An Alternating Sum of Alternating Sign Matrices

Nathan F. Williams
will3089@umn.edu

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Abstract

An alternating-sign matrix (ASM) is a square matrix with entries from {-1, 0,1}, row and column sums of 1, and in which the nonzero entries in each row and column alternate in sign. ASMs have many non-trivial parameters and symmetries that reveal their significant combinatorial structure. In this note, we will prove an identity that relates one parameter and one symmetry.

An alternating-sign matrix (ASM) is a square matrix with entries from {-1, 0,1}, row and column sums of 1, and in which the nonzero entries in each row and column alternate in sign. In the early 1980s, D. Robbins and H. Rumsey discovered ASMs, which can be thought of as generalizations of permutation matrices, through their work on λ-determinants [2]. With W. Mills, they advanced several conjectures related to determining the number A_n of $n \times n$ ASMs—but their relatively simple formulas turned out to be surprisingly difficult to prove [9]. In fact, it wasn't until 1995 that D. Zeilberger published a proof of the ASM Conjecture, confirming their formula for A_n [3, 10]. Amusingly, Zeilberger organized his paper into lemmas, sublemmas, subsublemmas, and so on to (sub)^{6}lemmas, and recruited 88 people (as well as his computer, Shalosh B. Ekhad) as referees. Each referee was assigned a single (sub)^{i}lemma with the charge to make sure that it followed from any (sub)$^{i+1}$lemmas [10].

The sequence A_n begins $1, 2, 7, 42, 429, 7436, \ldots$ (Sloane's A005130), and the general formula is given below.

Theorem 1 (The ASM Conjecture) *For $n \geq 1$,*

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

It would be difficult to guess this elegant formula for A_n based only on the first few terms of the sequence, but the ASM Conjecture was systematically developed once Robbins introduced a useful parameter to better attack the problem. Figure 1 lists the seven 3×3 ASMs, of which six are the 3×3 permutation matrices. The reader may notice that they each have exactly one 1 in their top row.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Figure 1. The seven 3×3 ASMs, arranged by the position of the 1 in their top row.

It turns out that this is no accident, and there is always exactly one 1 in the the top row of an ASM. For there certainly must be at least one 1 in the top row for the row sum to be 1. But if there is more than one 1, then because nonzero entries alternate there must also be a -1 in the top row—and the column sum of the column containing that -1 can be at most 0. This contradicts the condition that the column sums of an ASM have to be 1.

Thus, Robbins partitioned the $n \times n$ ASMs by the column k of the 1 in their top row. He arranged the sizes of these partitions, denoted by the numbers $A_{n,k}$, into rows to form a triangle [2]. The first few rows of this ASM triangle are given in Figure 2.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 2 & & 3 & & 2 & & \\ & 7 & & 14 & & 14 & & 7 & \\ 42 & & 105 & & 135 & & 105 & & 42 \\ & \vdots & & \vdots & & \vdots & & \vdots & \end{array}$$

Figure 2. The first few rows of the ASM triangle. The kth entry of the nth row is $A_{n,k}$.

By studying this triangle, Mills, Robbins, and Rumsey conjectured a formula for the ratio $\frac{A_{n,k}}{A_{n,k+1}}$, which became known as the Refined ASM Conjecture. Once this ratio is known, a formula for $A_{n,k}$ can be found inductively, using the fact that $A_{n,1} = \sum_{k=1}^{n-1} A_{n-1,k}$ [2]. This leads to the following formula for $A_{n,k}$.

Theorem 2 (The Refined ASM Conjecture) *For $n \geq 1$ and $1 \leq k \leq n$,*

$$A_{n,k} = \binom{n+k-2}{k-1} \frac{(2n-k-1)!}{(n-k)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!}.$$

In 1996, Zeilberger published a proof of the Refined ASM conjecture [11] drawing from techniques G. Kuperberg introduced in his proof of the ASM conjecture [4]. For more information about ASMs, see [2], [3], and [7].

We will also use vertically symmetric ASMs (VSASMs) in our identity. Robbins again conjectured a formula for the number of VSASMs [8], which Kuperberg then proved [5]. We will use an equivalent formula that was later proved by A. Razumov and Y. Stroganov [6]. By considering the position of the 1 in the top row, it is clear that there are no $2n \times 2n$ VSASMs. The 5×5 VSASMs are given in Figure 2.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Figure 2. The three 5×5 VSASMs

Theorem 3 (Kuperberg) *The number of $2n+1 \times 2n+1$ VSASMs, with $n \geq 0$, is*

$$V(2n+1) = \frac{1}{2^n} \prod_{k=1}^{n} \frac{(6k-2)!(2k-1)!}{(4k-1)!(4k-2)!}.$$

It is well-known that the sum of the entries in the nth row of Pascal's triangle is 2^n, while the alternating sum is 0 if $n \geq 1$. Since we know the sum of the entries in the nth row of the ASM triangle is A_n, it is therefore natural to ask for the corresponding alternating sum. In fact, the original form of the Refined ASM Conjecture says that the ratio of entries in the ASM triangle looks like the ratio of a sum of entries from Pascal's triangle—so our analogy seems appropriate [2].

