$L(d, 2, 1)$-Labeling of Simple Graphs

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Abstract

Radio signal interference can be modeled using distance labeling where the labels assigned to each vertex depend on the distance between vertices and the strength of the radio signal. This paper assumes three levels of signal interference within a graph, G. In the graph, if the distance between any two vertices, v_x, v_y, is 1 ($D(v_x, v_y) = 1$) then the difference of their labels must be at least d ($|f(v_x) - f(v_y)| \geq d$). Similarly, if $D(v_x, v_y) = 2$, then $|f(v_x) - f(v_y)| \geq 2$; and if $D(v_x, v_y) = 3$, then $|f(v_x) - f(v_y)| \geq 1$. The $L(d, 2, 1)$-labeling number $k_d(G)$ of G is the smallest positive integer k_d such that G has an $L(d, 2, 1)$-labeling with k_d as the maximum label.

This paper expands on previous results for the case of $d = 3$ and presents a general k_d-value for paths, bipartite graphs, complete graphs, and cycles that is d-dependent for any $d \geq 3$.

Keywords: distance labeling, distance labelling, radio labeling

1 Introduction

The channel assignment problem is an engineering problem in which the task is to assign a channel (non-negative integer) to each FM radio station in a set of given stations such that there is no interference between stations and the span of the assigned channels is minimized. The level of interference between any two FM radio stations correlates with the geographic locations of the stations. Closer stations have a stronger interference, and thus there must be a greater difference between their assigned channels.

In 1980, Hale introduced a graph theory model of the channel assignment problem where the problem was represented using the idea of vertex coloring. [4]. Vertices on the graph correspond to the radio stations and the edges show the proximity of the stations.

In 1991, Roberts proposed a variation of the channel assignment problem in which the FM radio stations were considered either "close" or "very close." "Close" stations were vertices of distance two apart on the graph and were assigned channels that differed by two; stations that were considered "very close" were adjacent vertices on the graph and were assigned distinct channels [6].

More precisely, Griggs and Yeh defined the $L(2,1)$-labeling of a graph as a function f which assigns to every vertex a label from the set of positive integers such that the following conditions are satisfied: $|f(v_x) - f(v_y)| \geq 2$ if the distance between v_x, v_y, $D(v_x, v_y)$, $= 1$ and $|f(v_x) - f(v_y)| \geq 1$ if $D(v_x, v_y) = 2$ [3]. $L(2,1)$-labeling has been studied in recent years.

In 2001, Chartrand et al. introduced the radio-labeling of graphs; this was motivated by the regulations for the channel assignments in the channel assignment problem [1]. Radio-labeling takes into consideration the diameter of the graph, and as a result, every vertex is related.

Practically, interference among channels may go beyond two levels. $L(3,2,1)$-labeling naturally extends from $L(2,1)$-labeling by taking into consideration vertices which are within a distance of three apart, but it remains less difficult than radio-labeling. In this paper the $L(d,2,1)$-labeling number for paths, cycles, complete graphs and complete bipartite graphs is determined. The results of Clipperton et al [2] are used as a basis for the unknown value d. [Introduction adapted slightly from that by Clipperton et al [2].]

2 Definitions and Notation

The definitions and some notations used in this section are adopted from those used by Clipperton, Gehrtz, Torkornoo, and Zsanizslo [2].

Definition 1. Let $G = (V, E)$ be a graph and f be a mapping $f : V \longrightarrow \mathbb{N}$. The distance between two such vertices is represented by $D(v_x, v_y)$ and the mapping of f is an $L(d,2,1)$-*labeling* of G if for all vertices $v_x, v_y \in V$,

$$|f(v_x) - f(v_y)| \geq \begin{cases} d, & \text{if } D(v_x, v_y) = 1; \\ 2, & \text{if } D(v_x, v_y) = 2; \\ 1, & \text{if } D(v_x, v_y) = 3. \end{cases}$$

Definition 2. The $L(d,2,1)$-*number*, $k_d(G)$, of a graph G is the smallest natural number k_d such that G has an $L(d,2,1)$-labeling with k_d as the maximum label. An $L(d,2,1)$-labeling of a graph G is called a *minimal* $L(d,2,1)$-*labeling* of G if, under the labeling, the highest label of any vertex is $k_d(G)$.

If 1 is not used as a vertex label in an $L(d,2,1)$-labeling of a graph, then every vertex label can be decreased by one to obtain another $L(d,2,1)$-labeling of the graph. Therefore in a minimal $L(d,2,1)$-labeling, 1 will necessarily appear as a vertex label.

Definition 3. A graph G, where $G = (V, E)$, is called a *complete graph* on n vertices, denoted by K_n, if for all vertices $v_x, v_y \in V$, $(v_x, v_y) \in E$.

Definition 4. G is called a *complete bipartite graph*, denoted by $K_{m,n}$, if the following conditions are satisfied:

1. The set of vertices, V, can be partitioned into two disjoint sets of vertices, A and B, such that $|A| = m$, $|B| = n$, and $|V| = m + n$
2. For all $a_i, a_j \in A$, $(a_i, a_j) \notin E$ and for all $b_i, b_j \in B$, $(b_i, b_j) \notin E$

3. For all $a_i \in A$ and $b_j \in B$, $(a_i, b_j) \in E$.

A *star*, denoted by S_n, is the complete bipartite graph $K_{1,n-1}$. A star can also be denoted by $K_{1,\Delta}$, where Δ represents the degree of the graph.

