A BECKMAN-QUARLES TYPE THEOREM FOR LAGUERRE TRANSFORMATIONS IN THE DUAL PLANE

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Abstract. We prove that any bijective transformation on the space of parabolas in the dual plane which preserves distance 1 between the parabolas must necessarily be a Laguerre transformation.

1. Introduction

Rigid motions are known to be isometries, that is, they preserve distances between points. The converse of this statement is also known to be true. In 1953, Beckman and Quarles published a theorem proving a stronger version of the converse. In particular, any transformation of Euclidean space with dimension greater than one which preserves a distance ρ between points must necessarily be a rigid motion [1]. Since then, many variations of this theorem exist in many contexts by various authors. For examples, see [2, 3, 5, 7].

One such example is Lester's characterization of Möbius transformations in the complex plane. In complex analysis, Möbius transformations are the fractional linear transformations, and they are known to preserve the space of circles and lines. See Figure 1.

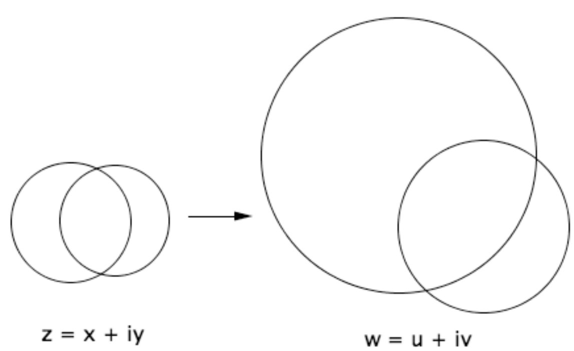

Figure 1. Möbius transformations map circles and lines to other circles and lines. They also preserve the angle of intersection.

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In addition, the Coxeter distance defined as the angle of intersection between two intersecting circles (or lines) is preserved. Using the Coxeter distance δ and the space of circles and lines $\mathscr{C}$, Lester proved the following result:

Theorem 1 (Lester [6]). *For a fixed real $\rho > 0$, let $X \to \bar{X}$ be a bijective mapping from $\mathscr{C}$ to itself such that, for all A, B in $\mathscr{C}$,*

$$\delta_{AB} = \rho \text{ if and only if } \delta_{\bar{A}\bar{B}} = \rho.$$

Then the mapping is induced on $\mathscr{C}$ by a Möbius transformation of $\mathbb{C}$.

As in complex analysis, we found that in the dual plane, the fractional linear transformations preserve the space that includes parabolas with vertical axes of symmetry and non-vertical lines. The dual numbers are given by $\mathbb{D} := \{z = x + yj : x, y \in \mathbb{R}, j^2 = 0\}$, and the fractional linear transformations are called Laguerre transformations. The space of parabolas, denoted by $\mathscr{P}$, contains all objects of the form $y = ax^2 + bx + c$ for $a, b, c \in \mathbb{R}$.

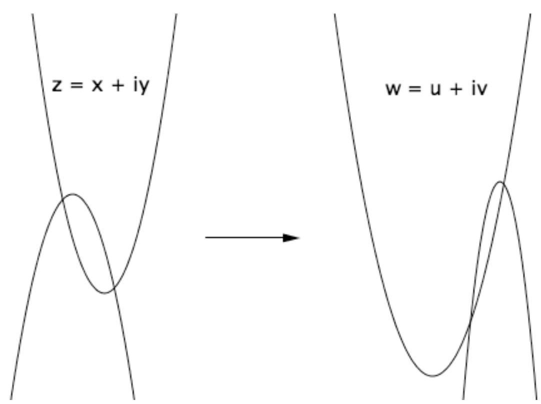

FIGURE 2. Laguerre transformations map parabolas and lines to other parabolas and lines. They also preserve the difference of slope at the intersection point(s).

Then, we found that the distance δ between two parabolas, here defined to be the difference of slope at the intersection point(s), is also preserved by the Laguerre transformations. So, modeling Lester's result, we give the following Beckman-Quarles type result:

Theorem 2. *Suppose* T *is a bijective mapping from the space of parabolas $\mathscr{P}$ to itself so that, for every $A, B \in \mathscr{P}$,*

$$\delta(A, B) = 1 \text{ if and only if } \delta(T(A), T(B)) = 1.$$

Then T *induces a Laguerre transformation of the dual plane $\mathbb{D}$.*

Of particular interest is the final section of the proof where we demonstrate that a map of the plane which preserves parabolas and distance one must be a Laguerre transformation. The corresponding fact in the Möbius case is classical and was therefore omitted from Lester's proof. This step can be viewed as a special case of the following more general result:

Theorem 3 (Bolt, Ferdinands, and Kavlie [4])**.** *Every injective map from a region bounded by a vertical parabola that maps vertical parabolas and nonvertical lines to other vertical parabolas and nonvertical lines must be the composition of a non-isotropic dilation* $d_\lambda : (x, y) \to (\lambda x, \lambda^2 y), 0 \neq \lambda \in \mathbb{R}$ *with a direct or indirect Laguerre transformation.*

2. Laguerre Geometry in the Dual Plane

2.1. The Dual Plane $\mathbb{D}$.

Definition 1. *A dual number is a number of the form* $x + yj$ *where* $x, y \in \mathbb{R}$ *and* $j^2 = 0$ *(*j *is known as a nilpotent). The dual plane* $\mathbb{D}$ *is the set of ordered pairs* (x, y) *taken from the dual number* $x + yj$*.*

As in the complex numbers, any dual number $z = x + yj$ has real part x, dual part y, and conjugate $\bar{z} = x - yj$.

