

ON INVOLUTIONS WITH MANY FIXED POINTS IN GASSMANN TRIPLES

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ABSTRACT. We show that in a non-trivial Gassmann triple (G, H, H') of index n there does not exist an involution $\tau \in G$ such that the value of the permutation character on τ is $n - 2$. In addition we describe a GAP program designed to search for examples of Gassmann triples and give a brief summary of the results of this search.

INTRODUCTION

Adolf Hurwitz was a German mathematician, a student of Felix Klein and a teacher of David Hilbert. When Frobenius left the Eidgenössische Technische Hochschule Zürich it was Hurwitz who took his chair and he remained there until his death in 1919. Fritz Gassmann, a student at the ETH Zürich under George Pólya and Hermann Weyl, was asked to look at a notebook of Hurwitz's containing unpublished work. In 1926 Gassmann published a section of this notebook along with an article explaining what he believed to be the point of Hurwitz's work [Gas26].

In his article Gassmann introduced the following condition on two subgroups H and H' of a group G : Each conjugacy class of G intersects H and H' in the same number of elements, that is for every $g \in G$ we have $|g^G \cap H| = |g^G \cap H'|$. Today we say two such subgroups are *Gassmann equivalent* and call (G, H, H') a *Gassmann triple*.

The motivation for studying Gassmann triples comes from several fields of mathematics. Most recently Terras and Stark showed in [TS00] that Gassmann equivalent subgroups can be used to create non-isomorphic graphs whose Ihara zeta functions are equal. In [Sun85] Sunada described how Gassmann equivalent subgroups of a group G can be used to construct riemannian manifolds that are isospectral but not isometric. Lastly Perlis has shown in [Per77] that two algebraic number fields F and F' share the same Dedekind zeta function precisely when the Galois groups $H = \text{Gal}(L/F)$ and $H' = \text{Gal}(L/F')$ are Gassmann equivalent subgroups of $G = \text{Gal}(L/\mathbb{Q})$ where L is a common normal extension. It is this last topic in which we find the motivation for this paper.

Cosets in the coset space G/H correspond to embeddings $F \hookrightarrow \mathbb{C}$. The group $G = \text{Gal}(L/\mathbb{Q})$ of automorphisms of L acts on these embeddings by left multiplication $G \curvearrowright G/H$. If $\tau \in G$ is the restriction of complex conjugation on \mathbb{C} to the

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subfield L then a fixed point of τ corresponds to a real embedding $F \hookrightarrow \mathbb{R}$. An element of order 2, such as τ , is called an *involution* and in section 3 we prove the following theorem.

Main Theorem. *If (G, H, H') is a Gassmann triple of index n and if there exists an involution $\tau \in G$ such that $\chi_{G/H}(\tau) = n - 2$ then the triple (G, H, H') is trivial.*

We call (G, H, H') a *trivial* Gassmann triple when H and H' are conjugate in G . The two fields F and F' are isomorphic iff their Galois groups H and H' are conjugate in G . Thus this main theorem proves that an algebraic number field F with only 2 non-real embeddings is uniquely determined up to isomorphism by its Dedekind zeta function. We also give an example of a non-trivial Gassmann triple (G, H, H') and an involution $\tau \in G$ such that $\chi_{G/H}(\tau) = n - 4$; thus the number $n - 2$ of fixed points of an involution cannot be reduced in a linear fashion.

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1. DEFINITIONS

Let G be a group and $H \leq G$ a subgroup of index n . G acts on the coset space G/H by left multiplication. The *fixed point character* of this action is the function $\chi_{G/H}: G \rightarrow \mathbb{N}_0$ defined by $g \mapsto |\text{Fix}(g)|$ where

$$\text{Fix}(g) = \{xH \in G/H \mid gxH = xH\}$$

is the set of cosets fixed under the action of the element g . The following proposition gives us a formula for the fixed point character. We will need this to prove proposition 5 in section 3.

Proposition 1 (Proposition 1.6 in [Bea91]). *Let $H \leq G$ be any subgroup. For all $g \in G$ we have $\chi_{G/H}(g) = \frac{|C_G(g)| \cdot |g^G \cap H|}{|H|}$.*

The connection between the permutation characters $\chi_{G/H}$ and $\chi_{G/H'}$ and the Gassmann equivalence of H and H' is given in the following proposition.

Proposition 2 (Lemma 1.9 in [Bea91]). *Let $H, H' \leq G$. The following conditions are equivalent:*

- (i) $\chi_{G/H} = \chi_{G/H'}$.
- (ii) $|g^G \cap H| = |g^G \cap H'| \quad \forall g \in G$.
- (iii) $\exists \phi \in \text{Bij}(H, H')$ satisfying $\phi(g) \in g^G \quad \forall g \in G$.

