Extensions of Extremal Graph Theory to Grids

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ABSTRACT

We consider extensions of Turán's original theorem of 1941 to planar grids. For a complete k x m array of vertices, we establish in Proposition 4.3 an exact formula for the maximal number of edges possible without any square regions. We establish with Theorem 4.12 an upper bound and with Theorem 4.15 an asymptotic lower bound for the maximal number of edges on a general grid graph with n vertices and no rectangles.

1. INTRODUCTION

The field of extremal graph theory began in 1940 when Paul Turán [12] answered the following question: given natural numbers $r < n$, how many edges can a graph G with n vertices have if it contains no complete subgraphs K_r? A complete subgraph on n vertices is just a graph for which every pair of vertices shares an edge. Turán was able to show that the maximal number of edges is attained through a complete ($r-1$)- partite graph. It follows that the number E of edges satisfies

(1) $$E \le \left(\frac{r-2}{r-1}\right)\frac{n^2}{2},$$

with equality if and only if $(r-1)|n$. This result opens a world of new Turán-type problems. The general problem can be formulated as follows: given a natural number n and a graph G', what is the maximal number $ex(n,G')$ of edges that can be put on a graph G of order n with no subgraph isomorphic to G'?

We consider extensions to planar grids with vertices at integer points and horizontal or vertical edges. Further, every point in the grid that an edge of the graph passes through does not have to be designated a vertex. If the n vertices form a complete k x m array, $V = \{(x,y) \in \mathbb{Z}^2 : 1 \le x \le k, 1 \le y \le m\}$, then the maximal number of edges $ex(n)$ is of course

$$ex(n) = (k-1)m + (m-1)k .$$

If furthermore the graph has no square regions, our Proposition 4.3 shows the maximal number of edges $ex_S(n)$ is exactly

$$ex_S(n) = \left\lfloor \frac{1}{2}(3km - k - m - 1) \right\rfloor.$$

For an arbitrary set of n vertices in $\mathbb{Z}^2$, the maximum number of edges $ex(n)$ is asymptotic to $2n$. If the graph further has no rectangles, that is no rectangular unions of regions, Theorem 4.15 establishes the asymptotic lower bound

$$\frac{7}{5}n + o(n) \leq ex_R(n).$$

Theorem 4.12 establishes the upper bound

$$ex_R(n) < \frac{3}{2}n.$$

Interestingly, the construction used in Theorem 4.15 uses 7-sided polygonal regions. We find other ways of putting many edges on a grid graph through recursive construction techniques that are outlined in Remark 4.16, which are useful for comparison. We conjecture in 4.24 that in fact the lower bound from Theorem 4.15 is the maximal number of edges for a grid graph with no rectangles:

$$ex_R(n) = (7/5)n + o(n).$$

2. HISTORY

Since Turán introduced the original problem, significant progress has been made on the more general problem. In 1946 Erdös and Stone [6] extended the result, letting G' be $K_r(t)$, the complete r-partite graph with exactly t vertices in each class. They showed that

(2) $$ex(n, K_r(t)) \leq \left(\frac{r-2}{r-1} + o(1)\right)\frac{n^2}{2}.$$

For $t = 1$, this gives a weaker form of Turán with an error term. Of course, for $t > n/r$, the hypothesis is vacuous and the maximum number of edges $E = \binom{n}{2}$.

Erdös-Stone holds for any graph with chromatic number r. Indeed, any graph G' of chromatic number r is contained in some $K_r(t)$, for t sufficiently large, and thus any graph that does not contain $K_r(t)$ also does not contain a G' subgraph. An alternate form of the result, the original statement that Erdös and Stone proved, is that for every natural number t and $\varepsilon > 0$, if the maximal size of a graph of order n is εn^2 greater than the maximal size of a graph G without a complete graph K_r, then

G contains a $K_r(t)$, for n sufficiently large [4]. For more general multipartite subgraphs the problem becomes much harder, and remains open.