Arithmetic in $\mathbb{D}$ is accomplished similarly to arithmetic in the complex plane $\mathbb{C}$. Geometrically, addition in $\mathbb{D}$ is vector addition as shown in Figure 3.

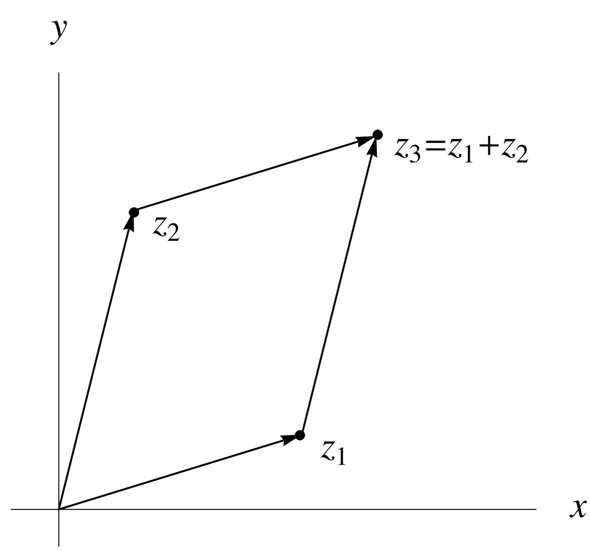

FIGURE 3. Addition of dual numbers is analogous to vector addtion.

Algebraically,

$$(2 + 4j) + (3 - 6j) = (2 + 3) + (4 - 6)j = 5 - 2j.$$

Multiplication in $\mathbb{C}$ can be explained using polar coordinates. This is done by multiplying their magnitudes and adding the angles. On the other hand, in $\mathbb{D}$, multiplication is done by multiplying the real parts and adding their corresponding slopes as shown in Figure 4.

Algebraically,

$$(2 + 4j) \cdot (3 - 6j) = 2 \cdot 3 + 4 \cdot 3j + 2 \cdot (-6j) + 4 \cdot (-6)j^2 = 6.$$

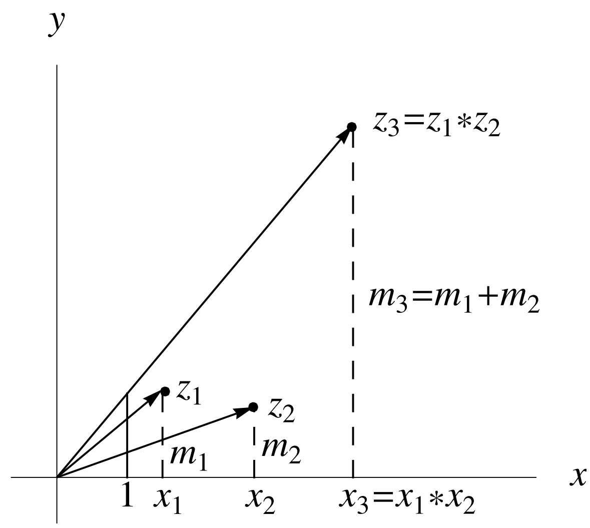

FIGURE 4. Multiplication of dual numbers is accomplished by multiplying the real parts and adding their corresponding slopes.

Further investigating $\mathbb{D}$, we find that $\mathbb{D}$ is a commutative ring with unity. However, it is not a field. It has a one parameter family of zero divisors, namely kj for $k \in \mathbb{R}$. (Notice, also, that division by kj is not defined for any value of k.)

The extended complex plane $\hat{\mathbb{C}}$ is defined as $\mathbb{C} \cup \{\infty\}$. Similarly, the extended dual plane $\hat{\mathbb{D}}$ is defined by $\mathbb{D} \cup \{\frac{1}{aj} : a \in \mathbb{R}\}$. In other words, this is the union of $\mathbb{D}$ with a line at infinity.

2.2. **Laguerre Transformations in $\mathbb{D}$.**

Definition 2. *A Laguerre transformation in the dual plane is a function $\mu\colon \hat{\mathbb{D}} \to \hat{\mathbb{D}}$ given by:*

$$\mu(z) = \tfrac{\alpha z+\beta}{\gamma z+\Delta} \quad \text{or} \quad \mu(z) = \tfrac{\alpha \bar{z}+\beta}{\gamma \bar{z}+\Delta}.$$

for $\alpha, \beta, \gamma, \delta \in \mathbb{D}$ with $\text{real}(\alpha\delta - \beta\gamma) \neq 0$. *These are called direct and indirect Laguerre transformations, respectively.*

The restriction $\text{real}(\alpha\Delta - \beta\gamma) \neq 0$ avoids the transformations which take the entire extended dual plane to a single point or vertical line. We may further restrict $\text{real}(\alpha\Delta - \beta\gamma) = 1$. These particular Laguerre transformations form a group under function composition that is isomorphic to the group $SL_2(\mathbb{D})$.

The space of parabolas is preserved by a Laguerre transformation (the word preserved refers to properties that remain constant under a transformation T and T^{-1}). This is verified by using the transformations $\mu(z) = \alpha z, z + \beta, \frac{1}{z}$ from which any Laguerre transformation is derived. $a = 0$ is acceptable which implies that these parabolas include all non-vertical lines.

