Proof of Solvability for the Generalized Oval Track Puzzle

Sam Kaufmann and Andreas Kavountzis
Research Advisor: Dr. Richard Statman
Department of Mathematics
Carnegie Mellon University

Introduction

The oval track puzzle (also known as *Top Spin*) is a game consisting of 20 numbered tiles in an oval shaped track. Also, there is a fixed window (the *swapping window*) of 4 tiles that reverses the order of the tiles within the window, leaving the other 16 tiles fixed. The object of the puzzle is to reorder the tiles into counting order using the mechanisms of the puzzle. Our paper presents conditions for both solvability and non-solvability for the general oval track puzzle with n total tiles and k tiles in the swapping window.

Terminology

To facilitate easy demonstration of game movements we shall employ a linear notation *(i.e. a string of numbers)* that begins with the left-most member of the *swapping window* in considering permutations. Here we tacitly use the notion of *position* when determining permutations by using this notation. However at times, mostly for the purpose of illustration, we will not fix the *location* of the swap window on the string. Finally, elements in the swap window will be enclosed by parentheses.

Definition: *We represent the puzzle consisting of n total tiles and k tiles in the swapping window by the 2-tuple (n, k).*

Remark: We assume that the bounds $2 \leq k < n$ must hold for the mechanism of the puzzle to work.

Definition: *The puzzle (n, k) is solved if the puzzle's configuration is in the identity configuration (i.e. 1 2 ... n).*

Definition: *A swap is the operation defined by the reflection of the k tiles in the swapping window. Since the value of k can be odd or even, we have two cases for the way in which the k tiles in the swapping window can be rearranged:*

- **Even Case:**
  Let $a_1, a_2, \ldots, a_k$ be the k tiles inside the swapping window, in the order shown, with a_1 in position 1. For k even, we have that one swap will yield a new permutation equal to $(a_1 a_k)(a_2 a_{k-1}) \ldots (a_{\frac{k}{2}} a_{\frac{k}{2}+1})$.

- **Odd Case:**
  Let $a_1, a_2, \ldots, a_k$ be the k tiles inside the swapping window, in the order shown, with a_1 in position 1. For k odd, we have that one swap will yield a new permutation equal to $(a_1 a_k)(a_2 a_{k-1}) \ldots (a_{\lceil \frac{k}{2} \rceil})$.

Definition: *A translation is the movement defined by sliding the tiles within the track such that any tile in position i is moved to position i + 1 for i = 1, ..., n - 1. In other words, if we let 1, 2, ..., n indicate the positions of the tiles in the track, then a translation can be described by the n-cycle (1 2 ... n).*

Remark: Note that applying a swap *twice* or applying a translation n times both leave the puzzle unchanged. Furthermore, due to the circular nature of the puzzle, we consider permutations to be equivalent up to translation.

Solvability

Before discussing the solvability of the puzzle, we need to give the following definitions:

Definition: *The family of all permutations of {1, 2, ..., n} is called the symmetric group on n-letters, denoted by S_n* [3, pg. 107].

Definition: *The subset of S_n consisting of all even permutations is called the alternating group, denoted by A_n* [3, pg. 150].

Throughout this paper, we will be presenting cases for which the puzzle is solvable and cases for which the puzzle is not solvable. However, what does it mean for the puzzle to be solvable? The following theorem addresses this question.

Theorem 0: *The puzzle (n, k) is solvable if and only if we can generate all permutations $\pi \in S_n$.*

Proof:
1. If we can generate all permutations $\pi \in S_n$, then the puzzle (n, k) is solvable.
Assume we can generate all permutations in S_n. If this were the case, then we could go from any puzzle configuration x to the identity configuration by applying a permutation. This permutation is characterized by undoing the moves that it took to get to x. Thus, we can solve the puzzle.

2. If the puzzle (n, k) is solvable, then we can generate all permutations $\pi \in S_n$.
If the puzzle is solvable, then we have to be able to switch adjacent tiles. For instance, if the configuration was **1 3 2 4 . . . n**, then we would have to switch tiles 2 and 3 in order for the puzzle to be solved. Thus, if we can switch adjacent elements, we can perform disjoint transpositions, and since any permutation $\pi \in S_n$ is the product of disjoint transpositions, we can generate all permutations in S_n.✓

Thus, for any puzzle *(n, k)*, we will prove that it is solvable by generating S_n. To prove it is unsolvable, we will show that it is not possible to generate S_n. We now examine the possible cases for values of n and k.

