Counting Containment Partitions

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Abstract

The study of integer partitions has wide applications to mathematics, mathematical physics, and statistical mechanics. We consider the problem of finding a generalized approach to counting the partitions of an integer n that contain a partition of a fixed integer k. We use generating function techniques to count containment partitions and verify experimental results using a self-made in program *Mathematica*. We have found explicit solutions to the problem for general n with k=1, 2, 3, 4, 5, and 6. We also discuss open questions and ideas for future work.

1 Introduction to Partitions

Integer partitions have been studied for hundreds of years dating back to Leonhard Euler. They have applications in mathematics, mathematical physics, and statistical mechanics. For instance, nonparametric statistics use ideas involving restricted partition problems while particle physics use partition asymptotics and partition identities [1].
We begin with the definition of a partition.

Definition 1.1. A **partition** $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a weakly decreasing sequence of nonnegative integers where $\lambda_1, \lambda_2, \ldots, \lambda_k$ sum to a positive integer, n. The λ_i are called the **parts** of a partition.

A simple example of a partition with $n = 16$ is $\lambda = (7, 5, 3, 1)$. Each partition also has a graphical representation known as a Young diagram or Ferrers graph, which provide additional methods for studying partitions.

Definition 1.2. For a given partition λ, where $\sum \lambda_i = n$ the **Young diagram** Y_λ of shape λ is a left-justifed diagram of n boxes, with λ_i boxes in the i-th row.

Example 1.3. Below is an example of a Young diagram for a partition of 16.

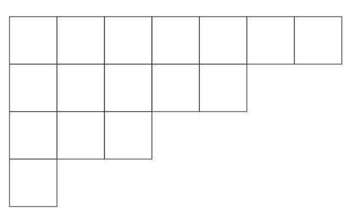

Figure 1: Young diagram for the partition of 16, $\lambda = (7, 5, 3, 1)$.

The most notable function relating to partitions, is the partition function. This function is simple to define, but mathematicians continue to find new properties with further study.

Definition 1.4. Let $p(n)$ equal the number of partitions of n. This is called the **partition function**.

Example 1.5. The partitions of 4 are:

4
3, 1
2, 2
2, 1, 1
1, 1, 1, 1

Thus, there are 5 partitions, so $p(4) = 5$.

We note that when $n = 0$, $p(0) = 1$. It is simple to explicitly count the number of partitions for small n, but as n increases slightly, $p(n)$ grows quite quickly. For example, $p(5) = 7$, $p(20) = 627$, and finally $p(100) = 190,569,292$, far too many to simply be counted. Our research looked at a subset of the partition function and took into account a number of its properties.

2 Containment Partitions

In this paper, we will consider the problem of enumerating containment partitions. A function for containment partitions became useful when counting the number of non-isomorphic subgraphs in a given graph G. We define containment partitions.

Definition 2.1. A partition $\lambda \vdash n$ **contains** a partition of k if there exist parts $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_j} \subset \lambda_1, \lambda_2, \ldots, \lambda_{|\lambda|}$ with $j \leq n$ such that $\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_j} = k$. Let $p(n \mid k)$ equal the number of partitions of n that contain some partition of k.

The definition of a containment partition can be easily understood through an example.

Example 2.2. The containment partition $p(4|2)$ can be counted directly. Again, the partitions of 4 are:

4
3, 1
2, **2**
2, **1**,**1**
1, 1, **1**, **1**

The partitions of 2 are (2) and $(1, 1)$, thus the last three partitions of 4 contain some partition of 2. So, $p(4|2) = 3$. Notice the last three partitions of 4 contain multiple partitions of 2, but each is counted only once.

We created a program in *Mathematica* to exhaustively count the number of containment partitions for any $p(n|k)$. We obtained the data in Table 1 with this program (See Appendix for more values of n and k). This allowed us to look for any patterns or trends in the containment partition numbers. Theorem 2.3 outlines the first pattern observed in Table 1.

Table 1: Containment partitions $p(n|k)$ for $n \leq 10$ and $k \leq 10$.