Definition 5. A graph G, where $G = (V, E)$, is called a *path*, denoted by P_n, if $V = \{v_1, v_2, \dots, v_n\}$ such that only $(v_i, v_{i+1}) \in E$ where $1 \le i \le n-1$. A graph G is called a *cycle*, denoted by C_n, if $V = \{v_1, v_2, \dots, v_n\}$ such that only $(v_i, v_{i+1}) \in E$ where $1 \le i < n-1$ and $(v_1, v_n) \in E$.

3 Complete and Complete Bipartite Graphs

In this section we will find the minimal $L(d, 2, 1)$-number, $k_d(K_n)$ or $k_d(K_{a,b})$, for complete and complete bipartite graphs.

Theorem 1. *For complete graphs, $k_d(K_n) = d(n-1) + 1$ where n is the number of vertices in the graph.*

Proof. Let K_n be a complete graph with n vertices. One vertex in K_n must be labeled with 1. As all other vertices are adjacent to each other, no two vertex labels may have a difference less than d. Thus, $k_d(K_n) = d(n-1) + 1$. □

Theorem 2. *Let $K_{a,b}$ be a complete bipartite graph with partitions A and B, where $|A| = a$ and $|B| = b$. Then, $k_d(K_{a,b}) = d + 2(a+b) - 3$.*

Proof. Every vertex in A is distance two from every other vertex in A and every vertex in B is distance two from every other vertex in B. Thus, the label on every vertex in partition A must differ by at least two from the label on every other vertex in partition A. Similarly, the vertex labels in partition B must also differ by at least two. Additionally, the difference between the maximum value of A and the minimum value of B must differ by at least d. Since the label 1 must be used, $k_d(K_{a,b}) = d+2(a-1)+2(b-1)+1 = d+2(a+b)-3$. □

Corollary 3. *For a star, $K_{1,\Delta}$, $k_d(K_{1,\Delta}) = d + 2\Delta - 1$.*

Proof. Since a star is a complete bipartite graph with $a = 1$ and $b = \Delta$, it follows that the maximum $k_d(K_{1,\Delta}) = d + 2\Delta - 1$. □

4 Paths

In this section we will find repeatable labeling patterns for paths of length n that minimize $k_d(P_n)$. We will first look at cases of $k_3(P_n)$ before generalizing to $k_d(P_n)$. If $d \ge 6$, we will find that the length of repeatable patterns will be 4 vertices. Lemma 4 and Theorem 5 have been adapted from the work presented by Clipperton et al [2].

Lemma 4. *For a path on n vertices, P_n, with $n \ge 8$, $k_3(P_n) \ge 8$.*

Proof. Let f be a minimal $L(3, 2, 1)$-labeling for a path on n vertices, P_n. Let v_i be a vertex with label 1. There is an induced subpath of at least 5 vertices with v_i as an end vertex. Let $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\}$ be this subpath. We will continue by considering the possibilities for $f(v_{i+1})$.

Case I: $f(v_{i+1}) = 4$:
Then $f(v_{i+2}) = 7$, $f(v_{i+3}) = 2$, and $f(v_{i+4}) = 5$. As there are 3 vertices in the path not yet labeled, we know that either v_{i+5} or $\{v_{i-3}, v_{i-2}, v_{i-1}\}$ exist. If v_{i+5} exists, $f(v_{i+5})$ must be greater or equal to 8. If $\{v_{i-3}, v_{i-2}, v_{i-1}\}$ exist, then $f(v_{i-1}) = 6$ and $f(v_{i-2}) = 3$. The final label for the path, $f(v_{i-3})$, must be at least 8 to satisfy labeling requirement.

Case II: $f(v_{i+1}) = 5$:
Then $f(v_{i+2}) \ge 8$.

Case III: $f(v_{i+1}) = 6$:
Then $f(v_{i+2}) = 3$, forcing $f(v_{i+3}) \ge 8$.

Case IV: $f(v_{i+1}) = 7$:
Then $f(v_{i+2}) = 3$ or 4. Either possibility for $f(v_{i+2})$ forces $f(v_{i+3}) \geq 9$.

Therefore we can conclude that $k_3(P_n) \geq 8$, when $n \geq 8$. □

Theorem 5. *For any path, P_n,*

$$k_3(P_n) = \begin{cases} 1, & \text{if } n = 1; \\ 4, & \text{if } n = 2; \\ 6, & \text{if } n = 3, 4; \\ 7, & \text{if } n = 5, 6, 7; \\ 8, & \text{if } n \geq 8. \end{cases}$$

Proof. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices of P_n such that v_i is adjacent to v_{i+1} for $1 \leq i < n-1$. For each P_n we proceed by cases.

Case I: $n = 1$.
This is trivially true.

Case II: $n = 2$.
The labeling pattern $\{1, 4\}$ shows that $k_3(P_2) = 4$.