When using the transformation $\mu(z) = \frac{1}{z}$, we can see that the parabola $y = ax^2 + bx + c$ must go through the point $-\frac{1}{aj}$. Hence, the only parabolas which go through the point $\frac{1}{0j}$ are lines.

When characterizing the relationship between two parabolas, we refer to those that intersect 0, 1, or 2 times as non-intersecting, tangent, or intersecting, respectively. This number of intersection points is preserved under a Laguerre transformation.

Another way of representing the parabola $y = ax^2 + bx + c$ is by the ordered triplet (a, b, c). We then denote the space of all such parabolas by $\mathscr{P}$. In this way, the space $\mathscr{P}$ can be identified with $\mathbb{R}^3$.

2.3. **An Invariant Distance in $\mathbb{D}$.**

Definition 3. *The distance between two parabolas $A := (a, b, c)$, and $B := (d, e, f)$ is the difference of their slopes at their intersection points. Algebraically, $\delta(A, B) = \sqrt{(b-e)^2 - 4(a-d)(c-f)}$.*

This has many "distance-like" properties. For example:

(1) If A and B are intersecting, $\delta(A, B) > 0$.
(2) If $A = B$, then $\delta(A, B) = 0$.
(3) $\delta(A, B) = \delta(B, A)$.
(4) Laguerre transformations preserve $\delta(A, B)$.

However, this distance is a not a distance in the usual sense:

(1) If A and B are tangent but $A \neq B$, it is still the case that $\delta(A, B) = 0$.
(2) There is no triangle inequality.
(3) If A and B are non-intersecting, the distance is undefined. (We may think of this as an imaginary distance.)

For computational reasons, we find it best to parametrize (a, b, c) as $t + (at^2 + bt + c)j$ when taking the Laguerre transformation of a parabola.

2.4. **Canonical Forms.** For simplicity, we use the canonical forms of two parabolas (reaching them by Laguerre transformations).

- Non-intersecting: $y = 0$ and $y = kx^2 + 1$ for $k > 0$.
- Tangent: $y = 1$ and $y = -1$. Equivalently, $y = 0$ andt $y = k$ for $k \neq 0$ (They intersect only at $\frac{1}{0j}$).
- Intersecting: $y = kx$ and either $y = -kx$ or $y = 0$, for $k \neq 0$.

3. Proof of Theorem 2

3.1. **T Preserves Tangent Parabolas.** We now show that T preserves tangency.

Proposition 1. *Two parabolas are tangent if and only if their images under* T *are tangent.*

Proposition 1 follows immediately from the following lemmas. It is important to notice that since T is a bijection and preserves distance 1, T and T^{-1} can neither create nor destroy parabolas a distance 1 apart from each other.

Lemma 1. T *(and* T^{-1}*) cannot take tangent parabolas to intersecting parabolas.*

Proof. Suppose A, B are tangent parabolas. In canonical form, $A := (0, 0, 1)$ and $B := (0, 0, -1)$. Now take an arbitrary $C := (a, b, c) \in \mathscr{P}$. We characterize all $D := (d, e, f) \in \mathscr{P}$ a distance 1 away from A, B, and C. This requires:

$$1 = e^2 - 4d(f - 1)$$

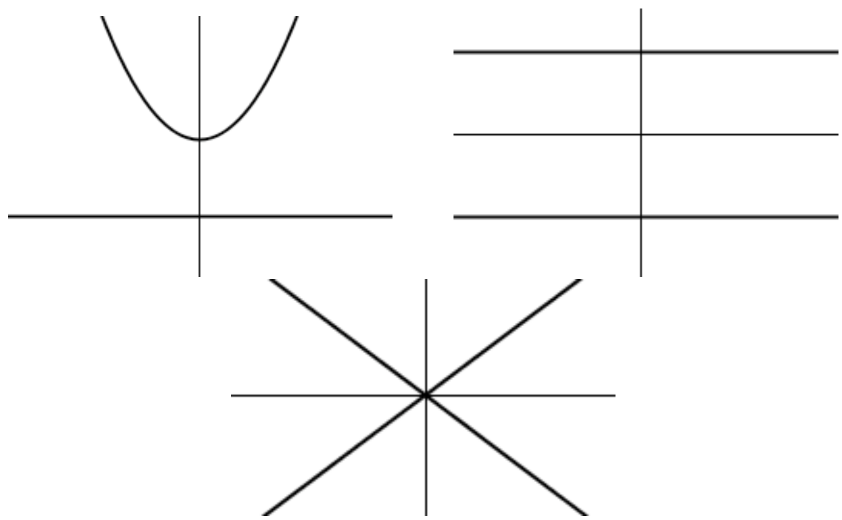

Figure 5. Canonical forms for (a)non-intersecting, (b)tangent, and (c)intersecting parabolas.

$$1 = e^2 - 4d(f+1)$$
$$1 = (e-b)^2 - 4(d-a)(f-c)$$

Solving these equations, for d, e, f we find:

(1) If $a = 0$, there is either a 1-parameter family $D := (0, \pm 1, f)$ for arbitrary f, or no D's since the equations put a restriction on b ($e = 1$ implies $b = 2, 0$ and $e = -1$ implies $b = 0, -2$).
(2) If $a \neq 0$, there are exactly 2 D's. Namely, $D := (0, \pm 1, c + \frac{-b^2 \pm 2b}{4a})$ where the $\pm$'s match.

In contrast, we now show that if A, B are intersecting, there exists a C so that there is a unique solution for D.