$k \setminus n$	1	2	3	4	5	6	7	8	9	10
1	1	1	2	3	5	7	11	15	22	30
2	0	2	2	3	5	8	11	17	23	33
3	0	0	3	3	5	6	11	15	23	30
4	0	0	0	5	5	8	11	14	22	33
5	0	0	0	0	7	7	11	15	22	25
6	0	0	0	0	0	11	11	17	23	33
7	0	0	0	0	0	0	15	15	23	30
8	0	0	0	0	0	0	0	22	22	33
9	0	0	0	0	0	0	0	0	30	30
10	0	0	0	0	0	0	0	0	0	42

Theorem 2.3. *For all integers $n > 0$, $p(n|n) = p(n+1|n)$.*

Theorem 2.3 is proved using a bijection. Obviously, the number of partitions of n that contain a partition of n is just the number of partitions of n. Adding a single 1 to each partition of n will result in all the partitions of $n+1$ that contain some partition of n. Conversely, by subtracting a 1 from all the partitions of $n+1$ that have a part of a single 1 results in all the partitions of n.

The second theorem observed in Table 1 outlines the symmetry seen between $p(n|k)$ and $p(n|n-k)$.

Theorem 2.4. *For all integers $n > 0$, $p(n|k) = p(n|n-k)$.*

Again we prove bijectively. A partition of n that contains some partition of k will have remaining parts equal to $n-k$. Thus, the partition of n will contain some partition of $n-k$. Conversely, a partition of n that contains some partition of $n-k$ will have remaining parts equal to k.
This result is useful; once values for $p(n|k)$ for k from 1 to $\frac{n}{2}$ are found, by the symmetry property the values of $p(n|k)$ for k from $\frac{n}{2}+1$ to $n-1$ are also found. Now we turn to the main focus of our study, looking for a way to enumerate containment partitions.

3 Generating Functions for Containment Partitions

In mathematics, a generating function is a formal power series whose coefficients represent information about a sequence in question. Thus, generating functions are particularly useful in transforming problems about sequences to problems about functions. Leonhard Euler discovered a generating function for the partition function in the 18th century. To construct this generating function Euler began by taking the infinite product of geometric sequences with a common factor q^k for all k from 1 to ∞. This is illustrated below:

$$(1 + q^1 + q^2 + q^3 + \cdots)(1 + q^2 + q^4 \cdots)(1 + q^3 + q^6 \cdots)(1 + q^4 + q^8 \cdots)$$

This product can be grouped as follows:

$$(1+q^1+q^{1+1}+q^{1+1+1}+\cdots)(1+q^2+q^{2+2}+q^{2+2+2}\cdots)(1+q^3+q^{3+3}+\cdots)(1+q^4+q^{4+4}\cdots)\cdots$$

Next we can rearrange the q terms so that all the terms with equal exponents are grouped together. Notice how every possible combination of these integers is present:

$$1+q^1+(q^{1+1}+q^2)+(q^{1+1+1}+q^{2+1}+q^3)+(q^{1+1+1+1}+q^{2+1+1}+q^{2+2}+q^{3+1}+q^4)\cdots$$

Finally, by summing like powers we get a series in which the coefficient of the q^{nth} term equals $p(n)$:

$$1+q+2q^2+3q^3+5q^4\cdots \tag{1}$$

For instance, the $5q^4$ term in equation (1) represents that there are five partitions of the integer four. Since a geometric series with common factor q converges to $1/(1-q)$; the definition of the generating function for the partition function is as follows.

Definition 3.1. The generating function for the partition function $p(n)$, is

$$\sum_{n=1}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k} \tag{2}$$

To create generating functions for containment partitions we use the relationship $p(n|k) = p(n) - p(n|\bar{k})$, where the notation $p(n|\bar{k})$ signifies the number of partitions of some integer n that do no contain any partition of some integer k. To construct a generating function for $p(n|\bar{k})$, restrictions were put on the generating function for $p(n)$ such that no combination of positive integers that sum to k are included. In the following sections we show you how to construct the generating functions for $p(n|1)$, $p(n|2)$, and $p(n|3)$. We also list the generating functions for $p(n|4)$, $p(n|5)$, and $p(n|6)$.

3.1 Generating Function for $p(n|1)$

The generating function $\sum_{n=1}^{\infty} p(n|1)$ is simple to create. Note that $\sum_{n=1}^{\infty} p(n|1) = \sum_{n=1}^{\infty} p(n) - \sum_{n=1}^{\infty} p(n|\bar{1})$. We know $\sum_{n=1}^{\infty} p(n) = \prod_{k=1}^{\infty} \frac{1}{1-q^k}$, thus, it remains to find the generating function $\sum_{n=1}^{\infty} p(n|\bar{1})$. The function $\sum_{n=1}^{\infty} p(n|\bar{1})$ is constructed by placing restrictions on the the generating function for partitions. Obviously the only partition of 1 is (1). So clearly, $p(n|\bar{1})$ equals all partitions of n with parts of size 2 or greater. Thus, the generating function $\sum_{n=1}^{\infty} p(n|\bar{1})q^n$ is shown below:

$$\sum_{n=1}^{\infty} p(n|\bar{1})q^n = \sum_{n=1}^{\infty} p(n \text{ with parts size 2 or greater})q^n = \prod_{k=2}^{\infty} \frac{1}{1-q^k} \tag{3}$$