Theorem 4 *For $n \geq 1$, let $S(n) = \sum_{k=1}^{n} (-1)^{k+1} A_{n,k}$. Then $S(n) = V(n)^2$.*

Proof. First, we note that the alternating sum $S(2n)$ is zero, by symmetry of the $A_{n,k}$. Therefore we have only to consider $S(2n+1)$.

Starting with the Refined ASM Conjecture we can write

$$S(2n+1) = \sum_{k=1}^{2n+1} (-1)^{k+1} A_{2n+1,k}$$

$$= \sum_{k=1}^{2n+1} \left((-1)^{k+1} \binom{2n+k-1}{k-1} \frac{(4n-k+1)!}{(2n-k+1)!} \prod_{j=0}^{2n-1} \frac{(3j+1)!}{(2n+1+j)!} \right)$$

$$= \left(\sum_{k=1}^{2n+1} (-1)^{k+1} \binom{2n+k-1}{k-1} \binom{4n-k+1}{2n-k+1} \right) \left((2n)! \prod_{j=0}^{2n-1} \frac{(3j+1)!}{(2n+1+j)!} \right)$$

.

Reindexing to have the sum begin at zero, and writing the product $(2n)! \prod_{j=0}^{2n-1} \frac{(3j+1)!}{(2n+1+j)!}$ as Π, we have

$$\frac{S(2n+1)}{\Pi} = \sum_{k=0}^{2n} (-1)^k \binom{2n+k}{k} \binom{4n-k}{2n-k}.$$

If we apply the identity $\binom{b+i-1}{i} = (-1)^i \binom{-b}{i}$ to both binomial coefficients in the sum, we get

$$\frac{S(2n+1)}{\Pi} = \sum_{k=0}^{2n} (-1)^k \binom{-(2n+1)}{k} \binom{-(2n+1)}{2n-k}.$$

Now, the coefficient of x^{2n} in the product $(1-x)^{-(2n+1)}(1+x)^{-(2n+1)}$ is exactly this sum. Since $(1-x)^{-(2n+1)}(1+x)^{-(2n+1)} = (1-x^2)^{-(2n+1)}$, we can equate coefficients of x^{2n} to obtain the identity

$$\sum_{k=0}^{2n} (-1)^k \binom{-(2n+1)}{k} \binom{-(2n+1)}{2n-k} = \binom{3n}{n}.$$

This is actually a special case of Kummer's ${}_2F_1$ identity—the proof we show above is from [1, Remark 3.4.1]. Therefore,

$$S(2n+1) = \binom{3n}{n} \Pi = \binom{3n}{n} (2n)! \prod_{j=0}^{2n-1} \frac{(3j+1)!}{(2n+1+j)!} = \frac{(3n)!}{n!} \prod_{j=0}^{2n-1} \frac{(3j+1)!}{(2n+1+j)!}.$$

It is not immediately clear that this number is actually a square. Regardless, we must now only check that this formula really is the same as that for $V(2n+1)^2$.

We proceed by induction. For both sides, evaluation at $n = 1$ gives one. Now, by our above simplification, we have

$$\frac{S(2n-1)}{S(2n+1)} = \left(\frac{n}{(3n)(3n-1)(3n-2)} \right) \left(\frac{(4n)!(4n-1)!(4n-2)!(4n-3)!}{(6n-2)!(6n-5)!(2n)!(2n-1)!} \right).$$

On the other hand, we also have

$$\frac{V(2n-1)^2}{V(2n+1)^2} = \left(\frac{2(4n-1)!(4n-2)!}{(6n-2)!(2n-1)!} \right)^2.$$

We leave it to the reader to check that these two expressions are equal, completing the proof.□

We conclude by providing a combinatorial interpretation of this identity. Because the central column of a VSASM has no zeros, we can represent an ordered pair of VSASMs as a single ASM with a non-zero middle column by choosing the left half from the first VSASM and the right half from the second. Thus, $V(2n+1)^2$ as the number of ASMs that have a central column with no zeros. It would therefore be interesting to see a direct bijection from $n \times n$ ASMs with their top-most one in an odd numbered column to the disjoint union of $n \times n$ ASMs with top-most one in an even numbered column and $n \times n$ ASMs whose middle column contains no zeros.

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