Case III: $n = 3$ or 4.
There is a vertex v_i such that $f(v_i) = 1$. If v_i has degree 2, then vertices v_{i-1} *and* v_{i+1} exist such that $f(v_{i-1}) \geq 4$ and $f(v_{i+1}) \geq 6$. If v_i has degree 1, let $v_i = v_1$, then the possibilities for $f(v_2)$ are 4 and 5. In both cases, we would need $f(v_3) > 6$, requiring $k_3(P_n) > 6$. The labeling pattern $\{3, 6, 1, 4\}$ shows that $k_3(P_n) = 6$ for $n = 3, 4$.

Case IV: $n = 5, 6$, or 7.
There is a vertex $v_i \in V$ such that $f(v_i) = 1$ and vertices v_{i+1} and v_{i+2} exist or v_{i-1} and v_{i-2} exist. Without loss of generality, suppose v_{i+1} and v_{i+2} exist. The possibilities for $f(v_{i+1})$ are 4, 5, and 6. If $f(v_{i+1})$ is 4 or 5, then $f(v_{i+2}) \geq 7$. If $f(v_{i+1}) = 6$, then $f(v_{i+2}) = 3$. Now if v_{i+2} has degree 2, then v_{i+3} exists and $f(v_{i+3}) > 7$. If v_{i+2} has degree 1 then vertices v_{i-1} and v_{i-2} exist. The only possibility for $f(v_{i-1})$ is 4. But this forces $f(v_{i-1}) \geq 7$, so we know $k_3(P_n) \geq 7$. The labeling pattern $\{3, 6, 1, 4, 7, 2, 5\}$ shows that $k_3(P_n) = 7$ for $n = 5, 6, 7$.

Case V: $n \geq 8$.
Define f such that $f(\{v_1, v_2, \ldots, v_8\}) = \{1, 4, 7, 2, 5, 8, 3, 6\}$ and $f(v_i) = f(v_j)$ if $i \equiv j \pmod 8$. By definition of f we can conclude that $k_3(P_n) \leq 8$ for $n \geq 8$. We combine this result with that of Lemma 4 to get $k_3(P_n) = 8$ for $n \geq 8$. □

Lemma 6. *For a path on n vertices, P_n, with $n \geq 5$ and $d \geq 4$, $k_d(P_n) = d + 5$.*

Proof. Let f be a minimal $L(d, 2, 1)$-labeling for a path on n vertices, P_n. Let v_i be a vertex with label 1. There is an induced subpath of at least 3 vertices with v_i as an end vertex. Let $\{v_i, v_{i+1}, v_{i+2}\}$ be this path. We will continue by considering the possibilities for $f(v_{i+1})$.

Case I: $f(v_{i+1}) = d + 1$:
Then $f(v_{i+2}) \geq 2d + 1$.

Case II: $f(v_{i+1}) = d + 2$:
Then $f(v_{i+2}) \geq 2d + 2$.

Case III: $f(v_{i+1}) = d + 3$:
Then $f(v_{i+2}) = 3$. As there are two vertices yet unlabeled we know that either v_{i-1} or v_{i+3} exists. If v_{i+3} exists, $f(v_{i+3})$ must be greater than or equal to $d + 5$. If v_{i-1} exists and v_{i+3} does not, then we know that v_{i-2} must exist as well. The only possible value for $f(v_{i-1})$ is $d + 1$, which forces $f(v_{i-2}) \geq 2d + 1 \geq d + 5$.

Case IV: $f(v_{i+1}) = d + 4$:

Then $f(v_{i+2}) = 3$ or 4. As there are two vertices yet unlabeled we know that either v_{i-1} or v_{i+3} exists. If v_{i+3} exists, $f(v_{i+3})$ must be greater than or equal to $d+6$. If v_{i-1} exists and v_{i+3} does not, then we know that v_{i-2} must exist as well. The only possible value for $f(v_{i-1})$ is $d+1$, which forces $f(v_{i-2}) \geq 2d+1 \geq d+5$.

Therefore we can conclude that $k_d(P_n) \geq d + 5$, when $n \geq 5$ and $d \geq 4$. □

Theorem 7. *For any path, P_n, when $d \geq 4$*

$$k_d(P_n) = \begin{cases} 1, & \text{if } n = 1; \\ d+1, & \text{if } n = 2; \\ d+3, & \text{if } n = 3, 4; \\ d+5, & \text{if } n \geq 5; \end{cases}$$

Proof. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices of P_n such that v_i is adjacent to v_{i+1} for $1 \leq i < n-1$.

For each P_n we proceed by cases using.

Case I: $n = 1$.

This is trivially true.

Case II: $n = 2$.

The labeling pattern $\{d + 1, 1\}$ shows that $k_d(P_n) = d + 1$ for $n = 2$.

Case III: $n = 3$ or 4.

The labeling pattern $\{d + 1, 1, d + 3, 3\}$ shows that $k_d(P_n) = d + 3$ for $n = 3$ or 4. Note that this pattern is not repeatable.

Case IV: $n \geq 5$.