Cases

Theorem 1: *If* $n \equiv 0$ *(mod 2) and* $k \equiv 0$ *(mod 4) or* $k \equiv$ *2 (mod 4), then the puzzle is solvable.*

Proof:
To prove this, we will present an algorithm that switches adjacent tiles in the puzzle. If we can create an adjacent transposition, then we can go from any configuration to the identity configuration by simply transposing the elements pairwise into their correct positions. The algorithm uses a special move called a *swap-translation*. Like the name describes, a swap-translation is a swap immediately followed by a translation. Note that when using a swap-translation, one needs to specify which direction the translation is going in. Using this special move, we can go through the steps of the algorithm. Assume without loss of generality that the starting configuration of the puzzle is the identity configuration

(1 2 . . . k) k+1 . . . n

Our goal is to swap the elements 1 and 2, fixing all other elements, so that our final configuration after applying the algorithm is

(2 1 . . . k) k+1 . . . n

Now we start the algorithm:

(1) First, we fix 1 and 2 to the left of the swapping window, making the configuration

1 2 (3 . . . k k+1 k+2) k+3 . . . n

(2) Next, we fix the rightmost k - 1 elements of the swap window and perform $n-(k-1)-2 = n-k-1$ counterclockwise swap-translations. This will bring us to the configuration

(k+1 k . . . 1) 2 . . . n

(3) Now we perform $\frac{k}{2}+1$ swap-translations, alternating the direction we translate, beginning with a counterclockwise translation. This step moves 1 and 2 into the two center positions of the swap window. This gives us the configuration

(. . . 2 1 . . .) . . . n

if $k \equiv 0$ (mod 4) and the following configuration

$$\textbf{(\ldots 1 2 \ldots) \ldots n}$$

if $k \equiv 2 \pmod 4$.

(4) Now we perform $\frac{k}{2}$ swap-translations, alternating the direction we translate, beginning with a clockwise translation if $k \equiv 0 \pmod 4$ and a counterclockwise translation if $k \equiv 2 \pmod 4$. This brings us to the configuration

$$\textbf{2(1 \ldots k k+1) k+2 \ldots n}$$

which, if we perform one more clockwise translation, we get

$$\textbf{(2 1 \ldots k) k+1 \ldots n}$$

Thus, we can solve the puzzle.✓

Theorem 2: *If $n \equiv 1 \pmod 2$ and $k \equiv 1 \pmod 4$ or $k \equiv 3 \pmod 4$, then the puzzle is solvable.*

Proof:
We show this case by first showing the existence of a $k-cycle$ which we will use to find a $3-cycle$. Define τ to be a clockwise translation, and σ to be a swap. Now applying the following sequence of moves to the puzzle (read from left to right) we will arrive at a $k-cycle$.

$$(\tau\sigma)^{n-k}\tau = (\tau\sigma\tau\sigma\ldots\tau\sigma)\tau$$

Written linearly as:

(1 ... k) k+1 ... n
(n 1 ... k-1) k ... n-1
(k-1 ... 1 n) k ... n-1
(n-1 k-1 ... 1) n k ... n-2
(1 ... k-1 n-1) n k ... n-2
⋮
(k 1 ... k-1) k+1 ... n

Informally, in employing this sequence of moves we essentially fix elements **1 ... k-1** and move the remaining elements "around" them. Since $n-k$ is *even* and the order of σ is 2 (i.e. σ^2 leaves the puzzle unchanged) it follows that **1 ... k-1** will be in the correct numerical ordering after this sequence of moves.
Using this $k-cycle$ we will now exhibit a $3-cycle$. We shall apply the following moves to the puzzle:

$$\sigma\tau^{-1}\sigma\tau^{-1}$$

then we apply our $k - cycle$ twice. Written linearly this is:

(1 . . . k) k+1 . . . n
(k . . . 1) k+1 . . . n
(k-1 . . . 1 k+1) k+2 . . . n k
(k+1 1 . . . k-1) k+2 . . . n k
(k k+1 1 . . . k-2) k-1 k+2 . . . n
(k+1 1 . . . k-2 k) k-1 k+2 . . . n
(1 . . . k-2 k k+1) k-1 k+2 . . . n

which yields the *consecutive 3-cycle* **(k k+1 k-1)**.
Now we shall apply the following Lemma's (Lemma 1 is borrowed from [2]).