After complete graphs and multipartite graphs, perhaps the most natural type of subgraph to exclude is cycles. For odd cycles, there is a comprehensive result. The complete bipartite graph $K_{n/2,n/2}$ [1] does not contain any odd cycles. A complete bipartite graphIt has been shown by Balister, Bollobás, Riordon and Schelp [3] that this is in fact the maximal case when n is even, and thus that $E \le n^2/4$ for $G' = C_{2r+1}$ [2]. When n is odd, there is a similar upper bound. Specifically, we consider a complete bipartite graph $K_{m,m+1}$, and we see that $E = (n^2-1)/4$. As with the Turán bound (1), the closer the cardinality of the parts of the bipartite graph are to equal, the more edges can be put on the graph [3]. For even cycles considerably less is known. When $G' = C_{2r}, r \ge 3,$ Bondy and Simonovits [10] have shown the asymptotic upper bound

(3) $$ex(n, C_{2r}) = \mathrm{O}\left(n^{1+1/r}\right).$$

For $G' = C_6$ or C_{10}, Mellinger and Mubayi [10] after [9] used higher dimensional projective planes to explicitly construct graphs establishing the lower bounds $\Omega\left(n^{4/3}\right)$ and $\Omega\left(n^{6/5}\right)$ and showing that the Bondy-Simonovits bound can sometimes be reached asymptotically. Bondy and Simonovits [10] found an upper bound for even cycles,

$$ex(n, C_{2r}) \le 100rn^{1+1/r}.$$

Verstraëte [13] subsequently showed that the constant can be lowered from $100r$ to 8(r -1). Note that for $G' = C_4$, the smallest even cycle, we can apply Erdös-Stone, as $C_4 = K_{2,2}$. For a graph on n vertices and no C_4, Erdös-Stone (2) says

(2a) $$ex(n, C_4) = \left(\frac{2-2}{2-1} + o(1)\right)\frac{n^2}{2} = o\left(n^2\right),$$

weaker than the Bondy- Simonovits upper bound, $\mathrm{O}(n^{3/2})$.

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[1] . A complete bipartite graph is composed of two separate parts such that vertices in the same part are not connected by an edge, while those in different parts are connected by an edge. Thus, a complete bipartite graph $K_{a,b}$ has $n = a + b$ vertices divided into two parts of order a and order b.

[2] A cycle C_n is a graph on n vertices where each vertex is connected to exactly two other vertices by an edge. Thus, it takes on the visual form of a closed loop.

Another means of modifying the original Turán problem is to fix not only G', but to start with a certain graph G and ask how many edges can be put on such a graph with no subgraph isomorphic to G'. Given $m \geq n \geq 1$ and $r \geq s \geq 1$, the Zarankiewicz Problem considers bipartite graphs with m vertices in one part and n vertices in the other that avoid a particular bipartite subgraph $K_{r,s}$. Kövari, Sós, and Turán [7] proved that the number of edges E of G satisfies

(4) $$E < (r-1)^{1/s}(n-s+1)m^{1-1/s} + (s-1)m + 1.$$

This bound (4) is not comparable to Turán's bound (1). An optimal bound is not known. The restriction on the graph G is stricter than Turán's is, while the restriction on the subgraph G' is looser.

We can compare Zarankiewicz to Erdös-Stone in a simple example. Let $G = K_{3,3}$, $G' = K_{2,2}$.

Erdös - Stone (2) gives us

$$ex(6, K_{2,2}) = \left(\frac{2-2}{2-1} + o(1)\right)\frac{6^2}{2} = 18 \;\; o(1)$$

that is no information, while Kövari-Sós-Turán (4) gives us

$$E < (2-1)^{1/2}(3-2+1)3^{1-1/2} + (2-1)2 + 1 = 2\sqrt{3} + 3 \approx 6.47$$

Thus, we see that the Zarankiewicz bound is superior in this case. Figure 2.2 illustrates how $ex(G_{3,3}, K_{2,2}) = 6$.

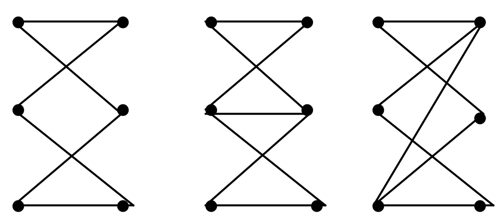

Figure 2.2. The arrangement on the left illustrates a $K_{3,3}$ with no subgraph isomorphic to $K_{2,2}$ and 6 edges. However, any graph on $K_{3,3}$ with 7 edges must either be isomorphic to the middle arrangement or to the right arrangement, either of which introduces a $K_{2,2}$. Thus we see that $ex(K_{3,3}, K_{2,2}) = 6$, [3]and that (4) is sharp in this case.

