Beyond Burnside's Lemma

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Abstract

An extension of Burnside's lemma is presented along with its suggested implementation in computer code. The extension is along the lines of de Bruijn's work, which itself is a generalization of Pólya's theory of counting. As an example, in addition to counting the number of distinguishable colorings of a checkerboard if rotations and reflections are allowed, our extension allows the colors themselves to be permuted. The historical context is briefly discussed. Examples are given along the way to illuminate the discussion.

1 Burnside's lemma and an example

Burnside's lemma is a way to count the number of distinguishable objects in a set, S. It is convenient for computational purposes to consider the objects of S as distinct colorings of some figure with n_v vertices, with n_c possible colors for each vertex. (A "vertex" might not actually be what we think of as a vertex of the figure, but it is a part of the figure that is to be colored. For example, if we color a 3×3 checkerboard, we consider all nine squares to be vertices.) Now let G be a group of elements that permute vertices of the objects in S such that G acts on S. Two colorings are considered indistinguishable with respect to G if there is some element $g \in G$ such that g sends one coloring to the other coloring. If g sends the coloring s_1 to the coloring s_2, we will write $g(s_1) = s_2$. Since G is a group, indistinguishability in this sense gives an equivalence relation on the colorings. If we number the vertices using the integers between 1 and n_v, we can consider G to be a subgroup of $\mathrm{Sym}(n_v)$, the symmetric group consisting of permutations of the first n_v natural numbers.

The equivalence class of a particular coloring, $s \in S$, which consists of all indistinguishable colorings to which s belongs is called the *orbit* of s, denoted $\mathrm{Orb}(s)$, and is defined as the following:

$$\mathrm{Orb}(s) = \{g(s) | g \in G\}.$$

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Let N be the number of distinguishable colorings in S. Then N is the total number of orbits that partition the set S. Burnside's lemma gives the result,

$$N = \frac{1}{|G|} \sum_{g \in G} \psi(g) \tag{1}$$

where the sum is over all the elements of G and $\psi(g)$ is the size of the set of colorings in S which are unchanged when acted upon by the element g.[2] Symbolically,

$$\psi(g) = |\{s \in S | g(s) = s\}|. \tag{2}$$

Consider the 2×2 checkerboard where each square could be painted with three colors, say black, white, and gray. In this example, $n_v = 4$ since there are four vertices, and since we have three possible color choices, $n_c = 3$. That makes the total number of colorings, $|S|$, equal to $3^4 = 81$.

The group we will consider is the symmetries of the square, which has eight elements. When the checkerboard's vertices are labeled as in Figure 1, the elements will appear as the permutations listed in Table 1. We define a new quantity, $x(g)$, which is the number of vertex cycles within the permutation g when it is written in disjoint cycle form and all numbers 1 through n_v can be seen. (Every permutation can be written in disjoint cycle form, and, for a given permutation, the number of disjoint cycles needed is always the same, so $x(g)$ is well-defined. We give an algorithm for writing a permutation in disjoint cycle form in Section 2.) For our example, the identity element is commonly written as (1), but it must be represented as (1)(2)(3)(4) for $x(g)$. Therefore $x = 4$ for the identity, whereas the 90° rotation (1 2 3 4) has $x = 1$.

The following is used to calculate $\psi(g)$ for Table 1:

$$\psi(g) = n_c^{x(g)}. \tag{3}$$

Recall the definition of $\psi(g)$ in Equation 2, which is the size of the set of colorings which are unchanged when acted upon by g. In order to preserve a coloring when g acts on it, all vertices within a vertex cycle must have the same color. Therefore Equation 3 is true because each vertex cycle in g can be painted with one of n_c color choices.

Description of g	Cycle form of g	$\psi(g)$
Identity	(1)(2)(3)(4)	3^4
90° Rotation	(1 2 3 4)	3^1
180° Rotation	(1 3)(2 4)	3^2
270° Rotation	(1 4 3 2)	3^1
Vertical reflection	(1 2)(3 4)	3^2
Horizontal reflection	(1 4)(2 3)	3^2
Diagonal reflection 1	(1)(2 4)(3)	3^3
Diagonal reflection 2	(1 3)(2)(4)	3^3

Table 1: Symmetries of the square applied to 2×2 checkerboard with 3 colors.