Define f such that $f(\{v_1, v_2, v_3, v_4\}) = \{1, d + 3, 3, d + 5\}$ and $f(v_i) = f(v_j)$ if $i \equiv j \pmod 4$. By definition of f we can conclude that $k_d(P_n) \leq d + 5$ for $n \geq 5$. We combine this result with Lemma 6 to get $k_d(P_n) = d + 5$ for $n \geq 5$. □

5 Cycles

Theorem 8. *For any cycle, C_n, where n is a positive integer greater than 3 and $d = 3$*

$$k_d(C_n) = \begin{cases} 2d + 2, & \text{if } 2|n; \\ 2d + 3, & \text{if } 2 \nmid n,\ n \neq 7; \\ 2d + 4 & \text{if } n = 7 \end{cases}$$

This result will be proved in section 5.1 (note that $d+5 = 2d+2$ for $d = 3$); proofs of $k_d(C_5)$ and $k_d(C_{11})$ may be found in the paper by Clipperton et al [2].

Theorem 9. *For any cycle, C_n, where n is a positive integer greater than 3 and $d = 4$*

$$k_d(C_n) = \begin{cases} 2d + 1, & \text{if } n \neq 6, 7, 11; \\ 2d + 2, & \text{if } n = 6, 11; \\ 2d + 3, & \text{if } n = 7 \end{cases}$$

This result will be proved in section 5.2 (note that $d + 5 = 2d + 1$ for $d = 4$).

Theorem 10. *For any cycle, C_n, where n is a positive integer greater than 3 and $d \geq 5$*

$$k_d(C_n) = \begin{cases} d + 5, & \text{if } 4|n; \\ d + 7, & \text{if } 2|n \text{ and } 4 \nmid n; \\ 2d + 2, & \text{if } n = 7 \text{ and } d = 5; \\ 2d + 1, & \text{if } 2 \nmid n \text{ (and } n \neq 7 \text{ for } d = 5) \end{cases}$$

This result will be proved in section 5.3 (note that $d+7 \le 2d+2$ and $d+5 < 2d+1$ for $d \ge 5$).

Lemma 11. *For a cycle with n vertices where $2 \nmid n$ and n is an integer greater than three, $k_d(C_n) \ge 2d+1$ when $d \ge 4$.*

Proof. Let f be a minimal $L(d,2,1)$-labeling for a cycle with n vertices, C_n. Due to the nature of odd cycles, two vertices with labels greater than or equal to d must be adjacent to one another in the graph. Thus, $k_d(C_n) \ge 2d$. Assume that $k_d(C_n) = 2d$. Then the two adjacent labels that are greater than or equal to d must be d and $2d$. Assume that there exists a vertex v_i such that v_i is labeled with $2d$ and that $f(v_{i+1}) = d$. Then, $f(v_{i+2}) \ge 2d+2$. Thus, regardless of the number of vertices in the cycle, $k_d(C_n) \ge 2d+1$ for $2 \nmid n$. ☐

5.1 Cycles where $d = 3$

In this section, we will prove results for cycles with 4, 6 and 7, vertices and cite results for cycles with 5 or 11 vertices. Results have been adapted from the paper of Clipperton et al [2] and only proofs relevant to larger values of d are given. All the original proofs (with no reference to varying d values) may be found in the Clipperton et al paper. [2]

Lemma 12. *For a cycle on 4 vertices, C_4, with $d \ge 3$, $k_d(C_4) = d+5$.*

Proof. Let f be a minimal $L(d,2,1)$-labeling for a cycle with 4 vertices, C_4. Let v_1 be a vertex with label 1. There is an induced subpath of at least 4 vertices with v_1 as an end vertex. Let $\{v_1, v_2, v_3, v_4\}$ be this path. We will continue by considering the possibilities for $f(v_2)$ to show $k_d(C_4) \ge d+5$.

Case I: $f(v_2) = d+1$:
Then $f(v_3) \ge 2d+1$. If $d > 3$, then $2d+1 \ge d+5$. If $d = 3$, then $f(v_3) = 2d+1$ and $f(v_4) \ge 3d+1$.

Case II: $f(v_2) = d+2$:
Then $f(v_3) \ge 2d+2$.

Case III: $f(v_2) = d+3$:
Then $f(v_3) = 3$, forcing $f(v_4) \ge d+5$.

Case IV: $f(v_2) = d+4$:
Then $f(v_3) = 3$ or 4. Either possibility for $f(v_3)$ forces $f(v_4) \ge d+6$.

The pattern $\{1, d+3, 3, d+5\}$ shows that $k_d(C_4) = d+5$, when $d \ge 3$. ☐

Lemma 13. *For a cycle on 6 vertices, C_6, when $3 \le d \le 4$, $k_d(C_6) = 2d+2$.*

Proof. Let f be a minimal $L(d,2,1)$-labeling for a cycle with 6 vertices, C_6. Let v_1 be a vertex with label 1. There is an induced subpath of 6 vertices with v_1 as an end vertex. Let $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ be this path. We will continue by considering the possibilities for $f(v_2)$ to show that $k_d(C_6) \ge 2d+2$.

Case I: $f(v_2) = d+1$:
Then $f(v_3) \ge 2d+1$ and $f(v_4) = a$ where $2 \le a \le d$. Then, $f(v_5) \ge d+a$ and $f(v_6)$ must be both greater than $d+3$ and less than $2d+a$, which is not possible when $k_d \le 2d+2$.

Case II: $f(v_2) = d+2$:
Then $f(v_3) \ge 2d+2$.

