

ON THE ORDER OF A GROUP CONTAINING NONTRIVIAL GASSMANN EQUIVALENT SUBGROUPS

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ABSTRACT. Using a result of de Smit and Lenstra, we prove that the order of a group containing nontrivial Gassmann equivalent subgroups must be divisible by at least five primes, not necessarily distinct. We then investigate the existence of Gassmann equivalent subgroups in groups with order divisible by exactly five primes.

INTRODUCTION

Let G be a finite group and let $H, H' \leq G$. Denote the conjugacy class of $g \in G$ by g^G . We say the triple (G, H, H') is a *Gassmann triple* if every conjugacy class intersects H and H' in the same number of elements, in other words, $|g^G \cap H| = |g^G \cap H'|$ for all $g \in G$. The subgroups H and H' in such a triple are called *Gassmann equivalent*. These triples are named after Fritz Gassmann, who first articulated the above condition in [5] while explaining the work of Adolf Hurwitz.

Since the time of Gassmann, these ‘almost conjugate’ triples have proved useful in a variety of areas. In [11] Terras and Stark show how Gassmann triples may be used to construct nonisomorphic graphs with the same Ihara zeta function. Sunada describes in [10] how Gassmann equivalent subgroups may give rise to Riemannian manifolds that are isospectral but not isometric. Finally, Perlis shows in [7] that two number fields have the same Dedekind zeta function precisely when associated Galois groups form a Gassmann triple.

In this paper we investigate the number of primes dividing the order of a group G containing non-conjugate Gassmann equivalent subgroups H and H' . Such Gassmann triples are called *nontrivial* triples, while triples in which H and H' are conjugate are called *trivial*. The motivation for investigating this topic comes from the research of Jim Stark, an REU student at Louisiana State University in 2007. He noted over six hundred cases in which the order of group containing nontrivial Gassmann equivalent subgroups was divisible by at least five primes, and he asks in [9] if this is true in general.

Our main result, Theorem 3.1, answers Stark’s question in the affirmative:

Main Theorem Let (G, H, H') be a nontrivial Gassmann triple. Then $|G|$ is divisible by at least five primes, not necessarily distinct.

Throughout this article, if $|G| = \prod_{i=1}^t p_i^{\alpha_i}$ with $m = \sum_{i=1}^t \alpha_i$, we say that $|G|$ is *divisible by m primes*. We will clearly indicate those cases in which we desire the group to be divisible by a number of distinct primes.

In proving the main theorem we rely on several powerful results which allow us to demonstrate the most difficult steps with ease. In Section 2 we show that the main theorem restricted to solvable groups is a corollary of a difficult theorem due to de Smit and Lenstra. We also use the Feit-Thompson Odd Order Theorem and two results of Burnside in Section 3.

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1. DEFINITIONS

We first introduce another criterion for Gassmann equivalence:

Proposition 1.1. *Let G be a finite group and $H, H' \leq G$. The following statements are equivalent (see Lemma 1.9 in [1]):*

- $|g^G \cap H| = |g^G \cap H'|$ for all $g \in G$. (*Gassmann's original criterion*)
- *There exists a bijection $\phi : H \rightarrow H'$ such that $\phi(h) \in h^G$ for all $h \in H$. (Sheng Chen's criterion)*

From Sheng Chen's criterion it is evident that two Gassmann equivalent subgroups $H, H' \leq G$ have the same order and hence the same index in G . A Gassmann triple (G, H, H') where $|G : H| = |G : H'| = n$ is called a Gassmann triple of index n . We give an example of nontrivial Gassmann equivalent subgroups of index 8 in a group with order 32.