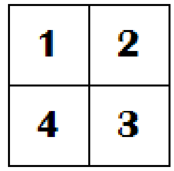

Figure 1: 2×2 checkerboard with vertices labeled.

Using Equation 1, the number of distinguishable boards is therefore

$$N = \frac{1}{|G|}\sum_{g\in G}\psi(g) = \frac{1}{8}\cdot\left(3^4+3^1+3^2+3^1+3^2+3^2+3^3+3^3\right) = \frac{1}{8}\cdot 168 = 21.$$

The boards are displayed in Figure 2.

2 Using computers to do Burnside calculations with larger groups

It would be tedious to do all the calculations for each $\psi(g)$ by hand when the group gets large. And since $\psi(g)$ has a nice formula, if we can find a way to represent g in cycle form, then we could automate the process. Not only that, we can also automate the creation of our group, G. If we list a couple of elements that represent the basic movements, we can expand the set through composing elements together. If composing two elements produces an element not within the set, we should add that into our set. Once all possible combinations have been checked and no new elements are found, the set is then closed, and it will form a group. The initial set is said to *generate* the group that we find. For example, the 90° rotation and any reflection are sufficient to generate the 8 symmetries of the square.

Lemma 1. *A closed, nonempty subset of a finite group G is a subgroup of G.*

Proof. A subgroup must be nonempty, closed, and it must contain the identity and have inverses. The first two are given. For every a in the set, a^k is also in the set for all k in the natural numbers, because the set is closed. Since G is finite, there must be some natural number m such that a^m is the identity. (The smallest such m is the *order* of a.) Therefore the subset contains the identity, a^m, and also a^{m-1}, the inverse of a. □

A convenient way for a computer to use permutations is for it to store them as ordered lists. For the 2×2 checkerboard example, the permutations would be represented as $a = [a_1, a_2, a_3, a_4]$, with each a_i taking on a value from 1 to 4 such that each number is represented in the set of a_is. Since this is an ordered list, it acts very much like a mapping, showing that 1 would be sent to a_1, and 2 to a_2, and so on. In most code there is a simple way to find out a_i by asking for $a(i)$. (As a side note, in some languages the first element in the list is $a(0)$, and for the purposes of permutations we might find it convenient to redefine the symmetric group as acting on the numbers 0 to $n_v - 1$.)

If we have two permutations, a as defined above and b defined in a similar manner, then the new element $c = a \circ b$ is another list whose elements can be computed by $c(i) = a(b(i))$. For expanding a set into a group, an algorithm can be written to compose elements and add the results into the set if they are different than the other elements of the set, then in turn compose the new elements with all the others, and repeat this process as many times as necessary to ensure closure.

Once the set is closed, it is a subgroup of Sym(n_v) by Lemma 1. If we figure out how to calculate $x(g)$, then $\psi(g)$ can be calculated for all group elements, given a particular n_c. An algorithm to create a list of the vertex cycles is given below. As an example, the permutation [1, 4, 3, 2] would be converted into cycle form as [[1], [2, 4], [3]]. Then $x(g)$ would be the number of elements in the cycle form list. In this example, $x = 3$, since there are three elements in the list ([1], [2, 4], and [3]). There is usually a list function that returns the length of a list, so $x(g) =$ length(cycle form of g).

The following pseudo-code requires a language where the manipulation of the size of a list can be done. Ways around this could be found, but will not be explored in this paper.

```
Algorithm: given pmtn, a permutation, return it in cycle form
    cycleform = [ [ 1 ] ]          :: Cycle form always starts with 1.
    nlist = [ 2, 3, 4, ..., n_v ]  :: The list of numbers that need to go into the cycle,
                                   :: will shrink as the numbers are added in.
    i = 1                          :: Indicates current location in cycleform.
    j = 1                          :: Indicates location within a given cycle in cycleform.
    while length( nlist ) > 0:     :: Go until all numbers are seen in the cycle.
        z = cycleform(i)(j)        :: The current number in the cycle form.
        if pmtn(z) == z or pmtn(z) == cycleform(i)(0):
            :: Then the current cycle i is finished, begin a new one
            :: starting with the next number in nlist.
            remove nlist(1) from nlist and append to cycleform
            i = i + 1              :: Since we start a new cycle in cycleform.
            j = 1                  :: Return to the beginning of the cycle.
        else:
            :: Then z goes to pmtn(z), something that we have not yet considered.
            append pmtn(z) to cycleform(i)
            j = j + 1
            remove pmtn(z) from nlist
    return cycleform
```