Case III: $f(v_2) = d+3$:
Then $f(v_3) = 3$, and $f(v_4) \ge d+5$. If $d = 3$, $d+5 \ge 2d+2$. If $d = 4$, then $f(v_5) \ge 5$, and $f(v_6) \ge d+7 \ge 2d+2$.

Case IV: $f(v_2) = d+4$:
Then $f(v_3) = 3$ or 4. Either possibility for $f(v_3)$ requires $f(v_4) \ge d+6$, which is greater than or equal to $2d+2$ when $d = 3$ or 4.

Case V: $f(v_2) = d + 5$ for $d = 4$:

Then $f(v_3) = 3$, 4, or 5. If $f(v_3) = 3$, then $f(v_4) \geq d + 3$, $f(v_5) = 5$, and $f(v_6) = d + 7$. If $f(v_3) = 4$ or 5 then $f(v_4) \geq d + 7 \geq 2d + 2$.

Thus $k_d(C_6) \geq 2d + 2$, when $3 \leq d \leq 4$. We can conclude that $k_d(C_6) = 2d + 2$ as the labeling pattern $\{1, d + 1, 2d + 1, 2, d + 2, 2d + 2\}$ exhibits a satisfactory labeling for C_6. □

Lemma 14. *For a cycle on 7 vertices $k_d(C_7) = 2d + 4$ for $d = 3$, $k_d(C_7) = 2d + 3$ for $d = 4$, and $k_d(C_7) = 2d + 2$ for $d = 5$.*

Proof. Let $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ be the set the vertices of C_7. By Lemma 11, $k_d(C_7) \geq 2d + 1$. Suppose $k_d(C_7) = 2d+1$ and let f be a minimal $L(d, 2, 1)$-labeling of C_7. The possible values for $f(V)$ must be in $S = \{1, 2, ..., 2d + 1\}$. Since the greatest distance between any two vertices in V is three, none of the possible values for $f(V)$ can be repeated. Also, only up to two consecutive labels may be used, as using three consecutive labels is not possible due to the distance constraints in V. Then at most $\frac{2}{3}(2d+1)$ labels can be used from S, which does not provide a sufficient number of available labels for C_7. Thus, $k_d(C_7) \geq 2d + 4$ for $d = 3$, $k_d(C_7) \geq 2d + 3$ for $d = 4$, and $k_d(C_7) \geq 2d + 2$ for $d = 5$. □

Lemma 15. *For a cycle on n vertices where n is an even positive integer and $n \geq 8$, $k_3(C_n) = 2d + 2$.*

The above proof can be found in Theorem 11 of Clipperton et al [2]. The proof relies on combinations of labeling patterns for cycles with 4 or 6 vertices. Every even integer greater than or equal to six can be written as $4m + 6n$ where m and n are positive integers. To obtain $k_d(C_n) = 2d + 2$, combine m paths labeled $\{1, d + 3, 3, d + 5\}$ with n paths labeled $\{1, d + 1, 2d + 1, 2, d + 2, 2d + 2\}$.

Lemma 16. *For a cycle on n vertices where n is an odd positive integer and $n \geq 8$, $n \neq 11$, $k_3(C_n) = 2d+3$.*

The above proof can also be found in Theorem 11 of Clipperton et al [2]. The proof relies on combinations of labeling patterns for cycles with 4 or 5 vertices. Every integer greater than or equal to eight with the exception of eleven can be written as $4p + 5q$ where p and q are positive integers. To obtain $k_d(C_n) = 2d+3$, combine p paths labeled $\{1, d + 3, 3, d + 5\}$ with q paths labeled $\{1, 2d + 1, d, 2d + 3, d + 2\}$.

5.2 Cycles for which $d = 4$

Before we begin with special cases, we can revisit what what we already know from section 5.1. We've established in Lemma 13 that for a cycle on 6 vertices, $k_4(C_6) = 2d + 2$. In addition to this, we know from Lemma 14 that $k_4(C_7) = 2d + 3$. We've also learned from Lemma 12 that $k_4(C_4) = d + 5$.

Lemma 17. *For a cycle on n vertices, C_n, where $d \geq 4$, $4|n$, and $n \geq 8$, $k_d(C_n) = d + 5 = 2d + 1$.*

Proof. We've already shown in Lemma 12 that for cycles of length 4, $k_d(C_n) = d + 5$. We know by Theorem 5, and Theorem 7 that for any path on 8 or more vertices $k_d(P_n) = d + 5$. The pattern $\{1, d + 3, 3, d + 5\}$ can be repeated infinitely many times. Thus for $4|n$, $n \geq 8$, $k_d(C_n) = d + 5$. □

Lemma 18. *For a cycle with 5 vertices, $k_d(C_5) = 2d + 1$ when $d \geq 4$.*

Proof. Let f be a minimal $L(d, 2, 1)$-labeling for a cycle with 5 vertices, C_5. We know by Lemma 11 that $k_d(C_5) \geq 2d + 1$. The pattern $\{1, d + 1, 2d + 1, 3, d + 3\}$ yields $k_d(C_5) = 2d + 1$ when $d \geq 4$ and illustrates a valid labeling for a 5-cycle graph. Thus, $k_d(C_5) = 2d + 1$. □

Lemma 19. *For a cycle with 11 vertices, $k_d(C_{11}) = 2d + 2$ when $d = 4$.*

Proof. Let f be a minimal $L(d, 2, 1)$-labeling for a cycle with 11 vertices, C_{11}. We know by Lemma 11 that $k_d(C_{11}) \geq 2d + 1$. Let $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}\}$ be a subpath of 11 vertices. We will suppose $f(v_1) = 1$ and consider the possibilities for $f(v_2)$, showing that $k_d(C_{11}) \geq 2d + 2$.