Example: Let $G = (\mathbb{Z}/8\mathbb{Z})^\times \rtimes \mathbb{Z}/8\mathbb{Z}$ (where $(\mathbb{Z}/8\mathbb{Z})^\times$ denotes the multiplicative group of units mod 8) with multiplication defined by $(a, b) * (c, d) = (ac, bc + d)$. Define H and H' as follows:

$$H = \{(1, 0), (3, 0), (5, 0), (7, 0)\}$$

$$H' = \{(1, 0), (3, 4), (5, 4), (7, 0)\}$$

Define a bijection $\phi : H \rightarrow H'$ as below:

$$\begin{aligned} \phi((1, 0)) &= (1, 0) \\ \phi((3, 0)) &= (3, 4) = (3, 0)^{(1,2)} \\ \phi((5, 0)) &= (5, 4) = (5, 0)^{(1,1)} \\ \phi((7, 0)) &= (7, 0) \end{aligned}$$

ϕ satisfies Sheng Chen's criterion since $(1, 0)$ conjugates $(1, 0)$ and $(7, 0)$ to themselves, $(1, 2)$ conjugates $(3, 0)$ to $(3, 4)$, and $(1, 1)$ conjugates $(5, 0)$ to $(5, 4)$. It is not difficult to show that H and H' are not conjugate subgroups, hence (G, H, H') is a nontrivial Gassmann triple, as claimed.

2. THE SOLVABLE CASE

In this section we prove that the main theorem is true for solvable groups. We begin with a few simple results.

Lemma 2.1. *Let (G, H, H') be a Gassmann triple and let $t \in H$. There exists $g \in G$ such that $(G, H, g^{-1}H'g)$ is a Gassmann triple and $t \in H \cap g^{-1}H'g$. Moreover, $(G, H, g^{-1}H'g)$ is trivial if and only if (G, H, H') is trivial.*

Proof: By Sheng Chen's criterion, there exists a bijection $\psi : H' \rightarrow H$ with $\psi(h')$ conjugate to h' in G for all $h' \in H'$. Hence there is $g \in G$ and $t' \in H'$ such that $t = \psi(t') = g^{-1}t'g$, implying $t \in H \cap g^{-1}H'g$. Now let $x \in g^{-1}H'g$. Then $\psi(gxg^{-1}) : g^{-1}H'g \rightarrow H$ is a bijection satisfying Sheng Chen's criterion, therefore $(G, H, g^{-1}H'g)$ is a Gassmann triple. The last statement is transparent, since $g^{-1}H'g$ is conjugate to H if and only if H' is conjugate to H .

Corollary 2.2. *Let (G, H, H') be a Gassmann triple such that $H = \langle t \rangle$ is cyclic. Then (G, H, H') is trivial.*

Proof: By Lemma 2.1 there exists $g \in G$ so that $t \in g^{-1}H'g$. Since $|H| = |H'|$, $g^{-1}H'g = H$.

Corollary 2.3. *Let (G, H, H') be a nontrivial Gassmann triple. Then at least two primes divide $|H| = |H'|$.*

Proof: Suppose $|H| = |H'| = p$, p a prime. It follows that H, H' are cyclic and hence trivial by Corollary 2.2. So at least two primes divide $|H| = |H'|$.

In the following, the Gassmann triple (G, H, H') is said to be a *solvable Gassmann triple* if the group G is solvable. We now present a theorem of de Smit and Lenstra (see [2]) which gives our main theorem for solvable Gassmann triples more or less directly.

Theorem 2.4. *(B. de Smit and H.W. Lenstra, Jr.) For every positive integer n the following are equivalent:*

- (1) *There exists a nontrivial solvable Gassmann triple (G, H, H') of index n .*
- (2) *There are prime numbers p, q, r (not necessarily distinct) with $pqr|n$ and $p|q(q-1)$.*

The forward implication of this theorem gives us everything we need:

Corollary 2.5. *Let (G, H, H') be a nontrivial solvable Gassmann triple of index n . Then $|G|$ is divisible by at least five primes.*

Proof: By Theorem 2.4, n is divisible by at least three primes. Furthermore, by Corollary 2.3, at least two primes divide $|H|$. Hence $|G| = n|H|$ is divisible by at least five primes.

3. THE GENERAL CASE

In this section we prove the main result of this paper:

Theorem 3.1. *Let (G, H, H') be a nontrivial Gassmann triple. Then $|G|$ is divisible by at least five primes, not necessarily distinct.*

The outline of our proof for Theorem 3.1 is as follows. We first claim that all groups divisible by four or fewer primes are either solvable or are isomorphic to A_5 (Claim 1). It then follows from Corollary 2.5 that if $G \not\cong A_5$ and $|G|$ is divisible by four or fewer primes, then G contains no nontrivial Gassmann equivalent subgroups. We finish the proof by showing that A_5 also has no nontrivial Gassmann equivalent subgroups.