3 Introducing color cycling into the elements of the group with two examples

The cycles that we have considered so far have permuted vertices. We thought to add the idea of switching colors into the elements of the group. We depict these new and improved group elements as [vertex cycles, color cycles]. To compose two elements, compose the vertex cycles like one normally would, and separately compose the color cycles. This group is then a subgroup of the direct product of $\mathrm{Sym}(n_v)$ and $\mathrm{Sym}(n_c)$. If the use of a computer is desired to speed things up, the colors can also be thought of as numbers, which will allow for easy composition.

These ideas have been thought of before. In 1964, N. G. de Bruijn considered a group of permutations on the vertices, V, and a group of permutations on the colors, C, though de Bruijn did not use these terms. De Bruijn used $V \times C$ as the group under consideration.[1] This $V \times C$ is similar to the improved G that we are studying, with one difference. $V \times C$ would by definition include every element of V with every element of C, whereas we can generate our G however we like, making G a subgroup of $V \times C$. (De Bruijn's techniques will work if G is the direct product of a subgroup of V and a subgroup of C, of course, but we also allow G to be a subgroup of $V \times C$ not formed in this way.)

As a historical aside, de Bruijn's work was an extension of G. Pólya's work. Pólya had generalized Burnside's lemma in 1937 in an elegant way, which not only expressed the number of distinguishable colorings but also abstractly represented them. But we shall seek only a number. We will consider some examples where $G = V \times C$ and where G is a subgroup of $V \times C$. For another perspective on these topics, see Reference [3].

Let us extend our previous example with the 2×2 checkerboard so that it deals with color cycling. Let G be

$$\langle\ [\ (1\ 2\ 3\ 4),\ (\mathrm{W})(\mathrm{G})(\mathrm{B})\ ],\ [\ (1\ 2)(3\ 4),\ (\mathrm{W})(\mathrm{G})(\mathrm{B})\ ],\ [\ (1)(2)(3)(4),\ (\mathrm{W\ B})(\mathrm{G})\ ]\ \rangle,$$

where

$$\langle g_1, g_2, \ldots, g_k \rangle$$

denotes the group generated by $g_1, g_2, \ldots, g_k$. The first two elements shown are ones we have encountered before. The first is a rotation of 90^o and the second is a reflection across the vertical; both leave the colors alone. The third element does nothing to the vertices, but switches white and black, leaving gray alone. Since we have separated the vertex movements from the color movements, our group will be exactly $V \times C$, where V is the symmetries of the square and $C = \{(\mathrm{W})(\mathrm{G})(\mathrm{B}), (\mathrm{W\ B})(\mathrm{G})\}$. Our group size is now 16, twice the original group size without the color cycling. The original elements now come in two flavors, one with an identity color cycle, and the other that switches white and black.

The trick now is to figure out what $\psi(g)$ is. Clearly, $\psi(g)$ should reduce to Equation 3 for elements whose color cycle is the identity. But using the same ideas that were used to derive Equation 3, one would argue that

$$\psi(g) = \prod_{i=1}^{x(g)} m_i(g) \tag{4}$$

where $m_i(g)$ is the number of allowed color choices for the i^{th} vertex cycle of g. Each $m_i(g)$ is just n_c if g has the identity color permutation. But if the color permutation is not the identity, $m_i(g)$ is more complicated. Just as $x(g)$ is the number of vertex cycles of g, let $r(g)$ be the number of color cycles of g. Also let $v_i(g)$ be the number of vertices in the i^{th}

vertex cycle, and $c_j(g)$ be the number of colors in the j^{th} color cycle. With this handful of notation, we can now write out an expression for $m_i(g)$.

