, \rho_0', \rho_1'$ in S_n arise from a pair of elements g_0, g_1 in a Gassmann triple (G, H, H') of finite groups, then the corresponding dessins are isomorphic: $\mathbb{D}(\rho_0, \rho_1) \cong \mathbb{D}(\rho'_0, \rho'_1)$.

References

[G] B. Gaines, LSU REU final project report, summer 2008, unpublished.

[M-P] M. Merling and R. Perlis, Gassmann Equivalent Dessins, Communications in Algebra, vol 36, (2009), to appear.

[Y] J. Yelton, LSU REU final project report, summer 2007, unpublished.

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