We break the proof of Claim 1 into three cases:

- (1) G has odd order or is divisible by at most two distinct primes $\Rightarrow G$ solvable.
- (2) G has order $2pqr$, where $2 < p < q \leq r \Rightarrow G$ solvable.
- (3) G has order 2^2pq , where $p < q \Rightarrow G$ solvable or $G \cong A_5$.

Case 1: By the Odd Order Theorem of Feit and Thompson every group of odd order is solvable (see [4]). By Burnside's $p^a q^b$ Theorem every group with order $p^a q^b$, where p, q are prime, is solvable (see [8] pp. 247-8).

In proving Cases 2 and 3, we use the following additional results due to Burnside. For proofs of these, consult [8], pp. 289-290.

Theorem 3.2. *Let G be a finite group and let S be a Sylow- p subgroup of G such that $S \leq Z(N_G(S))$. Then $G = N \rtimes S$ where $N \triangleleft G$.*

Here $N_G(S)$ refers to the normalizer of $S \leq G$ while $Z(N_G(S))$ denotes the center of $N_G(S)$. We also use the following corollary of Theorem 3.2.

Corollary 3.3. *Let G be a finite group and let S be a cyclic Sylow- p subgroup of G where p is the smallest prime divisor of $|G|$. Then S satisfies the conditions of Theorem 3.2 and $G = N \rtimes S$ where $N \triangleleft G$.*

Now we proceed with Cases 2 and 3.

Case 2: Let $|G| = 2pqr$, with $2 < p < q \leq r$, and let S be a Sylow-2 subgroup of G . Corollary 3.3 tells us that we may write $G = N \rtimes S$, where $N \triangleleft G$ and $|N| = pqr$. Since N has odd order, it is solvable. Since G/N has order 2, it is solvable. Hence G is solvable.

Case 3: Let $|G| = 2^2pq$, where $p < q$, excluding the case $p=3, q=5$. Let S be a Sylow-2 subgroup of G . S hence has order 4. If S is cyclic, then we can use Corollary 3.3 and proceed as in Case 2. Hence let $S \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. We first prove the case where $|G|$ is not divisible by 3. Let $N_G(S)$ act on S by conjugation, giving a homomorphism $\phi : N_G(S) \rightarrow \text{Aut}(S)$, where $\text{Aut}(S)$ is the automorphism group of S . Since $S \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is a vector space over \mathbb{Z}_2 , $\text{Aut}(S) \cong GL_2(\mathbb{Z}_2)$ and hence has order $(2^2 - 1)(2^2 - 2) = 6$. Let Q_{p_i} denote any Sylow- p_i subgroup of $N_G(S)$. Then $N_G(S) = \langle Q_{p_i} | p_i \text{ divides } |N_G(S)| \rangle$. $N_G(S)$ is hence mapped trivially under ϕ if Q_{p_i} is mapped trivially for every prime p_i dividing $|N_G(S)|$. We let $Q_2 = S$. Clearly, since S is abelian, it acts on itself trivially by conjugation and $\phi(S) = 1$. Since 3 does not divide $|N_G(S)|$ by hypothesis, $|Q_{p_i}|$ is relatively prime with $|\text{Aut}(S)| = 6$ for all primes p_i dividing $|N_G(S)|$ where $p_i \neq 2$. It follows that $\phi(Q_{p_i}) = 1$ for all of these. Hence $S \leq Z(N_G(S))$ and we use Theorem 3.2 to claim that $G = N \rtimes S$ where $N \triangleleft G$. As above, N and G/N are solvable, hence G is as well. Now suppose

$p = 3$ and $q > 5$ ($|G| = 2^2 \cdot 3q$, $q > 5$). Let n_q denote the number of Sylow- q subgroups of G and let Q be a Sylow- q subgroup of G . Basic results on Sylow subgroups give us: 1) $n_q \equiv 1 \pmod{q}$, and 2) $n_q = |G : N_G(Q)|$ divides $2^2 \cdot 3$. By 2), $n_q = 1, 2, 3, 4, 6$, or 12. Coupling this with 1) and the restriction $q > 5$ we have the following cases:

- (1) $q > 5$ and $n_q = 1$.
- (2) $q = 11$ and $n_q = 12$.