$$m_i(g) = \sum_{j=1}^{r(g)} \begin{cases} c_j(g) & \text{if } c_j(g) | v_i(g) \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

The meaning of this equation is that for every color cycle, if the size of that color cycle divides into the size of the considered vertex cycle, then that vertex cycle can be painted with the colors in that color cycle. The first vertex in vertex cycle j has one of $c_j(g)$ choices to be painted, and then the rest of the vertices in the vertex cycle must follow the sequence in the color cycle. But if the size of the color cycle does not divide into the vertex cycle, there is no way to paint the vertex cycle with those colors without the color cycling of g changing the overall coloring into something different; therefore that coloring does not get added into $m_i(g)$. Notice that if even one $m_i(g)$ is zero, which translates into being unable to paint a vertex cycle and still preserve the coloring when g acts, then $\psi(g) = 0$ as a result of Equation 4.

A few color cycle examples are given for calculating $\psi(g)$ for the checkerboard with the $m_i(g)$s left explicit. The first example is with the black-white color swap considered above. The results are in the first few columns of Table 2. Most illuminating are the diagonal reflections (e.g. (1)(2 4)(3)). Using Burnside's lemma, we find that $N = 13$, and Figure 2 encloses indistinguishable boards with blue boxes. The second example uses $C = \langle$ (W G B) $\rangle$ instead of the $\langle$ (W B)(G) $\rangle$; we have calculated $\psi(g)$ in the last column of Table 2. With this example, there are 24 group elements, because (W G B)$\circ$(W G B) = (W B G), but none of the $\psi(g)$s for the group elements containing either one of these cycles contribute to the sum. With Burnside's lemma, $N = 7$, and Figure 2 encloses indistinguishable boards with purple boxes. When both the (W B) and (W G B) cycles are included in C, the group size jumps up to 48, but we only decrease N to 6.

Vertex cycles of g	$\psi(g)$ when g has color cycle (W)(G)(B)	$\psi(g)$ when g has color cycle (W B)(G)	$\psi(g)$ when g has color cycle (W G B) or (W B G)
(1)(2)(3)(4)	$3 \cdot 3 \cdot 3 \cdot 3$	$1 \cdot 1 \cdot 1 \cdot 1$	$0 \cdot 0 \cdot 0 \cdot 0$
(1 2 3 4)	3	3	0
(1 3)(2 4)	$3 \cdot 3$	$3 \cdot 3$	$0 \cdot 0$
(1 4 3 2)	3	3	0
(1 2)(3 4)	$3 \cdot 3$	$3 \cdot 3$	$0 \cdot 0$
(1 4)(2 3)	$3 \cdot 3$	$3 \cdot 3$	$0 \cdot 0$
(1)(2 4)(3)	$3 \cdot 3 \cdot 3$	$1 \cdot 3 \cdot 1$	$0 \cdot 0 \cdot 0$
(1 3)(2)(4)	$3 \cdot 3 \cdot 3$	$3 \cdot 1 \cdot 1$	$0 \cdot 0 \cdot 0$

Table 2: $\psi(g)$ for various three color cycles for the 2×2 checkerboard.

In general, the less our color cycle lengths divide into the vertex cycle lengths, the fewer distinguishable objects we will get in the end, because we are adding more group elements

($|G|$ increases) and the sum of the $\psi(g)$s is not getting bigger. The smallest number of distinguishable colorings for a given number of vertices and colors will be found when $G = \mathrm{Sym}(n_v) \times \mathrm{Sym}(n_c)$, which we think produces the partition function $q(n_v, n_c)$, the number of partitions of the number n_v with n_c or fewer addends.[4]

For an example without checkerboards, consider a triangle whose vertices are to be painted with three colors: green, orange, and blue. We compare when our group is the simple rotation group of the triangle,

$$\langle\ [\ (1\ 2\ 3),\ (\mathrm{G})(\mathrm{O})(\mathrm{B})\ ]\ \rangle$$

to when it has a color cycle,

$$\langle\ [\ (1\ 2\ 3),\ (\mathrm{G\ O\ B})\ ]\ \rangle.$$

In both cases, $|G| = 3$ and $N = 11$. The difference between the groups is that they have different equivalence classes; this is depicted in Figure 3. In the notation we introduced earlier, $V = \langle\ (1\ 2\ 3)\ \rangle$ and $C = \langle\ (\mathrm{G\ O\ B})\ \rangle$, but neither group is G is $V \times C$, but rather both are proper subgroups of $V \times C$.