In canonical form, we take $A := (0, k, 0)$ and $B := (0, -k, 0)$ for $k \neq 0$. Let $C := (1, 0, 0)$ and $D := (d, e, f)$ be arbitrary. As before, this requires:

$$1 = (e-k)^2 - 4df$$
$$1 = (e+k)^2 - 4df$$
$$1 = e^2 - 4(d-1)f$$

Solving for d, e, f, we observe that D must be the unique parabola $(1 - \frac{1}{k^2}, 0, \frac{k^2}{4})$. Since parabolas distance 1 away can neither be created nor destroyed, T cannot take tangent parabolas to intersecting parabolas. The same argument can be applied to T^{-1}. □

Lemma 2. T *(and* T^{-1}*) cannot take tangent parabolas to non-intersecting parabolas.*

Proof. Suppose A, B are tangent parabolas. In canonical form, $A := (0, 0, k), k \neq 0$, and $B := (0, 0, 0)$. We characterize all $C := (a, b, c) \in \mathscr{P}$ a distance 1 away from A and B. This requires:

$$1 = b^2 - 4a(c-k)$$
$$1 = b^2 - 4ac$$

Solving for C, we get that $C := (0, \pm 1, c)$ for arbitrary c. Notice that all such C's are a distance 0 or 2 away from each other.

Suppose A, B are non-intersecting parabolas. In canonical form, $A := (0,0,0)$ and $B := (k, 0, 1)$ for $k > 0$. We once again let $C := (a, b, c)$ be arbitrary. Similarly as before we find:

$$1 = b^2 - 4(a-k)(c-1)$$
$$1 = b^2 - 4ac$$

Solving for a, b, c we get $C := (k - kc, \pm\sqrt{-4kc^2 + 4kc + 1}, c)$ for arbitrary c. We see that the range of appropriate c's must satisfy the inequality:

$$-4kc^2 + 4kc + 1 \geq 0.$$

We find that the acceptable c's are on the interval $\left[\frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{1}{k}}, \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{1}{k}}\right]$. Now, we show that there exists C_1, C_2 such that $\delta(C_1, C_2) = 1$.

When $C_1 := (\frac{k}{2}, \sqrt{k+1}, \frac{1}{2})$ (the midpoint of the interval) and $C_2 := (k - k\gamma, \sqrt{4k\gamma - 4k\gamma^2 + 1}, \gamma)$ for each k, we must show that such a γ exists. We get the following equation for the distance from C_1 to C_2:

$$1 = \left(\sqrt{4k\gamma - 4k\gamma^2 + 1} - \sqrt{k+1}\right)^2 - 4\left(k - k\gamma - \frac{k}{2}\right)\left(\gamma - \frac{1}{2}\right)$$

Solving for γ, we get $\gamma = \frac{1}{2} \pm \frac{\sqrt{3k + 7k^2 + 4k^3}}{4k + 4k^2}$. Since $\frac{1}{2}\sqrt{1 + \frac{1}{k}} \geq \frac{\sqrt{3k+7k^2+4k^3}}{4k+4k^2} \geq 0$ for $k > 0$, γ is in our acceptable interval and must exist.

Since parabolas a distance 1 away from a given parabola can be neither created nor destroyed, T cannot take tangent parabolas to non-intersecting parabolas. The same argument can be applied to T^{-1}. □

3.2. **T induces a Transformation of $\mathbb{D}$.** Here, we extend our work from the previous section to show that T induces a transformation of the dual plane.

Remark. *Any three mutually tangent parabolas meet at a single point.*

This can be seen by taking A, B to be tangent parabolas in canonical form. The only possibility for a third parabola tangent to both A and B is another horizontal line.

Definition 4. *Given a point p and a real number m, define $\mathscr{T}_{p,m}$ as the set of all parabolas with slope m at p.*

In particular, if $p = x_0 + y_0 j$, then $\mathscr{T}_{p,m}$ includes parabolas that have the form $y = a(x - x_0)^2 + m(x - x_0) + y_0$ for $a \in \mathbb{R}$. Any two parabolas of this form intersect only once (and are therefore tangent) at p. If $p = \frac{1}{aj}$, we denote by $\mathscr{T}_{p,m}$ the family of parabolas that have the form $y = -ax^2 + mx + c$ for $c \in \mathbb{R}$. Any two parabolas of this form again intersect only once (and are therefore tangent) at p.

Now, we show that T preserves these families of parabolas.

Lemma 3. *Given $p \in \hat{\mathbb{D}}$ and $m \in \mathbb{R}$, there exists $p' \in \hat{\mathbb{D}}$ and $m' \in \mathbb{R}$ so that* $\mathrm{T}(\mathscr{T}_{p,m}) = \mathscr{T}_{p',m'}$.

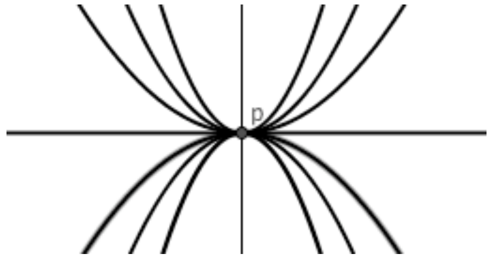

FIGURE 6. $\mathscr{T}_{p,0}$ with $p := (0,0)$.

Proof. Take $p \in \hat{\mathbb{D}}$ and $m \in \mathbb{R}$ and let $P, Q \in \mathscr{T}_{p,m}$. Since P and Q are tangent, Proposition 1 tells us that $\mathrm{T}(P)$ and $\mathrm{T}(Q)$ are tangent. Let $p' \in \hat{\mathbb{D}}$ be their point of tangency, and let m' be their slope at p'.

