An Alternating Sum of Alternating Sign Matrices

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September 23, 2008

Abstract

An alternating-sign matrix (ASM) is a square matrix with entries from {-1, 0,1}, row and column sums of 1, and in which the nonzero entries in each row and column alternate in sign. ASMs have many non-trivial parameters and symmetries that reveal their significant combinatorial structure. In this note, we will prove an identity that relates one parameter and one symmetry.

An alternating-sign matrix (ASM) is a square matrix with entries from {-1, 0,1}, row and column sums of 1, and in which the nonzero entries in each row and column alternate in sign. D. Robbins and H. Rumsey discovered ASMs, which can be thought of as generalizations of permutation matrices, through their work on λ-determinants [2]. With W. Mills, they formulated many conjectures in an attempt to count the number of $n \times n$ ASMs [9]. In approaching the problem, they considered the number of $n \times n$ ASMs with a 1 in the top row, column k, which they denoted $A_{n,k}$. It is easy to verify that there is exactly one 1 in the top row of an ASM. Thus, the numbers $A_{n,k}$ can be arranged in a triangle that partitions the total number of $n \times n$ ASMs across a row [2]. The first few rows of this ASM triangle are given in Figure 1.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 2 & & 3 & & 2 & & \\
 & 7 & & 14 & & 14 & & 7 & \\
42 & & 105 & & 135 & & 105 & & 42 \\
 & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}
$$

Figure 1. The first few rows of the ASM triangle

Mills, Robbins, and Rumsey conjectured a formula for $A_{n,k}$, which became known as the Refined ASM Conjecture. In 1992, D. Zeilberger proved the conjecture [10], using techniques G. Kuperberg introduced [4].

Theorem 0.1 (Zeilberger) *For $n \geq 1$ and $1 \leq k \leq n$,*

$$A_{n,k} = \binom{n+k-2}{k-1} \frac{(2n-k-1)!}{(n-k)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!}.$$

For more information about ASMs, see [2], [3], and [7].

We will also use vertically symmetric ASMs (VSASMs) in our identity. Robbins again conjectured a formula for the number of VSASMs [8], which Kuperberg then proved [5]. We will use an equivalent formula that was later proved by A. Razumov and Y. Stroganov [6]. By considering the position of the 1 in the top row, it is clear that there are no $2n \times 2n$ VSASMs. The 5×5 VSASMs are given in Figure 2.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Figure 2. The three 5×5 VSASMs

Theorem 0.2 (Kuperberg) *The number of $2n+1 \times 2n+1$ VSASMs, with $n \geq 0$, is*

$$V(2n+1) = \frac{1}{2^n} \prod_{k=1}^{n} \frac{(6k-2)!(2k-1)!}{(4k-1)!(4k-2)!}.$$

It is well-known that the sum of the entries in the nth row of Pascal's triangle is 2^n, while the alternating sum is 0 if $n \geq 1$. Since we know the sum of the entries in the nth row of the ASM triangle, it is therefore natural to ask for the corresponding alternating sum. In fact, ratios of entries in the ASM triangle look like ratios of sums of entries from Pascal's triangle—so our analogy seems appropriate [2].

Theorem 0.3 *For $n \geq 1$, let $S(n) = \sum_{k=1}^{n} (-1)^{k+1} A_{n,k}$. Then $S(n) = V(n)^2$.*

Proof. First, we note that the alternating sum $S(2n)$ is zero, by symmetry of the $A_{n,k}$. Therefore we have only to consider $S(2n+1)$.

Starting with the Refined ASM Conjecture we can write

$$S(2n+1) = \sum_{k=1}^{2n+1} (-1)^{k+1} A_{2n+1,k} = \sum_{k=1}^{2n+1} \left((-1)^{k+1} \binom{2n+k-1}{k-1} \frac{(4n-k+1)!}{(2n-k+1)!} \prod_{j=0}^{2n-1} \frac{(3j+1)!}{(2n+1+j)!} \right)$$

$$= \left(\sum_{k=1}^{2n+1} (-1)^{k+1} \binom{2n+k-1}{k-1} \binom{4n-k+1}{2n-k+1} \right) \left((2n)! \prod_{j=0}^{2n-1} \frac{(3j+1)!}{(2n+1+j)!} \right).$$

Reindexing to have the sum begin at zero, and writing the product $(2n)! \prod_{j=0}^{2n-1} \frac{(3j+1)!}{(2n+1+j)!}$ as Π, we have

$$\frac{S(2n+1)}{\Pi} = \sum_{k=0}^{2n} (-1)^k \binom{2n+k}{k} \binom{4n-k}{2n-k}.$$

If we apply the identity $\binom{b+i-1}{i} = (-1)^i \binom{-b}{i}$ to both binomial coefficients in the sum, we get

$$\frac{S(2n+1)}{\Pi} = \sum_{k=0}^{2n} (-1)^k \binom{-(2n+1)}{k} \binom{-(2n+1)}{2n-k}.$$

Now, the coefficient of x^{2n} in the product $(1-x)^{-(2n+1)}(1+x)^{-(2n+1)}$ is exactly this sum. Since $(1-x)^{-(2n+1)}(1+x)^{-(2n+1)} = (1-x^2)^{-(2n+1)}$, we can equate the coefficient of x^{2n} in both, obtaining the identity

$$\sum_{k=0}^{2n} (-1)^k \binom{-(2n+1)}{k} \binom{-(2n+1)}{2n-k} = \binom{3n}{n}.$$

This is actually a special case of Kummer's ${}_2F_1$ identity—the proof we show above is from [1, Remark 3.4.1]. Therefore,

$$S(2n+1) = \binom{3n}{n} \Pi = \binom{3n}{n} (2n)! \prod_{j=0}^{2n-1} \frac{(3j+1)!}{(2n+1+j)!} = \frac{(3n)!}{n!} \prod_{j=0}^{2n-1} \frac{(3j+1)!}{(2n+1+j)!}.$$

It is not immediately clear that this number is actually a square. Regardless, we must now only check that this formula really is the same as that for $V(2n+1)^2$.

We proceed by induction. For both sides, evaluation at $n = 1$ gives one. Now, by our above simplification, we have

$$\frac{S(2n-1)}{S(2n+1)} = \left(\frac{n}{(3n)(3n-1)(3n-2)} \right) \left(\frac{(4n)!(4n-1)!(4n-2)!(4n-3)!}{(6n-2)!(6n-5)!(2n)!(2n-1)!} \right).$$

On the other hand, we also have

$$\frac{V(2n-1)^2}{V(2n+1)^2} = \left(\frac{2(4n-1)!(4n-2)!}{(6n-2)!(2n-1)!} \right)^2.$$

We leave it to the reader to check that these two expressions are equal, completing the proof.□

We conclude by providing a combinatorial interpretation of this identity. Because the central column of a VSASM has no zeros, we can represent an ordered pair of VSASMs as a single ASM with a non-zero middle column by choosing the left half from the first VSASM and the right half from the second. Thus, $V(2n+1)^2$ as the number of ASMs that have a central column with no zeros. It would therefore be interesting to see a direct bijection from $n \times n$ ASMs with their top-most one in an odd numbered column to the disjoint union of $n \times n$ ASMs with top-most one in an even numbered column and $n \times n$ ASMs whose middle column contains no zeros.

With many thanks to Eric Egge for his mathematical and stylistic help, and to Greg Kuperberg for his thoughts on proving this identity.

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