Hence,

Theorem 3.2.

$$\sum_{n=1}^{\infty} p(n|1)q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k} - \prod_{k=2}^{\infty} \frac{1}{1-q^k} \tag{4}$$

To search for relationships in the containment partitions, we computed all of our generating functions with a common denominator. So, to find a common denominator for $\sum_{n=1}^{\infty} p(n|1)q^n$ we multiplied the $\sum_{n=1}^{\infty} p(n|\bar{1})q^n$ term by (1-q)/(1-q). With this common denominator our generating function simplifies to:

$$\sum_{n=1}^{\infty} p(n \mid 1)q^n = \frac{q}{\prod_{k=1}^{\infty} 1-q^k}$$

Recall our explanation of the generating function for all partitions. A single q represents a single 1. Thus, the function $\frac{q}{\prod_{k=1}^{\infty} 1-q^k}$ represents all partitions of n with at least a single 1. By removing a 1 from all the partitions of n that contain at least a single 1, we get all the partitions of $n-1$.

Corollary 3.3. *For all integers $n > 0$, then $p(n|1) = p(n-1)$.*

For the generating functions of $p(n|\bar{k})$ for k =2,3,4,5, and 6 we repeat this process of finding a common denominator. This simple algebra that can be run in *Mathematica* allows us to look at the containment partitions in a new perspective. We'll discuss an interesting pattern we found by listing the containment partitions in this format later in the paper.

3.2 Generating Function for $p(n|2)$

To find a generating function for $p(n|2)$, we again use the relationship that $p(n|2) = p(n) - p(n|\bar{2})$. Clearly, (2) and $(1,1)$ are the only partitions of 2. So to find $\sum_{n=1}^{\infty} p(n|\bar{2})$, restrictions must be put on $\sum_{n=1}^{\infty} p(n)$ such that no more than a single 1, and no 2's are represented. Thus, the generating function will represent all partitions of n that contain parts 3 or greater, or parts 3 or greater and a single 1. The generating function for partitions of n with parts 3 or greater is shown below:

$$\sum_{n=1}^{\infty} p(n \text{ with parts size 3 or greater})q^n = \prod_{k=3}^{\infty} \frac{1}{1-q^k} \tag{5}$$

If we multiply this generating function by q^1, the function for the partitions of n with parts size 3 or greater and a single 1 is formed. This function is shown below:

$$\sum_{n=1}^{\infty} p(n \text{ containing a single 1 and parts size 3 or greater})q^n = q\prod_{k=3}^{\infty} \frac{1}{1-q^k} \tag{6}$$

By summing equations (5) and (6), we get

$$\sum_{n=1}^{\infty} p(n|\bar{2})q^n = \prod_{k=3}^{\infty} \frac{1}{1-q^k} + q\prod_{k=3}^{\infty} \frac{1}{1-q^k}$$

which results in

$$\sum_{n=1}^{\infty} p(n|\bar{2})q^n = (1+q)\prod_{k=3}^{\infty} \frac{1}{1-q^k}$$

Since $p(n|2) = p(n) - p(n|\bar{2})$:

Theorem 3.4.

$$\sum_{n=1}^{\infty} p(n \mid 2)q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k} - (1+q)\prod_{k=3}^{\infty} \frac{1}{1-q^k} \tag{7}$$

With a common denominator the generating function reduces to

$$\sum_{n=1}^{\infty} p(n \mid 2)q^n = \frac{2q^2 - q^4}{\prod_{k=1}^{\infty} 1-q^k}$$

from which Corollary 3.5 arises.