To show $\mathrm{T}(\mathscr{T}_{p,m}) \subset \mathscr{T}_{p',m'}$, take $A \in \mathscr{T}_{p,m}$ with $A \neq P, Q$. A is tangent to both P and Q. $\mathrm{T}(A)$ is tangent to $\mathrm{T}(P)$ and $\mathrm{T}(Q)$ by Proposition 1. By the remark above about mutually tangent parabolas, $\mathrm{T}(A), \mathrm{T}(P)$, and $\mathrm{T}(Q)$ are mutually tangent at the same point, p'. Thus $\mathrm{T}(A) \in \mathscr{T}_{p',m'}$.

To show $\mathrm{T}(\mathscr{T}_{p,m}) \supset \mathscr{T}_{p',m'}$, take a parabola $A' \in \mathscr{T}_{p',m'}$. since T is surjective, there is an $A \in \mathscr{P}$ so that $\mathrm{T}(A) = A'$. As before, by the above remark and Proposition 1, $A \in \mathscr{T}_{p,m}$. □

We now introduce an induced map $\hat{\mathrm{T}}\colon \hat{\mathbb{D}} \to \hat{\mathbb{D}}$. Let $p \in \hat{\mathbb{D}}$ and let $m \in \mathbb{R}$. By Lemma 3, there exist $p' \in \hat{\mathbb{D}}$ and $m' \in \mathbb{R}$ so that $\mathrm{T}(\mathscr{T}_{p,m}) = \mathscr{T}_{p',m'}$. Set $\hat{\mathrm{T}}(p) := p'$. Next we show that this mapping is well-defined; that is, it does not depend on the choice of m as in Figure 7. The following lemma is necessary for that result.

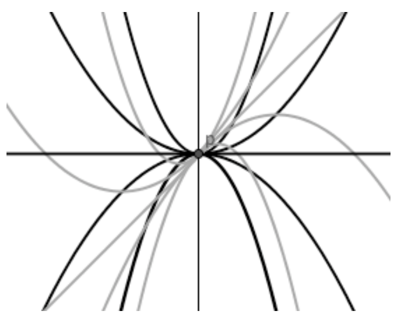

FIGURE 7. The intersecting objects $\mathscr{T}_{p,0}$ and $\mathscr{T}_{p,1}$ for $p := (0,0)$.

Lemma 4. *Suppose $A, B \in \mathscr{P}$ so that A, B are intersecting and p is a point on A, but p is not on B. Then, there exists a unique $C \in \mathscr{P}$ so that C is tangent to A at p, and C is tangent to B.*

Proof. Take $A, B \in \mathscr{P}$ to be intersecting in canonical form. So, $A := (0,0,0)$ and $B := (0,k,0)$ for $k > 0$.

First, we characterize all parabolas $C := (a,b,c)$ where C is tangent to both A and B. This requires:

$$0 = b^2 - 4ac$$
$$0 = (b-k)^2 - 4ac$$

Solving for a, b, and c, we get a one parameter family for C: $(\frac{k^2}{16\lambda}, \frac{k}{2}, \lambda)$. Now, we pick a point $(d, 0)$ on A ($d \neq 0$) and assume that C passes through $(d, 0)$. We then get the unique parabola, $C = (\frac{k}{-4d}, \frac{k}{2}, -\frac{kd}{4})$. □

Proposition 2. $\hat{\mathrm{T}}$ *is well-defined.*

Proof. Take two families of parabolas at a point p, namely $\mathscr{T}_{p,m_1}, \mathscr{T}_{p,m_2}$ so that $m_1 \neq m_2$. Now, let $T(\mathscr{T}_{p,m_1}) = \mathscr{T}_{p'_1,m'_1}$ and $T(\mathscr{T}_{p,m_2}) = \mathscr{T}_{p'_2,m'_2}$. Suppose $p'_1 \neq p'_2$.

Notice that each parabola in $\mathscr{T}_{p,m_1}$ must necessarily intersect each parabola from $\mathscr{T}_{p,m_2}$ twice. There exists $P \in \mathscr{T}_{p'_1,m'_1}$ so that $p'_2 \notin P$. Then, there is an $Q \in \mathscr{T}_{p'_2,m'_2}$ that intersects P twice (Q exists since the closure of $\mathscr{T}_{p'_2,m'_2}$ is all of $\hat{\mathbb{D}}$). Then, by Lemma 4, there is a $C \in \mathscr{P}$ so that C is tangent to Q at p'_2 and tangent to P. In particular, $C \in \mathscr{T}_{p'_2,m'_2}$ is tangent to $P \in \mathscr{T}_{p'_1,m'_1}$. This contradicts Proposition 1. □

3.3. **The Induced Mapping $\hat{\mathrm{T}}$.** We've based our definition of $\hat{\mathrm{T}}$ on T. However, in the next lemma, we show that T is also determined by $\hat{\mathrm{T}}$.