Case I: $f(v_2) = d + 1$

Then, $f(v_3) = 2d + 1$ and $f(v_4) = 2$ or 3. If $f(v_4) = 2$, then $f(v_5) = d + 2$ or $d + 3$ and $f(v_6) \geq 2d + 2$ or $2d + 3$. If $f(v_4) = 3$, then $f(v_5) = d + 3$ and $f(v_6) = 1$. Then, $f(v_7) = d + 1$ or $d + 5$. If $f(v_7) = d + 1$, then $f(v_8) = 2d+1$, $f(v_9) = 2$ or 3. If $f(v_9) = 2$, then $f(v_{10}) = d+2$ or $d+3$ and $f(v_{11}) \geq 2d+2$. If $f(v_9) = 3$, then $f(v_{10}) = d+3$ and $f(v_{11}) \geq 2d+3$. If $f(v_7) = d+5$, then $f(v_8) = 3, 4$, or 5. If $f(v_8) = 3$, then $f(v_9) = d+3$ and $f(v_{10}) \geq 2d+3$. If $f(v_7) = d+5$ and $f(v_8) = 4$ or 5, then $f(v_9) \geq d+7 = 2d+3$.

Case II: $f(v_2) = d+2$
Then, $f(v_3) \geq 2d+2$.

Case III: $f(v_2) = d+3$
Then, $f(v_3) = 3$, $f(v_4) = 2d+1$ and $f(v_5) = 1$ or $d+1$. If $f(v_5) = 1$, then $f(v_6) = d+1, d+2$ or $d+3$. If $f(v_6) = d+1$ or $d+2$, then $f(v_7) \geq 2d+3$. If $f(v_6) = d+3$, then $f(v_7) = 3$, $f(v_8) = 2d+1$, $f(v_9) = d+1$ and $f(v_{10}) \geq 2d+3$. If $f(v_5) = d+1$, then $f(v_6) = 1$ and $f(v_7) = d+3$ or $2d$. If $f(v_7) = d+3$, then $f(v_8) = 3$, $f(v_9) = 2d+1$, $f(v_{10}) = d+1$, and $f(v_{11}) \geq 2d+3$. If $f(v_7) = 2d$, $f(v_8) = 3$ or 4 and $f(v_9) \geq 2d+2$.

Case IV: $f(v_2) = 2d$
Then, $f(v_3) = d$ or $d-1$ and $f(v_4) \geq 2d+2$.

Case V: $f(v_2) = 2d+1$
Then, $f(v_3) = d+1, d$ or $d-1$. If $f(v_3) = d+1$ or d, then $f(v_4) \geq 2d+3$. If $f(v_3) = d-1 = 3$, then $f(v_4) = d+3$ and $f(v_5) = 1$. Then $f(v_6) = d+1$ or $2d+1$. If $f(v_6) = d+1$, then $f(v_7) = 2d+1$ and $f(v_8) = 2$ or 3. If $f(v_8) = 2$, then $f(v_9) = d+2$ and $f(v_{10}) \geq 2d+2$. If $f(v_8) = 3$, then $f(v_9) = d+3$ and $f(v_{10}) \geq 2d+3$. If $f(v_6) = 2d+1$, then $f(v_7) = d+1, d$ or $d-1$. If $f(v_7) = d+1$ or d, then $f(v_8) \geq 2d+3$. If $f(v_7) = d-1 = 3$, then $f(v_8) = d+3$ and $f(v_9) 2d+3$.

Thus, the pattern $f(V) = \{1, d+1, 2d+1, 2, d+2, 2d+2, 1, d+1, 2d+1, 3, d+3\}$ demonstrates a labeling that satisfies $k_d(C_{11}) = 2d+2$. □

Lemma 20. *For a cycle with n vertices where $n \geq 9$, $d = 4$, and $n \neq 11$, $k_d(C_n) = 2d+1 = d+5$.*

Proof. We know by Lemma 11 that $k_d(C_n) \geq 2d+1$. Every odd integer greater than or equal to nine with the exception of eleven can be written as $4p+5q$ where p and q are positive integers. To obtain $k_d(C_n) = 2d+1 = d+5$, combine p paths labeled $\{1, d+5, 3, d+3\}$ with q paths labeled $\{1, d+1, 2d+1, 3, d+3\}$. Thus, for any cycle with $n \geq 9$, $n \neq 11$, $k_d(C_n) = 2d+1$. □

5.3 Cycles for which $d \geq 5$

Before we examine cycles labeled with large values of d, we can revisit what we know from sections 5.1 and 5.2. In Lemma 14 we found that for a cycle on 7 vertices $k_5(C_7) = 2d+2$. Also, from Lemma 11, we know that for a cycle on n vertices, $k_d(C_n) \geq 2d+1$ for $d \geq 4$ and where n is an odd positive integer greater than three. Theorem 17 states that for cycles with vertices in multiples of four, $k_d(C_n) = d+5$.