Corollary 3.5. *For all integers $n > 0$, then $p(n|2) = 2p(n-2) - p(n-4)$.*

3.3 Generating Function for $p(n|3)$

To find the generating function for $p(n|3)$, we need to produce the generating function for $p(n|\bar{3})$. The three partitions of 3 include (3), $(2,1)$, and $(1,1,1)$. Thus, we put restrictions on the generating function of $p(n)$ such that no combination of these integers are included. With $k = 3$ we organize the production of $\sum_{n=1}^{\infty} p(n|\bar{k})$ into three parts. The first, being all partitions of n with all parts size greater than k. Thus, for $k = 3$ this is the generating function for partitions of n with all parts size 4 or greater. The function is shown below:

$$\sum_{n=1}^{\infty} p(n \text{ with parts size 4 or greater})q^n = \prod_{k=4}^{\infty} \frac{1}{1-q^k} \tag{8}$$

The second term encompasses partitions of n with parts greater than size k combined with at least one part less than size k that sum to less than k. Thus, for all k this second part is equal $(p(1)q^1 + p(2)q^2 + \cdots + p(k-1)q^{k-1})(\prod_{k=k+1}^{\infty} \frac{1}{1-q^k})$. Following this form for $k = 3$, the generating function for partitions of 3 with parts greater than size 3 and at least one part less than size 3 that does sums to less than 3 is shown below:

$$(q + 2q^2) \prod_{k=4}^{\infty} \frac{1}{1-q^k} \tag{9}$$

The final term for the generating function of $p(n|\bar{k})$ contains parts greater than size k and parts less than size k that sum to more than k. For $k = 3$, this term consists of partitions of n with parts size 4 or greater and more than a single 2. Recall that $1/(1-q^2)$ represents all possible quantities of 2's in a partition of n. Thus, $q^{2+2}/(1-q^2) = q^4/(1-q^2)$ represents all possible quantities greater than one, of 2's . The generating function for this term with respect to $k = 3$ is shown below:

$$\sum_{n=1}^{\infty} p(n \text{ with parts size 4 or greater and at least two 2's})q^n = q^4/(1-q^2) \prod_{k=4}^{\infty} \frac{1}{1-q^k} \tag{10}$$

Combining the three parts represented by equations (8), (9) and (10) we get the generating function for $p(\bar{3})$:

$$\sum_{n=1}^{\infty} p(n|\bar{3})q^n = [1 + (q + 2q^2) + (\frac{q^4}{1-q^2})] \prod_{k=4}^{\infty} \frac{1}{1-q^k}$$

Thus,

Theorem 3.6.

$$\sum_{n=1}^{\infty} p(n \mid 3)q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k} - (1 + q + 2q^2 + \frac{q^4}{1-q^2}) \prod_{k=4}^{\infty} \frac{1}{1-q^k} \tag{11}$$

We used *Mathematica* to find a common denominator that transforms the generating function to:

$$\sum_{n=1}^{\infty} p(n \mid 3)q^n = \frac{3q^3 - q^5 - 2q^6 + q^8}{\prod_{k=1}^{\infty} 1-q^k}$$

From which Corollary 3.7 arises.

Corollary 3.7. *For all integers $n > 0$, then $p(n|3) = 3p(n-3) - p(n-5) - 2p(n-6) + p(n-8)$.*

3.4 Generating Function for $p(n|4)$

Using similar techniques we found the generating functions for the containment partitions of $p(n|4)$, $p(n|5)$, and $p(n|6)$. These functions are shown on the next page.

Theorem 3.8.

$$\sum_{n=1}^{\infty} p(n \mid 4)q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k} - (1 + q + 2q^2 + 3q^3 + \frac{q^6+q^5}{1-q^3}) \prod_{k=5}^{\infty} \frac{1}{1-q^k} \tag{12}$$

With a common denominator Theorem 3.8 looks as so:

$$\sum_{n=1}^{\infty} p(n \mid 4)q^n = \frac{5q^4 - 2q^6 - 2q^7 - 5q^8 + 2q^9 + 2q^{10} + 2q^{11} + q^{12} - 2q^{13}}{\prod_{k=1}^{\infty} 1-q^k}$$

and its resulting corollary.

Corollary 3.9. *For all integers* $n > 0$, *then* $p(n|4) = 5p(n-4) - 2p(n-6) - 2p(n-7) - 5p(n-8) + 2p(n-9) + 2p(n-10) + 2p(n-11) + p(n-12) - 2p(n-13)$.

3.5 Generating Function for $p(n|5)$

The generating function for $p(n|5)$ and corollary.