Lemma 5. *If $P \in \mathscr{P}$, then $\mathrm{T}(P) = \{\hat{\mathrm{T}}(p) : p \in P\}$*

Proof. Take $p \in P$ and let m be the slope of P at p. Construct $\mathscr{T}_{p,m}$. Notice that $P \in \mathscr{T}_{p,m}$. Lemma 3 gives us $p' \in \hat{\mathbb{D}}, m' \in \mathbb{R}$ such that $\mathrm{T}(\mathscr{T}_{p,m}) = \mathscr{T}_{p',m'}$. Since $P \in \mathscr{T}_{p,m}$ and $T(\mathscr{T}_{p,m}) = \mathscr{T}_{p',m'}$, $\mathrm{T}(P) \in \mathscr{T}_{p',m'}$. That is, $\mathrm{T}(P) \in \mathscr{T}_{\hat{\mathrm{T}}(p),m}$. Therefore, $\hat{\mathrm{T}}(p) \in \mathrm{T}(P)$ and $\mathrm{T}(P) \supset \{\hat{\mathrm{T}}(p) : p \in P\}$.

Now take $p' \in \mathrm{T}(P)$. Let m' be the slope of $\mathrm{T}(P)$ at p'. Construct $\mathscr{T}_{p',m'}$. Notice that $\mathrm{T}(P) \in \mathscr{T}_{p',m'}$. Now, from Lemma 3 for T^{-1}, there exist $p \in \hat{\mathbb{D}}$ and $m \in \mathbb{R}$ so that $T(\mathscr{T}_{p,m}) = \mathscr{T}_{p',m'}$. By the definition of $\hat{\mathrm{T}}$, $\hat{\mathrm{T}}(p) = p'$. Moreover, since T is bijective, $p \in \mathscr{T}_{p',m'}$. Therefore, $p \in P$ and $\mathrm{T}(P) \subset \{\hat{\mathrm{T}}(p) : p \in P\}$. □

Hence $\hat{\mathrm{T}}$ maps parabolas to parabolas; it maps tangent parabolas to tangent parabolas; and it preserves distance 1 between parabolas.

Moreover, the second part of the proof of Lemma 5 guarantees that $\hat{\mathrm{T}}$: $\hat{\mathbb{D}} \to \hat{\mathbb{D}}$ is surjective since any $p' \in \hat{\mathbb{D}}$ belongs to $T(P)$ for some P. Surjectivity allows us to conclude that intersection points cannot be created by $\hat{\mathrm{T}}$. To show that intersection points cannot be destroyed by $\hat{\mathrm{T}}$, we must show that $\hat{\mathrm{T}}$ is injective. From this, it follows that the number of intersection points between parabolas is preserved.

Lemma 6. $\hat{\mathrm{T}}$ *is injective.*

Proof. Take $x, y \in \hat{\mathbb{D}}$ to be distinct. First, let x and y lie on distinct vertical lines. Let A, B, C be distinct parabolas so that $x, y \in A$; A is tangent to B at x and C is tangent to A at y, B and C are non-intersecting as in Figure 8. If $\hat{\mathrm{T}}(x) = \hat{\mathrm{T}}(y)$, then $\hat{\mathrm{T}}(B)$ and $\hat{\mathrm{T}}(C)$ must be tangent at $\hat{\mathrm{T}}(x) = \hat{\mathrm{T}}(y)$. This is impossible by Proposition 1 and Lemma 5.

Now, let x and y lie on the same vertical line. Once again, take A and B to be tangent parabolas so that $x \in A$ and $y \in B$ but neither x nor y is the point of tangency. Let z be the point of tangency. We know that $\hat{\mathrm{T}}(A)$ and $\hat{\mathrm{T}}(B)$ must

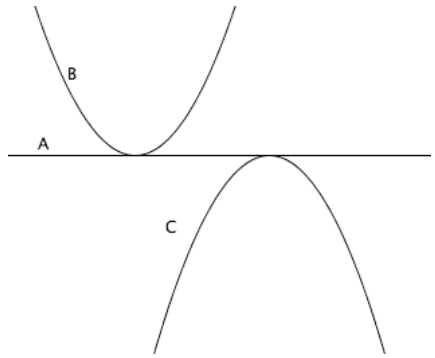

FIGURE 8. A tangent to B at x and A tangent to C at y.

be tangent at $\hat{\mathrm{T}}(z)$ by Proposition 1 and Lemma 5. Now, if $\hat{\mathrm{T}}(x) = \hat{\mathrm{T}}(y)$, then $\hat{\mathrm{T}}(x) = \hat{\mathrm{T}}(y) = \hat{\mathrm{T}}(z)$ is the point of tangency of $\hat{\mathrm{T}}(A)$ and $\hat{\mathrm{T}}(B)$. However, this is impossible by the first part of this lemma. □

3.4. **$\hat{\mathrm{T}}$ is Laguerre: a Grid Argument.** We now use a Laguerre transformation to normalize $\hat{\mathrm{T}}$. In particular, if $\hat{\mathrm{T}}(0) = w_1, \hat{\mathrm{T}}(1) = w_2, \hat{\mathrm{T}}(\frac{1}{0j}) = w_3$, we define

$$\mu(w) = \tfrac{w-w_1}{w-w_3} \cdot \tfrac{w_2-w_3}{w_2-w_1}.$$

It then follows that the composition $\mu\circ\hat{\mathrm{T}}$ sets $(\mu\circ\hat{\mathrm{T}})(0) = 0$, $(\mu\circ\hat{\mathrm{T}})(1) = 1$, and $(\mu\circ\hat{\mathrm{T}})(\frac{1}{0j}) = \frac{1}{0j}$. We will prove that, for $z \in \mathbb{D}$, $(\mu\circ\hat{\mathrm{T}})(z) = z$ or $(\mu\circ\hat{\mathrm{T}})(z) = \bar{z}$. Due to the group structure of the Laguerre transformations, it will then follow that $\hat{\mathrm{T}}$ is either a direct or indirect Laguerre transformation, respectively.