[3] The notation $ex(K_{a,b}, K_{c,d})$ indicates the maximum number of edges on a bipartite graph $K_{a,b}$ excluding any subgraphs isomorphic to the complete subgraph $K_{c,d}$.

Erdös, Györi, and Simonovits [5] considered graphs on n vertices that are triangle-free[4]. If the graph is further bipartite, then the following relation holds:

(5) $$\left\lfloor \frac{n^2}{4} \right\rfloor - \left(\left\lfloor \frac{n}{2} \right\rfloor - 2 \right) \leq ex(n,\Delta) \leq \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Of course the upper bound simply follows from Turán (1).

Turning the problem on its head is another means of extending Turán's result. Instead of asking how many edges can put on a graph without certain subgraphs, we ask how many of a certain subgraph are necessarily incurred in a graph of order n with E edges. Moon and Moser [9] provide us with the following lower bound for the number Δ of triangles in a graph.

$$\Delta \geq \left(\frac{E}{3n} \right) \left(4E - n^2 \right).$$

It is also interesting to consider how Turán can be extended to multigraphs. A multigraph is a graph which can have more than one edge between any two vertices. Aldea, Cruz, Gaccione, Jablonski and Shelton [2], an undergraduate research group, noticed that any result in the case of multiplicity 1 can be extended to a multigraph of maximum multiplicity u simply by giving each edge multiplicity u. From Turán (1), we know that any graph for which $E \geq \lfloor n^2/4 \rfloor + 1$ must contain at least one triangle. This simply follows from inserting $r = 3$ into (1). Extending this to multigraphs, we see that any graph for which $E \geq u \lfloor n^2/4 \rfloor + 1$ must have at least one triangle. The undergraduate group also extended the result of Erdös, Györi, and Simonovits (5), If a triangle-free graph of multiplicity u is also bipartite then the following holds:

$$u \left(\left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + 1 \right) < ex(n,\Delta) \leq u \left\lfloor \frac{n^2}{4} \right\rfloor.$$

3. TRIVIAL EXTENSIONS

There are some rather trivial ways in which Turán's premises can be changed to obtain new results. For example, consider the case where the restriction placed on the graph G is the degree of each vertex.

Proposition 3.1 *Given a graph G with order n (that is, a graph with n vertices) such that every vertex has degree $d(v) \leq m \leq n-1$,*

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[4] A triangle-free graph is one with no subgraph K_3.

$$E \leq \frac{nm}{2}.$$

Proof. The result follows directly from Euler's Theorem stating $2E = \sum_{k=1}^{n} m_k$, where m is the degree of vertex k.

Definition 3.2 A graph G is *regular* if every vertex has the same degree. A graph for which the degree of every vertex is k is said to be k*-regular*.

It is well known [14] that graphs of odd order cannot be odd-regular. Euler's formula gives us $2E = kn$, clearly impossible in this case. It is also well known that otherwise, a graph on n vertices can be k-regular for $k < n$.

Another type of subgraph which might be excluded is an induced subgraph. An induced subgraph G' is defined as a subset of vertices of G for which $g_1 g_2$ is an edge in G' if and only if $g_1 g_2$ is an edge in G.

Proposition 3.3 *Given a graph G', let $ex(n,G')$ be the maximal number of edges of a graph G with order n with no induced subgraph isomorphic to G'.*

If G' is a complete graph K_r, then

$$ex(n,G') \leq \left(\frac{r-2}{r-1}\right)\frac{n^2}{2},$$

with equality when $(r-1)\,|\,n$. Otherwise,

$$ex(n,G') = \binom{n}{2}.$$

Proof. Induced complete subgraphs are in fact the same as complete subgraphs in every case. Thus, Turán's Theorem holds when $G' = K_r$. If G' is an induced subgraph on r vertices that is not complete, we can simply add in the rest of the edges between those r vertices, eliminating G'. In this way, every edge can be added to the graph and thus $ex(n,G') = \binom{n}{2}$.