References

[1] De Bruijn, N. G. (1964). Pólya's Theory of Counting. In E. F. Beckenbach (Ed.), *Applied Combinatorial Mathematics*. New York: John Wiley and Sons.

[2] Durbin, John R. (2005). *Modern Algebra: An Introduction*. 5$^{\text{th}}$ edition. Hoboken, NJ: John Wiley & Sons.

[3] Liu, Chung Laung (1968). *Introduction to combinatorial mathematics.* New York: McGraw-Hill.

[4] Weisstein, Eric W. "Partition Function q." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/PartitionFunctionq.html

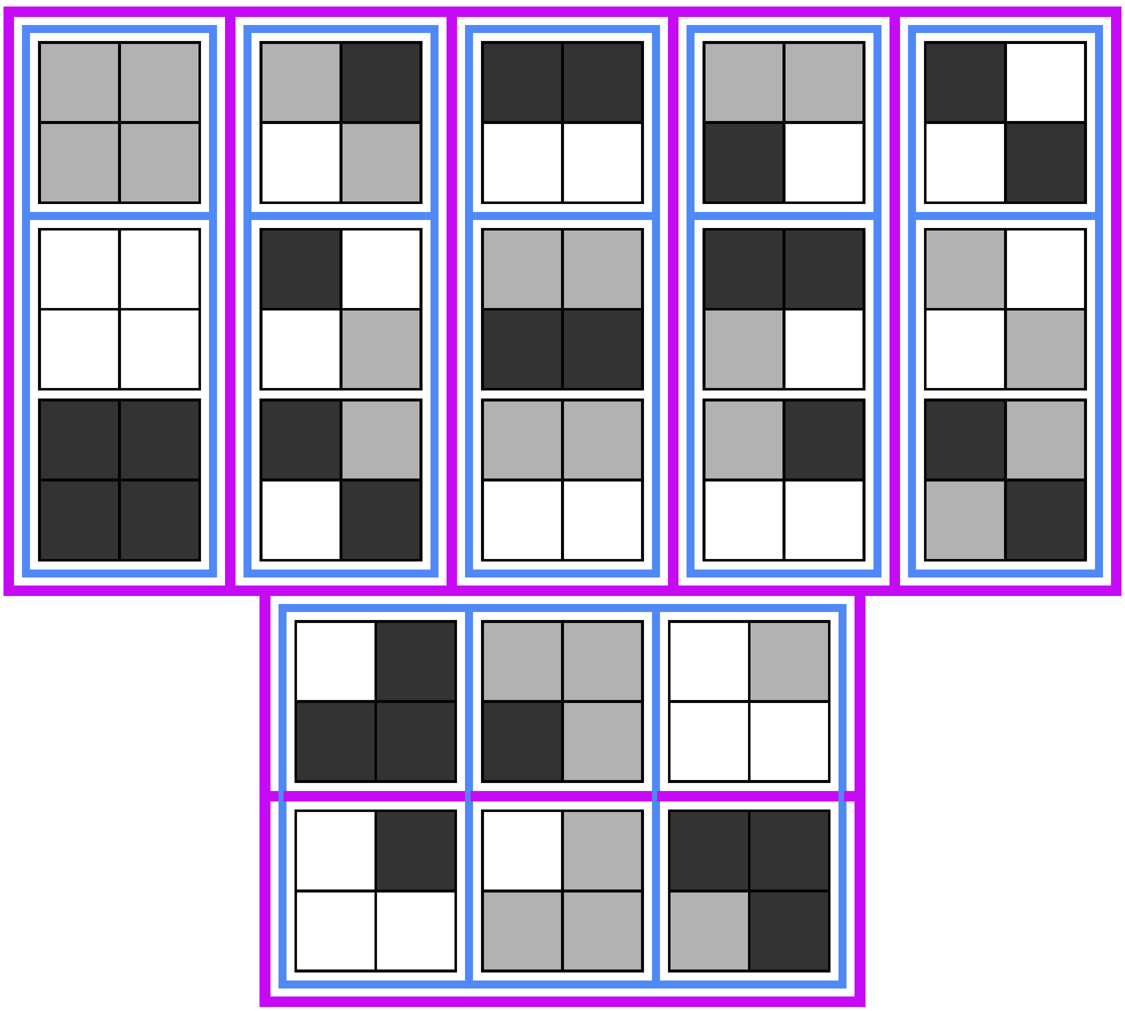

Figure 2: Distinguishable 2×2 checkerboards with 3 colors, 21 total. The blue boxes partition the boards into