Lemma 21. *For a cycle with 6 vertices, C_6, $k_d(C_6) = d+7$ when $d \geq 5$.*

Proof. Let f be a minimal $L(d,2,1)$-labeling for a cycle with 6 vertices, C_6. Let v_1 be a vertex such that $f(v_1) = 1$. There is an induced subpath of 6 vertices with v_1 as an end vertex. Let $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ be this path. We will continue by considering the possibilities for $f(v_2)$.

Case I: $f(v_2) = d+1$:
Then $f(v_3) \geq 2d+1 > d+6$ for $d \geq 6$. If $d = 5$, then $f(v_3) = 2d+1$, $2 \leq f(v_4) \leq 4$, $f(v_5) \leq d+4$ and $f(v_6) \geq 2d+2$ ($2d+2 = d+7$ for $d = 5$).

Case II: $f(v_2) = d+2$:
Then $f(v_3) \geq 2d+2 \geq d+7$ for $d \geq 5$.

Case III: $f(v_2) = d+3$:
Then $f(v_3) = 3$, and $f(v_4) \geq d+5$. Then, $f(v_5) \geq 5$ and $f(v_6) \geq d+7$.

Case IV: $f(v_2) = d+4$:
Then $f(v_3) = 3$ or 4. Either possibility for $f(v_3)$ requires $f(v_4) \geq d+6$. Then $f(v_5) \geq 5$ and $f(v_6) \geq d+8$.

Case V: $f(v_2) = d + 5$:
Then $f(v_3) = 3$, 4, or 5. If $f(v_3) = 3$, then $f(v_4) = d + 3$. Then $f(v_5) \geq 2d + 3$. Labels $f(v_3) = 4$ and 5 both require $f(v_4) \geq d + 7$.

Case VI: $f(v_2) = d + 6$:
Then $f(v_3) = 3$, 4, 5, or 6. If $f(v_3) = 3$ or 4, then $f(v_4) = d+3$ or $d+4$. It follows that $f(v_5) \geq 2d+3 \geq d+7$. If $f(v_3) = 5$ or 6, then $f(v_4) \geq d + 8$.

Thus $k_d(C_6) \geq d + 7$, when $d \geq 5$. We can conclude that $k_d(C_6) = d + 7$, as the labeling pattern $\{1, d + 3, 3, d + 5, 5, d + 7\}$ exhibits a satisfactory labeling for C_6 with $k_d(C_6) = d + 7$. □

Lemma 22. *For a cycle on 7 vertices, $d \geq 6$, $k_d(C_7) = 2d + 1$.*

Proof. We know from Lemma 11 that for any cycle with an odd number of vertices, $k_d(C_{2n+1}) \geq 2d + 1$. The pattern $\{1, d + 1, 2d + 1, 5, d + 5, 3, d + 3\}$ produces $k_d(C_7)$ for $d \geq 6$. □

Lemma 23. *For a cycle on 11 vertices, $d \geq 5$, $k_d(C_{11}) = 2d + 1$.*

Proof. We know from Lemma 11 that for odd cycles, $k_d(C_{2d+1}) \geq 2d+1$. The following pattern demonstrates that $k_d(C_{11}) = 2d + 1$: $f(V) = \{1, d + 1, 2d + 1, 2, d + 4, 4, 2d + 1, 1, d + 3, 3, 2d\}$. □

Lemma 24. *For a cycle on n vertices where $n > 6$, $2|n$ but $4 \nmid n$, and $d \geq 5$, $k_d(C_n) = d + 7$.*

Proof. From Lemma 6, we know that no label smaller than d+5 may be used as each graph contains a subpath of at least 12 vertices. Let v_1 be the vertex in the graph that has the smallest label and starts a six-vertex subpath $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ where $f(v_x) \neq f(v_y)$ if $x \equiv y \pmod 4$. Such a subpath must exist since $4 \nmid n$; thus the cycle cannot be labeled with a repeated pattern of length four. Note that $f(v_1) \in \{1, 2, 3, 4, 5, 6\}$ since the labels of any two adjacent vertices must differ by d and $f(v_1)$ is the smallest label.

The initial cases, denoted by lowercase numerals, represent values that cannot be adjacent. After two such labels, the third vertex must be labeled with a value greater than $d + 7$. The remaining cases represent the different ways the subpath might be labeled barring the initial three labeling combinations.

Case i: $f(v_i) = 1$ and $f(v_{i+1}) = d + 2$:
Then $f(v_{i+2}) \geq 2d + 1 \geq d + 7$.

Case ii: $f(v_i) = 2$ and $f(v_{i+1}) = d + 2$:
Then $f(v_{i+2}) \geq 2d + 2 \geq d + 7$.

Case iii: $f(v_i) = 2$ and $f(v_{i+1}) = d + 3$:
Then $f(v_{i+2}) \geq 2d + 3 \geq d + 7$.

Case I: $f(v_1) = 1$ and $f(v_2) = d + 1$:
Then $f(v_3) \geq 2d + 1 \geq d + 7$.

Case II: $f(v_1) = 1$ and $f(v_2) = d + 3$:
Then $f(v_3)$ must be 3 and $f(v_4)$ may be either $d + 5$ or $d + 6$. Then, since $f(v_1) \neq f(v_5)$, $f(v_5) = 5$ or 6. Thus $f(v_6)$ must be at least $d + 7$.