Since the points $0, 1, \frac{1}{0j}$ uniquely determine the parabola, $y = 0$, we know that $\mu\circ\hat{\mathrm{T}}$ preserves this line. Recall that a parabola is a line if and only if it goes through the point $\frac{1}{0j}$. Knowing this, we can conclude that our new transformation $\mu\circ\hat{\mathrm{T}}$ preserves lines. Since any line is either intersecting to the preserved line $y = 0$ or a "tangent" horizontal line (and tangency is preserved), we can conclude that horizontal lines are preserved.

In later arguments, it is important to note that Laguerre transformations preserve vertical lines. From the next lemma, we can also conclude that our induced transformation $\hat{\mathrm{T}}$ also preserves vertical lines.

Lemma 7. *Under $\hat{\mathrm{T}}$, vertical lines must go to vertical lines.*

Proof. Let A be a vertical line. Now, suppose $\hat{\mathrm{T}}(A)$ is not a vertical line. Then, there exists $x', y' \in \hat{\mathrm{T}}(A)$ so that x' and y' do not have the same real part. We may then draw a parabola $P' \in \mathscr{P}$ through x' and y'. Since T is surrjective, there exists $P \in \mathscr{P}$ so that $\mathrm{T}(P) = P'$. Notice that x', y' come from some $x, y \in A$. However, no parabola in $\mathscr{P}$ can have two distinct points which lie on the same vertical line. This is a contradiction. □

Lemma 8. *Suppose $(\mu\circ\hat{\mathrm{T}})(a + bj) = a + bj$, then $\mu\circ\hat{\mathrm{T}}$ preserves the vertical and horizontal lines $x = a$ and $y = b$.*

Proof. This follows immediately from the fact that that $\mu \circ \hat{\mathrm{T}}$ preserves both horizontal and vertical lines. □

Consider the line $y = x$. Since $\mu \circ \hat{\mathrm{T}}$ preserves $y = 0$ and the distance between $y = 0$ and $y = x$, we get that $\mu \circ \hat{\mathrm{T}}$ must take $y = x$ to either $y = x$, or $y = -x$. Assume $\mu \circ \hat{\mathrm{T}}$ takes $y = x$ to $y = x$. We will proceed to show $\hat{\mathrm{T}}$ is a direct Laguerre transformation. (In the case where $\mu \circ \hat{\mathrm{T}}$ takes $y = x$ to $y = -x$, we first reflect across $y = 0$. This corresponds with $\hat{\mathrm{T}}$ being an indirect Laguerre transformation.)

We make repeated use of the following rather obvious fact.

Lemma 9. *If L, M are lines so that $(\mu \circ \hat{\mathrm{T}})(L) = L$ and $(\mu \circ \hat{\mathrm{T}})(M) = M$, then $\mu \circ \hat{\mathrm{T}}$ preserves their intersection point(s) as well.*

Note that the case where L or M are vertical lines is not excluded in Lemma 9. From Lemmas 8 and 9 we see that the lines $x = 0$, $x = 1$, and $y = 1$ are preserved, and hence so are the points $1 + j$ and j.

Since the points j, 1, and $\frac{1}{0j}$ uniquely determine the line $y = -x + 1$, then $\mu \circ \hat{\mathrm{T}}$ preserves this line. We now have that $\mu \circ \hat{\mathrm{T}}$ preserves the "grid" as shown in Figure 9.

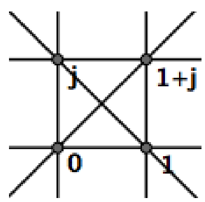

FIGURE 9. Lines and points that $\mu \circ \hat{\mathrm{T}}$ preserves.

The line $y = x - 1$ is distance 1 away from $y = 0$, and contains 1. Since $y = x - 1$ and $y = -x + 1$ are the only two such lines, $y = x - 1$ must go to one of these two. Since $\mu \circ \hat{\mathrm{T}}$ is surjective for parabolas, and $\mu \circ \hat{\mathrm{T}}$ preserves $y = -x + 1$, $y = x - 1$ must also be preserved. Using lemmas 8 and 9 we see that the points $-j$ and $2 + j$ and the lines $y = -1$ and $x = 2$ must be preserved. Applying lemma 9 once again tells us the points $1 - j$, 2, and $2 - j$ are preserved. Continuing in this manner (as in Figure 10), we find that for all $a, b \in \mathbb{Z}$ the point $a + bj$ is preserved.

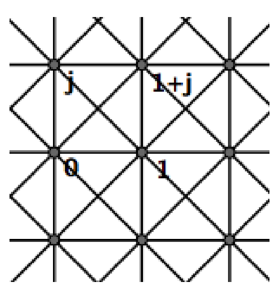

FIGURE 10. Extending the grid from Figure 9.

Consider the four points $0, 1, j$ and $1+j$. We already know that the lines $y = x$ and $y = -x+1$ are preserved. Lemmas 8 and 9 tell us that the point $\frac{1}{2} + \frac{1}{2}j$ and the lines $x = \frac{1}{2}$ and $y = \frac{1}{2}$ are preserved. Additionally the points $\frac{1}{2}j$, $\frac{1}{2}$, $1 + \frac{1}{2}j$, and $\frac{1}{2} + j$ are preserved. We have split the unit square into four equal squares of side length $\frac{1}{2}$. Continually repeating this process for each square, we see that for all $p, q, r, s \in \mathbb{Z}$, the point $\frac{p}{2^q} + \frac{r}{2^s}j$ is preserved. The numbers $\frac{p}{2^q}$ are known as the dyadic rationals, which are dense in $\mathbb{R}$

Now, we introduce the concept of betweenness of horizontal lines.