equivalence classes with respect to the group with the color cycle (W B)(G), giving a total of 13 distinguishable boards, whereas the purple boxes partition with respect to the cycle (W G B), giving 7 boards. With both color cycles included in the group, there are 6 distinguishable boards, with the lowermost six belonging to only one equivalence class.

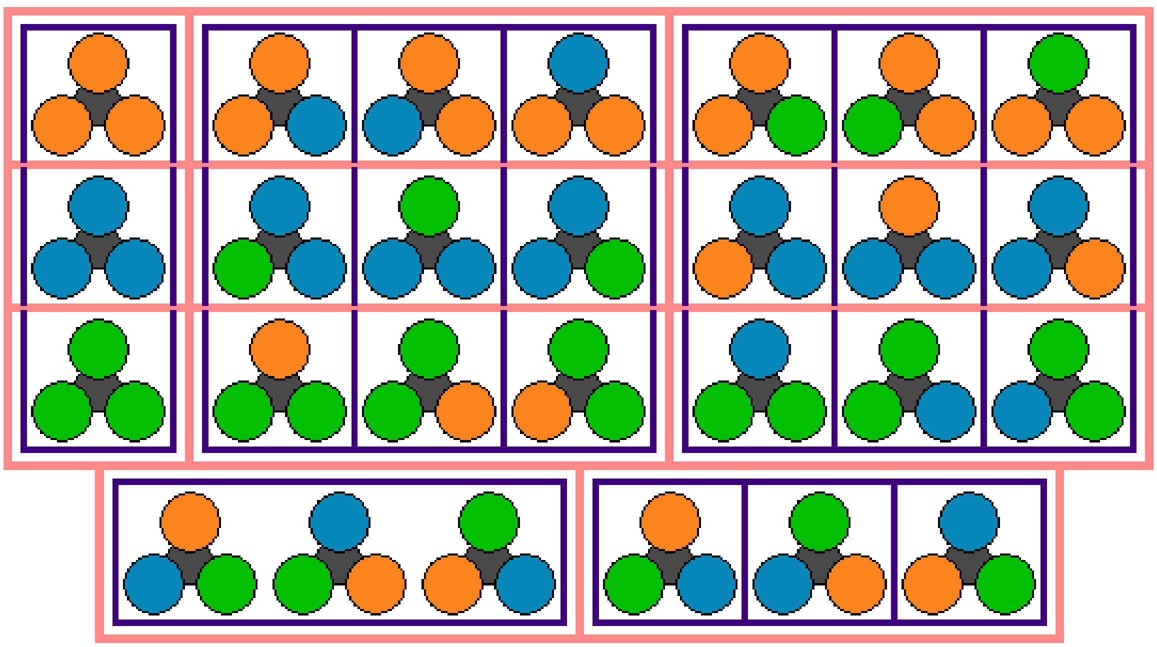

Figure 3: All 27 triangles with 3 colors: green, orange, and blue. The peach boxes partition the boards into equivalence classes with respect to the rotation group, and the violet boxes partition with respect to the group $\langle$ [(1 2 3), (G O B)] $\rangle$.