Case III: $f(v_1) = 1$ and $f(v_2) = d + 4$:
Then $f(v_3)$ may be 3 or 4 and $f(v_4)$ must be $d + 6$. As such, depending on the value assigned to $f(v_3)$, $f(v_5)$ may be either 2,5, or 6. If $f(v_3) = 4$ and $f(v_5) = 2$, then $f(v_6) = d + 2$. Otherwise, for $f(v_5) = 5$ or 6, $f(v_6) \geq d + 8$.

Case IV: $f(v_1) = 1$ and $f(v_2) = d + 5$:
Then $f(v_3)$ may be either 3, 4, or 5. If $f(v_3) = 3$, then $f(v_4) = d + 3$. Then, $f(v_5) \geq d + 7$. If $f(v_3) = 4$ or 5, then $f(v_4) \geq d + 7$.

Case V: $f(v_1) = 1$ and $f(v_2) = d + 6$:
Then $f(v_3)$ may be 3, 4, 5, or 6. If $f(v_3) = 3$, then $f(v_4) = d + 3$ or $d + 4$, and $f(v_5) \geq 2d + 3 \geq d + 7$.

If $f(v_3) = 4$, then $f(v_4) = d + 4$, $f(v_5) = 2$, and $f(v_6) = d + 2$. Note that $f(v_6) \neq d + 6$ as $2 = 6 \pmod 4$. By Case ii, we know that $f(v_7)$ must be greater or equal to $d + 7$. If $f(v_3) = 5$, or 6, then $f(v_4) \geq d + 8$.

Case VI: $f(v_1) = 2$ and $f(v_2) = d + 4$:
Then $f(v_3) = 4$, $f(v_4) = d + 6$, $f(v_5) = 6$ and $f(v_6) \geq d + 8$.

Case VII: $f(v_1) = 2$ and $f(v_2) = d + 5$:
Then $f(v_3) = 4$ or 5 and $f(v_4) \geq d + 7$.

Case VIII: $f(v_1) = 2$ and $f(v_2) = d + 6$:
Then $f(v_3) = 4, 5$, or 6. If $f(v_4) = 4$, then $f(v_5) = d + 4$, and $f(v_6) \geq 2d + 4 \geq d + 7$. If $f(v_3) = 5$ or 6, $f(v_4) \geq d + 8$.

Case IX: $f(v_1) = 3$ and $f(v_2) = d + 3$:
Then $f(v_3) \geq 2d + 3 \geq d + 7$.

Case X: $f(v_1) = 3$ and $f(v_2) = d + 4$:
Then $f(v_3) \geq 2d + 4 \geq d + 7$.

Case XI: $f(v_1) = 3$ and $f(v_2) = d + 5$:
Then $f(v_3) = 5$ and $f(v_4) \geq d + 7$.

Case XII: $f(v_1) = 3$ and $f(v_2) = d + 6$:
Then $f(v_3) = 5$ or 6. If $f(v_3) = 5$ or 6, then $f(v_4) \geq d + 8$.

Case XIII: $f(v_1) = 4$ and $f(v_2) = d + 4$:
Then $f(v_3) \geq 2d + 4 \geq d + 7$.

Case XIV: $f(v_1) = 4$ and $f(v_2) = d + 5$:
Then $f(v_3) \geq 2d + 5 \geq d + 7$.

Case XV: $f(v_1) = 4$ and $f(v_2) = d + 6$:
Then $f(v_3) = 6$ and $f(v_4) = d + 8$.

Case XVI: $f(v_1) = 5$ and $f(v_2) = d + 5$:
Then $f(v_3) \geq 2d + 5 \geq d + 7$.

Case XVII: $f(v_1) = 5$ and $f(v_2) = d + 6$:
Then $f(v_3) \geq 2d + 6 \geq d + 7$.

Case XVIII: $f(v_1) = 6$ and $f(v_2) = d + 6$:
Then $f(v_3) \geq 2d + 6 \geq d + 7$.

Hence, $k_d(C_n) \geq d + 7$. The pattern $\{1, d + 3, 3, d + 5, 5, d + 7\}$ can be repeated infinitely many times when $d \geq 5$ and illustrates a proper labeling of C_n with $k_d(C_n) = d + 7$. □

Lemma 25. *For a cycle with n vertices where $2 \nmid n$, $n \geq 9$, $n \neq 11$, and $d \geq 5$, $k_d(C_n) = 2d + 1$.*

Proof. We know by Lemma 11 that $k_d(C_n) \geq 2d + 1$. Every integer greater than or equal to nine with the exception of eleven can be written as $4p + 5q$ where p and q are positive integers. To obtain $k_d(C_n) = 2d + 3$, combine p paths labeled $\{1, d + 5, 3, d + 3\}$ with q paths labeled $\{1, d + 1, 2d + 1, 3, d + 3\}$. Thus, $k_d(C_n) = 2d + 1$. □

6 Future Work

Thus paper considered complete graphs, paths, and cycles and considered maximum labels containingin terms of d. Many other types graphs can still be considered. Additionally, a precise maximum k_d for any graph in terms of d and/or Δ remains to be determined.

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