4. GRID GRAPHS

Definition 4.1 A *grid graph* G is an embedded planar graph on n vertices on the infinite grid $\mathbb{Z}^2$, with horizontal or vertical edges. We further assume that every vertex has degree at least 2. Every point on the grid that an edge of the graph passes through does not have to be labeled a vertex. Of course any vertex has degree at most 4. We define $ex(n)$ to be the maximal number of edges on a grid graph with n vertices. If we do not allow square or rectangular regions, we define the maximal number of edges for a grid graph on n vertices to be $ex_S(n)$ and $ex_R(n)$, respectively. Accordingly, a maximal graph is one which contains the maximal number of edges. We define a rectangular region to be a region composed of two connected sets of parallel lines that meet at right angles. In other words, a rectangular region can be composed of more than 4 vertices.

Proposition 4.2 *For a grid graph G on n vertices, $E < 2n$.*

Proof. This result again follows directly from Euler's formula $2E = \sum_{k=1}^{n} m_k$.

Proposition 4.3 *For grid graphs arranged in a complete k x m array, so that the set of vertices $V = \{(x,y) \in \mathbb{Z}^2 : 1 \le x \le k, 1 \le y \le m\}$ with no square regions (we allow vertices of degree 1 for this proposition only),*

$$ex_S(n) = \left\lfloor \frac{1}{2}(3km - k - m - 1) \right\rfloor < \frac{3}{2}n.$$

Proof: Let G have $n = km$ vertices. Start with all $(m-1)k + (k-1)m$ edges. We need to remove at least one edge from each square. Since edges are shared by at most two squares, we know that we must remove a number of edges equal to at least half the number of squares, $(1/2)(m-1)(k-1)$. Therefore,

$$E \le (m-1)k + (k-1)m - \frac{1}{2}(m-1)(k-1)$$

$$= \frac{1}{2}(3km - k - m - 1).$$

If m or n is odd, we can reach this bound as in Figure 4.4. If m and n are even, we cannot, but can come within one edge as in Figure 4.5.

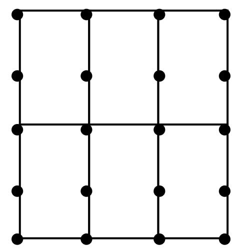

Figure 4.4 A maximal grid graph with no square regions when $k=4, m=5$. For example, substituting into the equation, we see that $.5(3\cdot4\cdot5-4-5-1)=25$. We see that thus we reach the optimal bound of edges on a graph with no squares when m is odd. The same construction applies when n is odd.

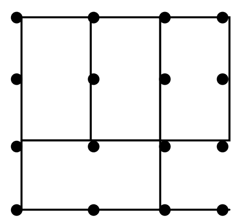

Figure 4.5 A maximal grid graph with no square regions when $k=4, m=4$. When m,k are both even, we cannot in fact reach the optimal bound.

Remark 4.6 Since forbidding rectangular regions is equivalent to forbidding C_4 s, a weaker result follows from the Bondy-Simonovits result for general graphs (3):

$$ex(n, C_4) = O\left(n^{3/2}\right).$$

The generalization of Proposition 4.3 to general grid graphs is more interesting.

Proposition 4.7 For *a grid graph G on $n \geq 6$ vertices with no rectangular regions, the maximal number of edges is at least n.*

Proof. For $n<6$, it can be easily checked that no arrangement of vertices and edges exists to allow for n edges. For $n=6$, we have n edges simply by forming a cycle, as shown in Figure 4.8. We can place one vertex on any edge to create a new edge and thereby a 7-cycle as in Figure 4.9. Similarly, we can attach any number of vertices to yield any arbitrary n edges and vertices.

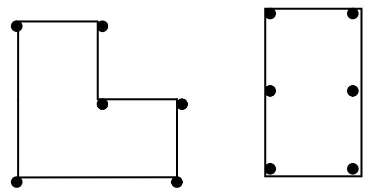

Figure 4.8 2 possible 6-cycles. Note that the one on the right is in fact a rectangular region.

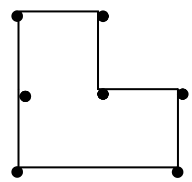

Figure 4.9 Adding one vertex and one edge to a 6-cycle to show that we can always have at least n edges.

Lemma 4.10 A maximal grid graph is connected.