Theorem 3.10.

$$\sum_{n=1}^{\infty} p(n \mid 5)q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k} - \frac{(1 + q + 2q^2 + 3q^3 + 5q^4 + \frac{q^6}{1-q^2} + \frac{q^6+q^7}{1-q^3} + \frac{q^8}{1-q^4} + \frac{q^6}{(1-q^2)(1-q^4)} + \frac{q^7}{(1-q^3)(1-q^4)})}{\prod_{k=6}^{\infty} \frac{1}{1-q^k}} \tag{13}$$

With a common denominator we have

$$\sum_{n=1}^{\infty} p(n \mid 5)q^n = \frac{7q^5 - 3q^7 - 3q^8 - 4q^9 - 5q^{10} + 3q^{11} + 6q^{12} + 3q^{13} + 2q^{14} - q^{15} - 3q^{16} - 3q^{17} + 2q^{19}}{\prod_{k=1}^{\infty} 1-q^k}$$

resulting in the corollary.

Corollary 3.11. *For all integers* $n > 0$, *then* $p(n|5) = 7p(n-5) - 3p(n-7) - 3p(n-8) - 4p(n-9) - 5p(n-10) + 3p(n-11) + 6p(n-12) + 3p(n-13) + 2p(n-14) - p(n-15) - 3p(n-16) - 3p(n-17) + 2p(n-19)$.

3.6 Generating Function for $p(n|6)$

The generating function for $p(n|6)$ and corollary.

Theorem 3.12.

$$\sum_{n=1}^{\infty} p(n \mid 6)q^n = \tag{14}$$

$$\prod_{k=1}^{\infty} \frac{1}{1-q^k} - \frac{(1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + q^7 + \frac{q^7+2q^8+q^9}{(1-q^4)} + \frac{q^7+q^8+q^9+2q^{10}+q^{12}}{(1-q^5)} + \frac{q^9}{(1-q^4)(1-q^5)} + \frac{q^{12}}{(1-q^4)(1-q^5)})}{\prod_{k=7}^{\infty} \frac{1}{1-q^k}}$$

Theorem 3.12 is eqivalent to

$$\sum_{n=1}^{\infty} p(n \mid 6)q^n =$$

$$\frac{11q^6-5q^8-5q^9-7q^{10}-2q^{11}-2q^{12}+11q^{13}+12q^{14}+7q^{15}+5q^{16}-2q^{17}-2q^{18}-13q^{19}-3q^{20}-2q^{21}+2q^{22}+4q^{23}+4q^{24}+2q^{25}-4q^{26}}{\prod_{k=1}^{\infty} 1-q^k}$$

and its corresponding corollary.

Corollary 3.13. *For all integers $n > 0$, then* $p(n|6) = 11p(n-6) - 5p(n-8) - 5p(n-9) - 7p(n-10) - 2p(n-11) - 12p(n-12) + 11p(n-13) + 12p(n-14) + 7p(n-15) + 5p(n-16) - 2p(n-17) - 2p(n-18) - 13p(n-19) - 3p(n-20) - 2p(n-21) + 2p(n-22) + 4p(n-23) + 4p(n-24) + 2p(n-25) - 4p(n-26)$.

4 Open Problems/Future Work

We were able to find the generating functions for partition containment for all n with k up to 6. Additionally, we are confident we could find generating functions for partition containment for any fixed k, however, it would become increasingly exhaustive as k gets larger. Finding a general way to count containment partitions is our ultimate goal. Future work could also look to other general forms of partition containment as precursors to the general form of $p(n|k)$. For instance, research into to $p(2n|n)$ or $p(2n+1|n)$ would be valuable functions to consider.

We think generating functions are the proper means of reaching our goal. The numerators of the generating functions for $k = 1, 2, 3, 4, 5, 6$ didn't simplify or factor to any simpler forms. That may have helped us find a pattern that could lead to a generalizable generating function of $p(n|k)$ for all k. Yet, we saw a few trends in our generating functions that need further investigation. When we looked at our generating functions with common denominators, the terms in the numerator followed an interesting inclusion/exclusion pattern that needs further study. Let us notice the first terms in the numerators of our generating functions with common denominators. For $p(n|1)$, this term is q, for $p(n|2)$ it is $2q^2$, $p(n|3)$ is $3q^3$, $p(n|4)$ is $5q^4$, $p(n|5)$ is $7q^5$, and $p(n|6)$ is $11q^6$. These terms follow the form $p(k)q^k$. We also see a pattern in the highest power terms in the numerator of these generating functions. The highest power term for $p(n|1)$ is q, for $p(n|2)$, $-q^4$; for $p(n|3)$, q^8; for $p(n|4)$, $-2q^{13}$; for $p(n|5)$, q^{19}; and finally for $p(n|6)$ the last term is $-4q^{26}$. Observe that for consecutive k, the difference in the powers, of the highest power terms is consecutive! Also, we notice that the signs of the highest power terms alternate for the consecutive k found thus far.

