A Combinatorial Proof of an Identity of Andrews

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Abstract

We give a combinatorial proof of an identity originally proved by G. E. Andrews in [1]. The identity simplifies a mock theta function first discovered by Rogers.

1 Background and Definitions

First we review the basics of partition theory.

Definition 1.1. A partition λ of a positive integer n is a finite nonincreasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i are called the *parts* of the partition and the notation $\lambda \vdash n$ denotes "λ is a partition of n." We call n the size of λ and λ_i the size of the i^{th} part.

Here we will expand this definition so that λ may include parts of size 0.

Graphically, a partition λ can be represented as a left-justified array of boxes called a Ferrers shape where the k^{th} row contains λ_k boxes. For example, the partition $\lambda = (6, 3, 2, 1, 1)$ of 13 can be represented by the Ferrers shape shown in Figure 1.

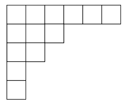

Figure 1: Ferrers shape for $\lambda = (6, 3, 2, 1, 1)$

We use standard generating functions as well as the standard notation for basic hypergeometric series as defined by Andrews in [2]:

Definition 1.2.
$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$,
$(a)_0 = 1$.

2 Introduction

We begin with the identity

$$\sum_{m=0}^{\infty} \frac{q^{m^2} x^m}{(y; q^2)_{m+1}} = \sum_{m=0}^{\infty} (-xq/y; q^2)_m y^m \tag{1}$$

which was introduced and proven algebraically by Andrews in [1]. When $x = 1$, the left-hand side of (1) becomes the mock theta function

$$\phi(a; q) = \sum_{m=0}^{\infty} \frac{q^{m^2}}{(-aq; q^2)_m}$$

where $y = -a/q$. Here we will give a combinatorial proof of (1) using integer partitions.

3 Proof

We will now give a combinatorial proof of (1). Both sides of the identity are generating functions for certain sets of partitions. We will construct a bijection between these sets of partitions that will establish (1).

For any partition λ let $a =$ the largest part of λ, $b =$ the number of non-empty parts, and $c =$ the number of empty parts (parts of size 0).

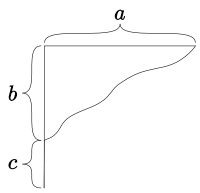

Figure 2: Labeled pieces of λ

3.1 Left-Hand Side of (1)

Let A be the set of partitions with m distinct odd parts, no even parts, and any number of empty parts. We will show that the left-hand side of (1) is the generating function for the partitions in set A.

Let $\lambda \in A, \lambda \vdash n$. Break λ into the three pieces $\lambda_1, \lambda_2,$ and λ_3 in the following way (see Figure 3):

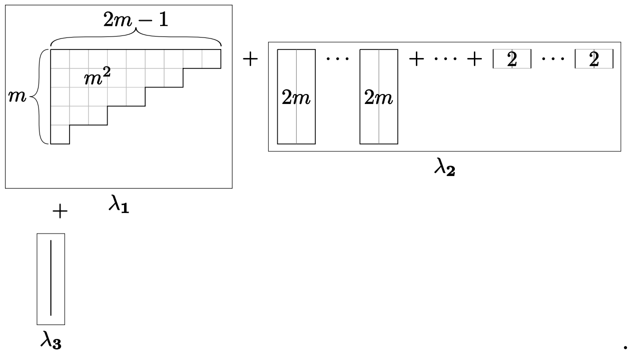

Figure 3: Ferrers shape for the left-hand side of (1)

1. The piece λ_1 consists of the partition $(1, 3, 5, \ldots, 2m-1) \vdash m^2$, a staircase Ferrers shape with exactly m odd parts. If we let x count the number of parts and q the size of the partition, λ_1 contributes $q^{m^2} x^m$ to the generating function for A.

2. To construct λ_2 we remove λ_1 and any empty parts from λ, and left-justify the remaining Ferrers shape. Now we separate λ_2 into pairs of columns. Since each part of λ is odd and λ_1 took away an odd number from each part, λ_2 consists of even-sized parts. When we let q count the

size of λ_2, and y count the number of pairs of columns, λ_2 contributes

$$(1+yq^2+(yq^2)^2+\cdots)(1+yq^4+(yq^4)^2+\cdots)\cdots(1+yq^{2m}+(yq^{2m})^2+\cdots) = \frac{1}{(1-yq^2)}\cdots\frac{1}{(1-yq^{2(m-1)})}\frac{1}{(1-yq^{2m})} = \frac{1}{(yq^2;q^2)_m}$$

to the generating function for A.

3. The final piece λ_3, then, consists of all of the empty parts of λ which are left after λ_1 and λ_2 have been removed. There can be any number of empty parts, and so λ_3 contributes

$$(1+y+y^2+y^3+\cdots) = \frac{1}{(1-y)} \tag{2}$$

to the generating function for A, where y counts the number of empty parts.

From the above it is clear that the left-hand side of (1) is the generating function

$$\sum_{\lambda\in A} q^{|\lambda|}x^b y^{c+\frac{a-(2b-1)}{2}} \quad (\text{where } |\lambda| = \text{ size of } \lambda)$$

for A, the set of partitions with distinct odd parts and any number of empty parts.

Example 3.1. Let $\lambda \in A = (15, 13, 9, 5, 0, 0, 0)$. Then λ contributes the term $q^{42}x^4y^{3+4})$ to the generating function for A.