Proof. Assume a grid graph G is the union of two disjoint subgraphs G_1 and G_2. We place a vertex $v_2 \in G_2$, the vertex farthest to the right on the bottom-most row occupied by G_2, directly above a vertex $v_1 \in G_1$, the vertex farthest to the left on the top-most row occupied by G_1, and connect them with an edge. The new graph G' has the same number of vertices as G but one more edge.

Proposition 4.11 *Consider a connected grid graph. Let $k_1,...,k_m$ be the number of edges of each polygonal region. Let m be the number of polygonal regions and N be the number of border edges, the edges which are bounded on one side by the plane as opposed to a polygonal region. Then the total number of edges E and vertices V satisfy*

$$E = \sum_{l=1}^{m} \frac{k_l}{2} + \frac{N}{2}$$

$$V = \sum_{l=1}^{m} \left(\frac{k_l}{2} - 1 \right) + \frac{N}{2} + 1.$$

Proof. The number of edges E follows directly from the fact that each edge occurs on two regions or one region and the exterior.

Let F be the number of faces or polygonal regions. We get the number of vertices V from Euler:

$$V - E + F = 1.$$

Substituting in for E and F, we get

$$V = \sum_{l=1}^{m} \frac{k_l}{2} + \frac{N}{2} - m + 1$$

$$= \sum_{l=1}^{m} \left(\frac{k_l}{2} - 1 \right) + \frac{N}{2} + 1.$$

We check this result for the simple example of the cross from Figure 4.16. We have four regions of 7 edges each, with 20 boundary edges. Thus, we get $E = (4 \bullet 7)/2 + 20/2 = 24$ and V = $(7/2 - 1)4 + 20/2 + 1 = 21$.

From this point forward, we replace the conjecture that there are no rectangular regions in a grid graph with the conjecture that there are no rectangles in a grid graph, i.e. no rectangular unions of regions.

Theorem 4.12 *Given n vertices on a grid, any grid graph* G *with no rectangles satisfies*

$$E < \frac{3}{2} n .$$

Proof. By Lemma 4.10, we may assume G is connected. Consider the result from Proposition 4.11. In a rectangle-free graph, every $k_l \geq 6$. When $k_l \geq 6$, we have that

$$\frac{3}{2}\left(\frac{k_l}{2} - 1 \right) \geq \frac{k_l}{2} .$$

Thus, $3V/2 > E$. This follows from examination of the formulas in Proposition 4.11.

Remark 4.13 It is interesting to see that it is impossible to have a rectangle-free grid graph where every $k_l = 6$. If we do not pack 6-sided regions, it is easy to see that regions with different numbers of vertices will necessarily arise. There are only two ways to pack 6-sided polygonal regions in the plane. If the regions are rectangular, you can pack them easily. However, we are interested in graphs with no rectangles. We consider the case when each region is shaped as in Figure 4.8. Because of the shape of the $3\pi/2$ radian angle, it and the adjacent $\pi/2$ radian angle can be shared with only one other region. Thus, we get a packing as in Figure 4.14. As this packing yields rectangles, we infer the final result $E < (3/2)n$.

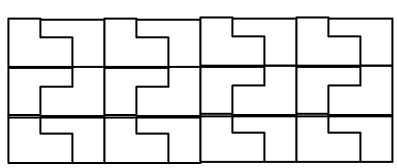

Figure 4.14. A packing of 6-sided regions with rectangles.

Having found an upper bound in Theorem 4.12, we are also interested in establishing a lower bound for the number of edges on a grid graph with n vertices and no rectangles.

Theorem 4.15: *For a grid graph G on n vertices with no rectangles, the maximum number of edges has the asymptotic lower bound*

$$ex_R(n) \geq \frac{7}{5}n + o(n).$$

Proof. We construct such a graph by packing 7-sided regions into symmetrical crosses as in Figure 4.17. These crosses are then packed in the plane, as in Figure 4.12, so that every edge is shared. From Proposition 4.11, we know that when every $k_i = 7$,

$$E = \frac{7}{2}m + \frac{N}{2}$$

$$V = \frac{5}{2}m + \frac{N}{2} + 1.$$

We see that if $N/m \to 0$, $ex_R(n) \geq (7/5)n + o(n)$. This can been seen by simply calculating $E/V = (7m/2)/(5m/2)$ when N is negligible, as demonstrated by the equations above. Further, referring to Figure 4.16, we see that each vertex with a square is shared with one other cross, and each vertex with a triangle is shared with three other crosses. Thus, counting vertices and attributing by number of crosses, we get $n = 1 + 16(1/2) + 4(1/4) = 10$. Each edge in the middle is attributable to only one cross, while all others are attributable to two crosses. Thus, we get $E = 4 + 20(1/2) = 14$. Asymptotically we see that