(1) $\Rightarrow Q \triangleleft G$. In this case Q is solvable and G/Q is divisible by exactly two distinct primes hence is solvable by Burnside's result. Hence G is solvable.

(2) $\Rightarrow |G : N_G(Q)| = 12 \Rightarrow |N_G(Q)| = 11 \Rightarrow N_G(Q) = Q$. Since Q is cyclic, hence abelian, this gives $Q = Z(N_G(Q)) = N_G(Q)$, hence Theorem 3.2 applies, giving $G = N \rtimes Q$, where $N \triangleleft G$. G/N is solvable, and so is N , since it has order $2^2 \cdot 3$. Hence G is solvable.

We are left with the case where G has order $2^2 \cdot 3 \cdot 5 = 60$. If G is not solvable, it must be simple, since the existence of $1 < N \triangleleft G$ would imply G and G/N solvable by order considerations. There is exactly one simple group of order 60 up to isomorphism, namely A_5 (see [3] pp. 145-6).

Up to this point we have shown that if $|G|$ is divisible by at most four primes and is not isomorphic to A_5 , then G is solvable. For all such groups G , Corollary 2.5 (or more precisely its contrapositive statement) guarantees that any Gassmann equivalent subgroups of G must be trivial. The only thing left to show is that A_5 contains no nontrivial Gassmann equivalent subgroups.

We prove this by showing that any two proper subgroups of A_5 with a particular order are either conjugate or not Gassmann equivalent. Due to Lagrange's theorem we need only check subgroups of order 2, 3, 4, 5, 6, 10, 12, 15, 20, and 30. Subgroups of order 2, 3, or 5 are cyclic and hence there cannot be any nontrivial Gassmann equivalent subgroups of A_5 with these orders by Corollary 2.2. Subgroups of order 4 are all conjugate because they are Sylow-2 subgroups of A_5 . Theorem 2, part (c) of [7] states that any two Gassmann equivalent subgroups of prime index p in A_p are conjugate; it follows that there are no Gassmann equivalent subgroups of A_5 with order 12. Furthermore, there are no subgroups of A_5 with order 15, 20, or 30 by the simplicity of A_5 . Any subgroup with order 30 would have index 2 and hence be normal. If there were a subgroup H of order 15 or 20, letting A_5 act on the coset space G/H would give a nontrivial homomorphism $\phi : A_5 \rightarrow S_3$ or $\phi : A_5 \rightarrow S_4$. The kernel of ϕ in either case would be a proper nontrivial normal subgroup of A_5 , hence no such H exists.

We are left with subgroups H of order $6 = 2 \cdot 3$ and $10 = 2 \cdot 5$. If such an H were abelian, it would be cyclic because it would be the direct product of two cyclic groups of relatively prime order. Hence by Corollary 2.2 any two abelian Gassmann equivalent subgroups of order 6 or 10 in A_5 are conjugate. It follows that we need only examine the case in which H is isomorphic to one of the dihedral groups D_6 or D_{10} . We consider the case $H \cong D_6$. Then H is given by the presentation $H = \langle r, s \mid r^3 = s^2 = 1, srs^{-1} = sr = r^{-1} \rangle$. It follows, since we are in A_5 , that r is a 3-cycle and s is a product of 2 disjoint 2-cycles (for example, $H = \langle (1, 2, 3), (1, 2)(4, 5) \rangle$). We claim that a choice of r determines precisely one subgroup $H \cong D_6$ of order 6 in A_5 . This will enable us to use Lemma 2.1 to

prove that any two Gassmann equivalent subgroups isomorphic to D_6 must be conjugate.