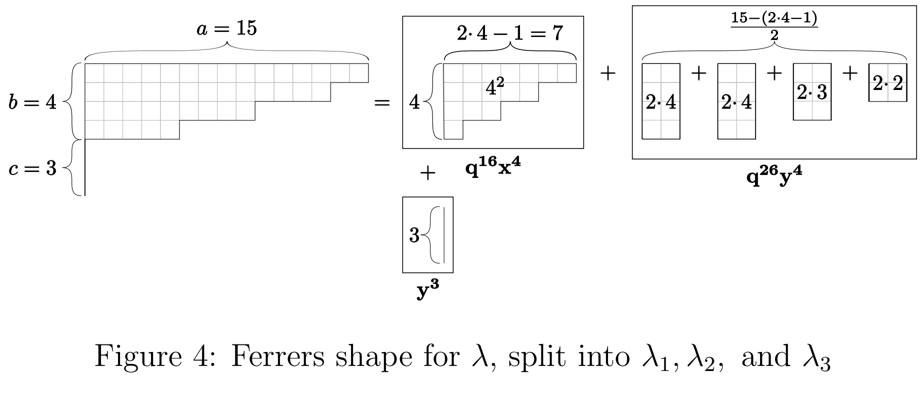

Figure 4: Ferrers shape for λ, split into $\lambda_1, \lambda_2,$ and λ_3

3.2 Right-Hand Side

Let B be the set of partitions with m parts which are either distinct odd parts or empty parts where the largest part is smaller than $2m$. We will show that the right-hand side of (1) is the generating function for these partitions. Let $\lambda \vdash n$ be a partition in B and let the generating function for B have the form

$$\sum_{\lambda \in A} q^{|\lambda|} x^b y^c.$$

Note that q and x count the same thing as in the left-hand side while y now counts only the empty parts.

To construct λ we start with a partition with no more than m distinct odd parts, which contributes $(1+xq)(1+xq^3)\cdots(1+xq^{2m-1}) = (-xq;q^2)_m$ to the generating function for B. We then add m empty parts, which contributes y^m. In order to keep a total of m parts we must take off an empty part for each distinct odd part added. To do this, we include a $1/y$ in the generating function for each of the distinct odd parts. Thus, the generating function for partitions in the set B is

$$F(q) = \sum_{m=0}^{\infty} (-xq/y;q^2)_m y^m$$

which is the right-hand side of (1).

3.3 Bijection

We will give a bijection between sets A and B that establishes (1).

Define a mapping $\phi : B \to A$ and let $\lambda \in B$. Construct $\phi(\lambda)$ by removing $(a-(2b-1))/2$ empty parts from λ.

Since $a \leq 2m-1$,

$$\begin{aligned} a-(2b-1) &\leq 2m-1-(2b-1) \\ a-(2b-1) &\leq 2(m-b) \\ \frac{a-(2b-1)}{2} &\leq c \end{aligned}$$

and so there will always be enough empty parts to strip off.

In order for ϕ to be a bijection we must show that it is weight-preserving and that its inverse is well-defined. The mapping ϕ is weight-preserving if for all $\lambda \in B$ the term that λ contributes to the right-hand side of (1) is the same as the term that $\phi(\lambda)$ contributes to the left-hand side of (1). As shown in Example 3.2, we are moving the y for each part that is stripped off to a pair of columns at the end of the partition. Since y in the left-hand side of (1) counts these columns as well as the empty parts, the exponent of y in the term in the right-hand side of (1) that counts λ will be the same as that in the left-hand side of (1) that counts $\phi(\lambda)$. Likewise, since the partition of non-empty parts in λ does not change, the exponents of x and q will remain constant under ϕ. Therefore, the terms contributed to the generating functions by λ and $\phi(\lambda)$ are identical and ϕ is weight-preserving.

The inverse of ϕ is very simple. Let $\lambda \in A$ and construct $\phi^{-1}(\lambda)$ by adding as many empty parts as there are pairs of columns in λ_2.

Example 3.2. Let $\lambda \in B = (15, 13, 11, 3, 0, 0, 0, 0)$. Then λ contributes the term $q^{42}x^4y^4$ to the generating function for B. Also, $\phi(\lambda) \in A = (15, 13, 11, 3)$ and $\phi(\lambda)$ contributes the same term to the generating function for A.

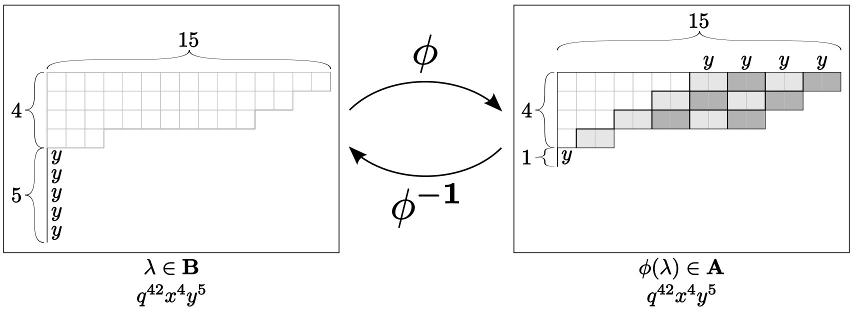

Figure 5: $\phi(\lambda)$

From the above it is clear that λ is a bijection and so

$$\sum_{\lambda \in A} q^{|\lambda|} x^b y^{c+\frac{a-(2b-1)}{2}} = \sum_{\lambda \in B} q^{|\lambda|} x^b y^c$$

establishing (1). ■

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[1] G.E. Andrews, On basic hypergeometric series, mock theta functions and partitions. II, *Quart. J. Math.* Oxford Ser.17 (1966), 132-143.

[2] G.E. Andrews, The Theory of Partitions, Addison-Wesley, Reading, Massachusetts, 1976.