$$ex_R(n) \geq \left(\frac{7/2}{5/2}\right)n + o(n) = \frac{7}{5}n + o(n).$$

We show that $N/m \to 0$ as the number of crosses goes to infinity. Let c be the number of crosses as in Figure 4.17. Let C be the greatest perfect square such that $C \leq c$. Let $s = c - C$, so that $c = C + s$. Arrange the crosses in diagonal rows and columns so that each row and each column has $\sqrt{C}$ crosses, as in Figure 4.17. Place the next two crosses at the top-right and bottom-left corners of the newest row and column. The rest of the crosses fill in the new row or the new column. A single cross has 20 border edges. Adding the top-right and bottom-left crosses adds an extra border edge, but filling in the rest of the crosses across the row or column does not affect the number of border edges. In fact, the number of border edges remains unchanged no matter how many crosses are added in the newly created row and column, until the formation becomes square again (i.e. the same number of crosses compose each row and each column). Thus, we get

$$N = \begin{cases} 20\sqrt{C} + 10s \text{ when } s = 0,1,2 \\ 20\sqrt{C} + 20 \text{ otherwise} \end{cases}$$

$$m = 4c\,.$$

Thus,

$$\frac{N}{m} \le \frac{20\sqrt{C} + 20}{4c}\,.$$

Since $c \ge C$ we know that $N / m \to 0$ as $c \to \infty$.

We must further show that the bound holds for all n. Clearly, it holds for all n that can be created by some arrangement of crosses. But, we can add vertices by inserting them into an edge, simultaneously adding edges. Call these added vertices v_0. We need $v_0 / m \to 0$. But, there are 21 vertices in a cross. Thus, $v_0 \le 21$. This implies the desired result, as any constant $v_0 / m \to 0$.

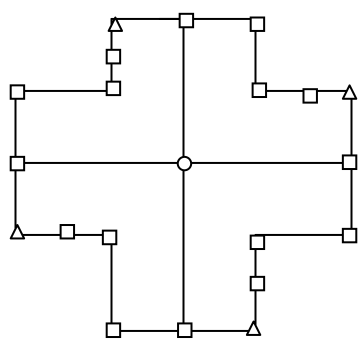

Figure 4.16 The most efficient means of arranging 7-cycles into one cross. Each vertex with a circle is shared with no other crosses.

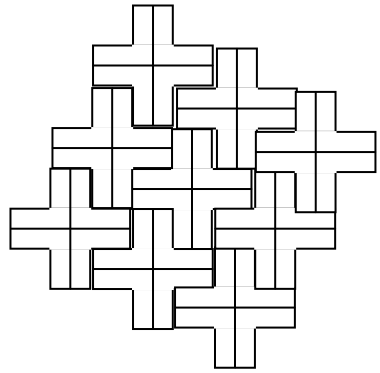

Figure 4.17 Illustrating how 3 rows and 3 columns of crosses fit together. The edge of every 7-sided region is shared with another, and no rectangular regions are created.

Remark 4.18 There are other ways of arranging a graph so that it has many edges for a given n. One reasonably successful method is to use recursive arrangements. We arrange 6-cycles such that they share a vertex with two other 6-cycles and such that the 6-cycles meet as in Figure 4.19. Then we make copies of this new arrangement to repeat the process. We are interested in the ratio of edges to vertices. Using this construction, each added 6-cycle adds 6 edges, and for every four cycles, there are four less vertices than there would be if the cycles were disconnected. We get

$$ex_R(n) \geq \lambda n + o(n),$$

where

$$\lambda = \lim_{k\to\infty} \frac{6(4)^k}{\Phi^{(k)}(6)}$$

where

$$\Phi(x) = 4x - 4$$

and equality when

$$n = \Phi^{(k)}(6), k \in N.$$

We can solve for $\Phi^{(k)}(6)$ and thus get a fractional bound using recursion techniques.