To see that a choice of r determines a unique subgroup isomorphic to D_6 , let a_1, a_2, a_3, a_4 , and a_5 represent the distinct integers 1 to 5, and set $r = (a_1, a_2, a_3)$. From $sr s = r^{-1}$ and $srs = (s(a_1), s(a_2), s(a_3))$ we see that s must be of the form $(a_1, a_2)(a_4, a_5)$, $(a_1, a_3)(a_4, a_5)$, or $(a_2, a_3)(a_4, a_5)$, since s cannot take one of a_1, a_2, a_3 to a_4 or a_5 (if it did then we would not have $sr s = r^{-1}$). However, each of these choices of s together with r generate precisely the same subgroup, namely $H = \{(), (a_1, a_2, a_3), (a_1, a_3, a_2), (a_1, a_2)(a_4, a_5), (a_1, a_3)(a_4, a_5), (a_2, a_3)(a_4, a_5)\}$. Hence a choice of r determines a unique subgroup of A_5 isomorphic to D_6 . Now suppose we are given two Gassmann equivalent subgroups $H, H' \leq A_5$ with $H, H' \cong D_6$, and let $H = \langle r, s \rangle$. By Lemma 2.1 there is a conjugate K of H' containing r . But we have seen that this implies $K = H$, hence the Gassmann triple (G, H, H') is trivial. A similar argument may be made for subgroups of order 10 isomorphic to D_{10} . A_5 therefore does not contain any nontrivial Gassmann triples.

This concludes the proof of Theorem 3.1. In the next section we address some questions that arise from this theorem.

4. GASSMANN TRIPLES IN GROUPS DIVISIBLE BY EXACTLY FIVE PRIMES

Having proved this result, a natural question to ask next is whether, given an integer m divisible by precisely five primes, there is a group G with $|G| = m$ containing nontrivial Gassmann equivalent subgroups. If $m = p^5$, with p prime, the answer is yes. Guralnick has constructed for $p > 2$ a group of order p^5 with nontrivial Gassmann equivalent subgroups of order p^2 in Example 4.1 of [6]. The case $p = 2$ is furnished by the group given in the example of Section 1.

However, one does not have to look very far to find several counterexamples to our question. Theorem 2.4 tell us that any nontrivial Gassmann triple of a solvable group G must have index divisible by three primes p, q , and r such that $p \mid q(q-1)$. We use this condition to set up several examples of groups with order divisible by precisely five primes which do not contain any nontrivial Gassmann equivalent subgroups simply because of their orders. In the following let m be an integer with $m = \prod_{i=1}^k p_i^{\alpha_i}$ such that $p_j \not\equiv 1 \pmod{p_i}$ for all pairs $p_i \leq p_j$ dividing m , $\sum_{i=1}^k \alpha_i = 5$, and $2 \leq k \leq 5$. This condition forces m to be odd since its order is divisible by at least 2 distinct primes and every odd prime is congruent to 1 mod 2. The Odd Order Theorem of Feit and Thompson then implies that any group G with $|G| = m$ is solvable and we can apply Theorem 2.4 to G .

First consider the case in which all five primes are distinct, say $m = \prod_{i=1}^5 p_i$. Since no choice of three primes from $\{p_i\}_{i=1}^5$ will satisfy the condition $p \mid q(q-1)$, no group of order m contains nontrivial Gassmann equivalent subgroups. An example of such an integer is $m = 5 \cdot 7 \cdot 13 \cdot 17 \cdot 23$.

Now suppose (1) $m = pqr^2s$ or (2) $m = pqr^3$ (p, q, r, s primes). Suppose there is some nontrivial Gassmann triple (G, H, H') with $|G| = m$. By Theorem 2.4, $|G : H| = n$ is divisible by three primes and two of them must be r . Consider Case (1) first. Without loss of generality, we let the other prime dividing n be s , so that $r^2s \mid |G : H|$. Then we have $|H| = |H'| = pq$. Suppose $p < q$. Then we have $q \not\equiv 1 \pmod{p}$ by the condition on m above, implying that H and H' are cyclic and (G, H, H') is a trivial triple by Lemma 2.2. For Case (2) we get the same

reduction. Examples of integers satisfying (1) and (2) are $m = 5 \cdot 7 \cdot 13^2 \cdot 17$ and $m = 5 \cdot 7 \cdot 13^3$.

We hence leave the reader with a slightly modified question: Given a set $\{p_i\}_{i=1}^5$ of five primes, what is the minimal order of a group G such that $\prod_{i=1}^k p_i$ divides $|G|$ and G contains nontrivial Gassmann equivalent subgroups?

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