Additional work could look into the asymptotics of the partition containment function with fixed k such as the $\lim_{n\to\infty} \frac{p(n|k)}{p(n)}$.

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References

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Appendix

The following is *Mathematica* code for the program that counts containment partitions $p(n|k)$. The function "func" indiciates with true or false whether a partition k is contained in some partition n.

```
func[n_, k_] :=
 Module[{p, i, nn, big, j, l, kk, x, y, z, q, r, s, t, part, test},
  p = n;
  l := {};
  nn = Length[p];
  big = p[[1]];
  Do[AppendTo[l, Count[p, i]],
   {i, 1, Sum[p[[j]], {j, 1, nn}]}];
  x = k;
  y := {};
  kk = Length[x];
  z = x[[1]];
  Do[AppendTo[y, Count[x, q]],
   {q, 1, Sum[x[[r]], {r, 1, kk}]}];
  t = PadRight[y, Sum[p[[j]], {j, 1, nn}]];
  part = l - t;
  test = True;
  Do[If[! NonNegative[part[[i]]], test = False], {i, 1,
    Length[part]}];
  If[test == False, Return[False], Return[True]];
  ]
```

The function "contain" indicates with a true or false whether a partition of some integer k is contained in the input partition, par.

```
contain[par_, kk_] := Module[{n, l, k, park, p, ans, i, j},
  p = par;
  l = Length[p];
  n = Sum[p[[i]], {i, 1, l}];
  k = kk;
  park = IntegerPartitions[k];
  ans = False;
  Do[If[func[p, park[[i]]], ans = True], {i, 1, Length[park]}];
  Return[ans]]
```

The function "finalcontain" counts the number of partitions of some integer k contained in the partitions of some integer n.

```
finalcontain[nn_, kk_] := Module[{n, l, k, parn, count, i, j},
  n = nn;
  k = kk;
```

```
  parn = IntegerPartitions[n];
  count = 0;
  Do[If[contain[parn[[i]], k] == True, count++], {i, 1,
    Length[parn]}];
  Return[count]]
```

The function "CP" will output a list of the number of partitions of some integer k contained in the partitions of all the integers up to n with the list starting from k.

```
CP[nn_, kk_] := Module[{n, k, i, x, j, l},
  k = kk;
  n = nn;
  x := {};
  Do[AppendTo[x, finalcontain[i, k]], {i, 1, n}];
  j = Drop[x, k - 1];
  Return[j]]
```

Table 2: Containment partitions $p(n|k)$ for $n \leq 21$ and $k \leq 21$.

k \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490	627
2	0	2	2	3	5	8	11	17	23	33	45	62	82	112	146	193	251	327	418	539	683
3	0	0	3	3	5	6	11	15	23	30	44	58	81	105	144	185	246	315	412	522	673
4	0	0	0	5	5	8	11	14	22	33	44	62	82	111	146	196	252	330	425	546	694
5	0	0	0	0	7	7	11	15	22	25	43	56	80	105	144	181	248	313	413	524	680
6	0	0	0	0	0	11	11	17	23	33	43	53	79	110	146	195	253	335	423	547	697
7	0	0	0	0	0	0	15	15	23	30	44	56	79	89	140	181	246	312	421	524	688
8	0	0	0	0	0	0	0	22	22	33	44	62	80	110	140	167	243	322	414	546	691
9	0	0	0	0	0	0	0	0	30	30	45	58	82	105	146	181	243	278	409	515	687
10	0	0	0	0	0	0	0	0	0	42	42	62	81	111	144	195	246	322	409	480	666
11	0	0	0	0	0	0	0	0	0	0	56	56	82	105	146	181	253	312	414	515	666
12	0	0	0	0	0	0	0	0	0	0	0	77	77	112	144	196	248	335	421	546	687
13	0	0	0	0	0	0	0	0	0	0	0	0	101	101	146	185	252	313	423	524	691
14	0	0	0	0	0	0	0	0	0	0	0	0	0	135	135	193	246	330	413	547	688
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	176	176	251	315	425	524	697
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	231	231	327	412	546	680
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	297	297	418	522	694
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	385	385	539	673
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	490	490	683
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	627	627
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	792