Let $x_2 = 4x_1 - 4$. Let $y_i = x_i + c$. Then $x_i = y_i - c$. Then $y_2 - c = 4(y_1 - c) - 4$. Simplifying, we get $y_2 - c = 4y_1 - 4c - 4$. Comparing back to the original equation $x_2 = 4x_1 - 4$, we see that we want $y_2 = 4y_1$, and thus $-c = -4c - 4$. This gives us $c = -4/3$. Substituting in for $y_k = 4^{k-1} y_1$, we get $y_k = 4^k (x_1 - 4/3)$. Since $x_1 = 6$, $y_k = 14/3(4)^k$ and $x_k = 14/3(4)^k + 4/3$. The constant $4/3$ becomes irrelevant as k gets big. Thus,

$$ex_R(n) \geq \frac{6}{14/3} n + o(n)$$

Simplification yields

$$ex_R(n) \geq \frac{9}{7} n + o(n) \approx 1.286\ n\,.$$

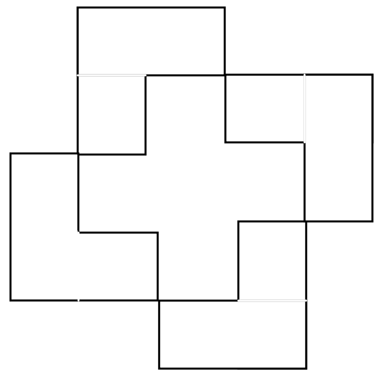

Figure 4.19 6-cycles can be arranged in such a way that rectangles are avoided.

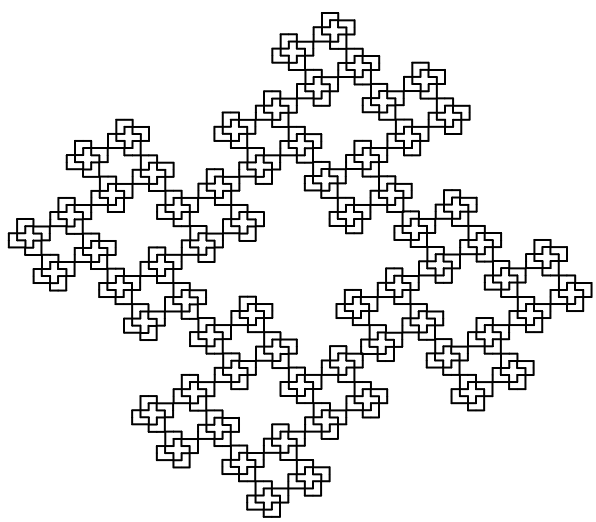

Figure 4.20 Iteration yields a grid graph with no rectangles and lots of edges.

We can further compare this bound with those found by similar recursive methods of connecting 6-cycles. If we connect 6-cycles at an edge and 2 vertices as in Figure 4.21, we cannot build arrange four 6-cycles as in the manner shown above in Figure 4.20.

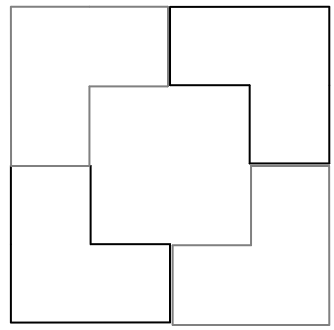

Figure 4.21 An arrangement of 6-cycles with a rectangular region.

Of course, we can connect 6-cycles without forming cycles. If we connect 6-cycles in an infinite string by an edge and two vertices, we find that each added 6-cycle adds 5 edges and only 4 vertices. Thus, we reach the lower bound

$$ex_R(n) \geq \lim_{k\to\infty}\left(\frac{6+5k}{6+4k}\right)n+o(n)=\frac{5}{4}n+o(n)=1.25n+o(n),$$

when

$$n=\frac{6+5k}{6+4k},\ k\in N,$$

a weaker bound. However, what if we arrange 6-cycles as in Figure 4.22, and iterate as before? Visually, we can see that this new shape actually resembles a larger 6-cycle itself. However, this larger figure an be arranged so that every group of cycles attaches to another by an edge and two vertices. So, we get a similar recursion, except that for every four added larger figures we drop 4 edges and 8 vertices. Note that each group of four larger figures is just four copies of the previous graph. Thus, the analysis is exactly the same, except that the initial figure has 30 edges and 24 vertices, and the recursion formulas are slightly different. We see that in fact

$$ex_R(n) \geq \lambda n+o(n),$$

where

$$\lambda=\lim_{k\to\infty}\frac{\mathrm{T}^{(k)}(30)}{\mathrm{M}^{(k)}(24)},$$

where

$$\mathrm{T}(x)=4(x-1) \text{ and } \mathrm{M}(x)=4(x-2),$$

with equality when

$$n = \frac{\mathrm{T}^{(k)}(30)}{\mathrm{M}^{(k)}(24)}, k \in N.$$

Again, we can find the fractional bound using recursion techniques.