Lemma 1: *For $n \geq 3$, the consecutive 3-cycles generate A_n.*

Lemma 2: *Given A_n and an odd permutation, we can generate all of S_n.*

Proof:
Since τ is an odd permutation, if we take $\pi \in A_n$ then

$$\tau \circ \pi \text{ and } \pi \circ \tau$$

are odd permutations. Therefore these permutations are not in A_n. This implies that any other subgroup we generate will be *larger* than A_n. By Lagrange's Theorem [3, pg. 156] we know that the order of this subgroup must divide the order of the group. Therefore since $|A_n|$ is the largest divisor of $|S_n|$ (other than $|S_n|$), the *larger* subgroup generated is S_n. ✓

Now that we have generated A_n by Lemma 1, it remains to show that we have an odd permutation. By Lemma 2, the composition of this odd permutation with elements of A_n will generate S_n and confirm the puzzle is solvable. The odd permutation we seek is τ. Therefore the game is solvable. ✓

Theorem 3: *If $n \equiv 1 \pmod 2$ and $k \equiv 0 \pmod 4$ or $k \equiv 2 \pmod 4$, then the puzzle is solvable.*

Proof: We shall proceed in a similar manner similar to that of Theorem 2. We begin by mentally *glueing* the first two tiles together and moving them both out of the window. This allows us to take tiles **3 . . . k+1** and view them as $fixed$ (i.e. never leaving) in the *swapping window*. Similar to the proof of Theorem 2 we now proceed to move the remaining tiles past the fixed elements in a series of $n-k+1$ $swap-translations$ (i.e. $(\sigma\tau)^{n-k+1}$). Since $n-k+1$ is even we know that after this sequence of $swap - translates$ that the fixed elements will remain in the correct counting order.
To better illustrate this we give the linear representation:

1 2 (3 . . . k+1 k+2) k+3 . . . n
1 2 (k+2 k+1 . . . 3) k+3 . . . n
1 2 k+2 (k+1 . . . 3 k+3) k+4 . . . n

1 2 k+2 (k+3 3 . . . k+1) k+4 . . . n
1 2 k+2 k+3 (3 . . . k+1 k+4) k+5 . . . n
⋮

We continue this a total of $n - k + 1$ times to get:

(3 4 . . . k+1 1 2) k+2 . . . n

This allows us to see that we now have the pair to which we can apply the same process (i.e. begin by mentally *glueing* them together). We repeat this process of pairing elements of the swapping window $\frac{k}{2}$ times, written linearly as:

(k+1 1 2 . . . k-1) k k+2 . . . n

Which yields the $(k+1) - cycle$ **(k+1 1 . . . k)**.
Using this $(k+1)-cycle$ we shall now exhibit a *consecutive 3-cycle* to generate A_n. We begin by performing a $swap-translate$, then a swap, (i.e. $\sigma\tau\sigma$), finally we apply our $(k+1)-cycle$ twice. Written linearly this is:

(1 . . . k) k+1 . . . n
(k . . . 1) k+1 . . . n
k (k-1 . . . 1 k+1) k+2 . . . n
k (k+1 . . . 1 k-1) k+2 . . . n
k (1 . . . k-1 k+2) k+1 . . . n
(1 . . . k-1 k+2) k k+1 . . . n

Which yields the *consecutive 3-cycle* **(k+2 k k+1)**. Again using Lemma 1 we can now generate the alternating group, A_n. Since τ is an odd permutation, we can now generate all of S_n by Lemma 2. Therefore the puzzle is solvable.✓

Theorem 4: *If* $n \equiv 0 \pmod 2$ *and* $k \equiv 1 \pmod 4$ *or* $k \equiv 3 \pmod 4$*, then the puzzle is not solvable.*

Proof:
Begin by coloring tiles alternately with two colors, we shall use red and green without loss of generality. Note that the tiles are now partitioned by color and parity, that is to say, the red tiles form the set of even numbers in $\{1 \ldots n\}$ and green tiles form the set of odd numbers. Since n is even it follows that there will be pairs of oppositely colored tiles. Furthermore, since k is odd any swap will result in an element remaining fixed (i.e. the center tile of the *swapping window*).
Next, enumerate the *positions* of tiles beginning with 1 on the game, again using the left-most position of the *swapping window* as a reference point. Since every swap fixes a center point, the only permutations generated transpose tiles of the same parity and therefore the same color. Thus, if two adjacent tiles are of the same color, we cannot create any permutation that will transpose the two. Therefore the puzzle is not solvable.✓

Bibliography

[1] E. Wilbur, *Topspin: Solvability of Sliding Number Games*, Rose-Hulman Undergraduate Math Journal; Vol. 2, Issue 2, 2001.

[2] A. F. Archer, *A Modern Treatment of the 15 Puzzle*, American Mathematical Monthly, November 1999, pp. 793-799

[3] J.J. Rotman, *A First Course in Abstract Algebra*, Prentice Hall, 2006.