Definition 5. *If A, B, C are the horizontal lines $y = a$, $y = b$, $y = c$, respectively, then B is between A and C if and only if $a < b < c$ or $c < b < a$.*

Lemma 10. $\mu \circ \hat{\mathrm{T}}$ *preserves betweenness of horizontal lines.*

Proof. Let A, B, C, be horizontal lines so that B is between A and C. Let $P \in \mathscr{P}$ be tangent with B, intersecting with A, and non-intersecting with C as in Figure 11. Since $\mu \circ \hat{\mathrm{T}}$ preserves horizontal lines, we know that $(\mu \circ \hat{\mathrm{T}})(A), (\mu \circ \hat{\mathrm{T}})(B), (\mu \circ \hat{\mathrm{T}})(C)$ all remain horizontal lines. Suppose $(\mu \circ \hat{\mathrm{T}})(B)$ is not between $(\mu \circ \hat{\mathrm{T}})(A)$ and $(\mu \circ \hat{\mathrm{T}})(C)$. Then either $(\mu \circ \hat{\mathrm{T}})(B)$ is intersecting to both $(\mu \circ \hat{\mathrm{T}})(A)$ and $(\mu \circ \hat{\mathrm{T}})(C)$, or non intersecting to both $(\mu \circ \hat{\mathrm{T}})(A)$ and $(\mu \circ \hat{\mathrm{T}})(C)$. Since $\mu \circ \hat{\mathrm{T}}$ preserves the number of intersection points, we have a contradiction. $\square$

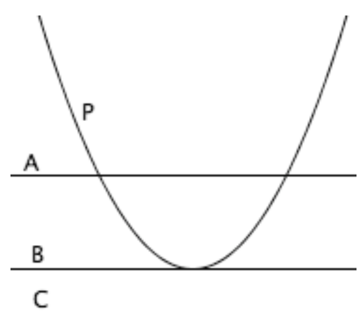

FIGURE 11. Horizontal lines with B between A and C. Also, a parabola P so that P is tangent to B and intersecting A.

Lemma 11. $\mu \circ \hat{\mathrm{T}}$ *preserves $y = k$ for each $k \in \mathbb{R}$*

Proof. Suppose $\mu \circ \hat{\mathrm{T}}$ takes $y = k$ to $y = k + \epsilon$ for some nonzero ϵ. Since the dyadic rationals are dense in $\mathbb{R}$, there exists two lines $y = b_1$ and $y = b_2$ so that $k < b_1 < b_2 < k+\epsilon$ or $k+\epsilon < b_2 < b_1 < k$ for dyadic rationals b_1, b_2. By definition, this implies $y = b_1$ is between $y = k$ and $y = b_2$, but $y = b_1$ is not between $y = b_2$ and $y = k + \epsilon$. As previously shown, $y = b_1$ and $y = b_2$ must be preserved. But, if $\mu \circ \hat{\mathrm{T}}$ takes $y = k$ to $y = k+\epsilon$ we get a contradiction since $\mu \circ \hat{\mathrm{T}}$ preserves horizontal betweenness. $\square$

Lemma 12. $\mu \circ \hat{\mathrm{T}}$ *preserves $x = k$ for each $k \in \mathbb{R}$*

Proof. $\mu \circ \hat{\mathrm{T}}$ preserves $y = x$ and the intersection point of $x = k$ and $y = 0$. Hence, Lemma 8 tells us the line $x = k$ is preserved. □

Completion of Theorem 2. Take $p = a + bj \in \mathbb{D}$. Then p is the intersection point of the vertical and horizontal lines $x = a$ and $y = b$, both of which are preserved from Lemmas 11 and 12. Lemma 9 then tells us that p is preserved. Therefore, $\mu \circ \hat{\mathrm{T}}$ acts identically on $\mathbb{D}$, and $\hat{\mathrm{T}}$ is a Laguerre transformation. □

References

[1] F. S. Beckman and D. A. Quarles, Jr. On isometries of Euclidean spaces. *Proc. Amer. Math. Soc.*, 4:810–815, 1953.

[2] Walter Benz. A Beckman Quarles type theorem for plane Lorentz transformations. *Math. Z.*, 177(1):101–106, 1981.

[3] Walter Benz. A Beckman-Quarles type theorem for finite Desarguesian planes. *J. Geom.*, 19(1):89–93, 1982.

[4] Michael Bolt, Timothy Ferdinands, and Landon Kavlie. The most general Planar Transformations that map Parabolas into Parabolas. *Involve, a Journal of Mathematics*, 2:79–88, 2009.

[5] J. A. Lester. The Beckman-Quarles theorem in Minkowski space for a spacelike square-distance. *Arch. Math. (Basel)*, 37(6):561–568, 1981.

[6] J. A. Lester. A Beckman-Quarles type theorem for Coxeter's inversive distance. *Canad. Math. Bull.*, 34(4):492–498, 1991.

[7] Joseph Zaks. A discrete form of the Beckman-Quarles theorem for rational spaces. *J. Geom.*, 72(1-2):199–205, 2001.

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