For $\mathrm{T}(x)$, we know from above that when k gets big, $x_k = 4^k(x_1 - 4/3)$. Since $x_1 = 30$, we get $x_k = 86/3(4)^k$.

For $\mathrm{M}(x)$, we use the same technique to see that $x_k = 4^k(x - 8/3)$. Since $x_1 = 24$, we get $x_k = 64/3(4)^k$.

We again plug back into the limit equation.

$$ex_R(n) \geq \lim_{k\to\infty} \frac{86/3(4)^k}{64/3(4)^k} n + o(n)$$

$$= \frac{86/3}{64/3} n + o(n) = \frac{43}{32} n + o(n) \approx 1.344 + o(n).$$

This is a better bound.

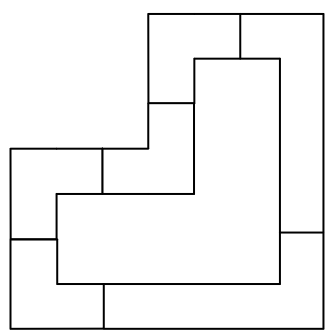

Figure 4.22 Another arrangement of 6-cycles without a rectangle.

There is one, final, natural way of piecing together 6-cycles recursively. We can arrange them as in Figure 4.23, so that as many edges as possible are shared without creating a rectangle, and again apply the same iterative method. However, this process starts with 21 edges and 17 vertices, and we lose 3 edges and 7 vertices with every newly introduced 4-cycle of figures. This process yields the bound

$$ex_R(n) \geq \lambda n + o(n),$$

where

$$\lambda = \lim_{k \to \infty} \frac{\mathrm{T}^{(k)}(21)}{\mathrm{M}^{(k)}(17)}$$

where

$$\mathrm{T}(x) = 4x - 3 \text{ and } \mathrm{M}(x) = 4x - 7,$$

with equality when

$$n = \frac{\mathrm{T}^{(k)}(21)}{\mathrm{M}^{(k)}(17)}, \text{ for } k \in N.$$

Applying recursion again, for $T(x)$, we see that for k big, $x_k = 4^k(x_1 - 1)$. Since $x_1 = 21$, we get $x_k = 20(4)^k$.

For $\mathrm{M}(x)$, we see that when k gets big, $x_k = 4^k(x_1 - 7/3)$. Since $x_1 = 17$, we see that $x_k = 44/3(4)^k$.

Plugging back into the limit equation,

$$ex_R(n) \geq \lim_{k \to \infty} \frac{20(4)^k}{44/3(4)^k} n + o(n)$$

$$= \frac{20}{44/3} n + o(n) = \frac{15}{11} n + o(n) \approx 1.367n + o(n).$$

This is the best bound that we have found recursively, but of course it is not as good as the packing of 7-cycles.

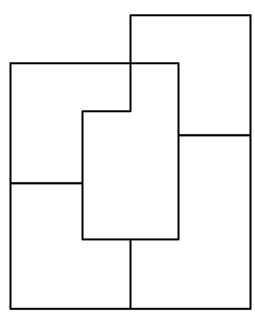

Figure 4.23 The most efficient recursive arrangement of 6-cycles found.

Putting Theorem 4.12 and Theorem 4.15 together, we see that we have reached the conclusion that for n vertices on a grid with no rectangular regions, the maximal number $ex_R(n)$ of edges satisfies

$$1.4n + o(n) \le ex_R(n) < 1.5n.$$

Conjecture 4.24 For *a grid graph on* n *vertices with no rectangular regions the maximal number of edges* $ex_R(n)$ *is asymptotic to* $1.4n$*,*

$$ex_R(n) = \frac{7}{5}n + o(n).$$

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