Note that we always have $\chi_{G/H}(1) = [G:H]$. We now expand the definition of Gassmann triple that was given in the introduction.

Definition 1. Let H and H' be subgroups of G . The triple (G, H, H') is called a *Gassmann triple* provided that any of the three equivalent conditions given in proposition 2 hold. Alternatively we may say that H and H' are *Gassmann equivalent subgroups* of the group G . As per the above note $\chi_{G/H}(1) = \chi_{G/H'}(1)$ implies that $[G:H] = [G:H']$. The *index* of the Gassmann triple (G, H, H') is defined to be this common index $[G:H] = [G:H']$.

Note that if H and H' are conjugate in G , then there is an inner automorphism of G taking H to H' . The restriction of this automorphism to H is a bijection satisfying condition (iii). Thus given any subgroup $H \leq G$, the triple (G, H, H^g) is Gassmann for all $g \in G$.

Definition 2. Let (G, H, H') be a Gassmann triple. We call (G, H, H') a *trivial* Gassmann triple if H and H' are conjugate in G . We call (G, H, H') a *faithful* Gassmann triple if the left multiplication action $G \curvearrowright G/H$ is a faithful action.

The kernel of the left multiplication action $G \curvearrowright G/H$ is exactly the set of elements that fix all $[G:H]$ cosets; that is, the elements $g \in G$ such that $\chi_{G/H}(g) = [G:H]$. In a Gassmann triple (G, H, H') we know that $\chi_{G/H} = \chi_{G/H'}$ thus the kernel of $G \curvearrowright G/H$ is equal to the kernel of $G \curvearrowright G/H'$. As elements in the kernel fix both H and H' the kernel is contained in $H \cap H'$.

Example 1. Consider the ring \mathbb{Z}_8 . The group of units \mathbb{Z}_8^* acts on the additive group \mathbb{Z}_8 by multiplication. Thus we can form the semidirect product $G = \mathbb{Z}_8^* \ltimes \mathbb{Z}_8$ whose group operation is $(a_1, b_1)(a_2, b_2) = (a_1 a_2, a_2 b_1 + b_2)$. Let H and H' be the subgroups $\{(1, 0), (3, 0), (5, 0), (7, 0)\}$ and $\{(1, 0), (3, 4), (5, 4), (7, 0)\}$ respectively. We claim that (G, H, H') is a non-trivial faithful Gassmann triple.

Proof of claim. Define the map $\phi: H \rightarrow H'$ as follows.

$$\begin{aligned} (1, 0) &\xrightarrow{\phi} (1, 0) \\ (3, 0) &\mapsto (3, 4) = (3, 0)^{(1,2)} \\ (5, 0) &\mapsto (5, 4) = (5, 0)^{(1,1)} \\ (7, 0) &\mapsto (7, 0) \end{aligned}$$

The map ϕ satisfies (iii) of theorem 2 thus (G, H, H') is Gassmann.

Assume that H and H' are conjugate, that is there is some element $(a, b) \in G$ such that $H^{(a,b)} = H'$. Conjugation in G fixes the first factor of any element, thus we must have $(3, 0)^{(a,b)} = (3, 4)$ and $(7, 0)^{(a,b)} = (7, 0)$. Note that $(a, b)^{-1} = (a, -ab)$. First we compute $(3, 4) = (a, b)(3, 0)(a, -ab) = (3, 2ab)$ which gives us $2ab \equiv 4 \pmod{8}$. But then we see that $(a, b)(7, 0)(a, -ab) = (7, 6ab) = (7, 4) \neq (7, 0)$. Thus the assumption that H and H' are conjugate is false. The Gassmann triple is non-trivial.

Finally observe that the only non-identity element in the intersection $H \cap H'$ is $(7, 0)$ so the kernel of $G \curvearrowright G/H$ is either $\{1\}$ or $\{1, (7, 0)\}$. But $(7, 0)(1, 1)H = (7, 1)H$ and we see that $(1, 1)H \neq (7, 1)H$ because $(7, 1)^{-1}(1, 1) = (7, 2) \notin H$. Thus $(7, 0)$ moves $(1, 1)H$ and so is not in the kernel. Hence the kernel is $\{1\}$; the triple is faithful. \square

We are interested in the involutions of G , that is the elements of order 2.

Example 2. Note that conjugation in a group G is an automorphism of G , thus every conjugate of an involution is also an involution. Hence when listing all involutions in a particular group it suffices to list them up to conjugation. Let G , H , and H' be as in example 1. The involutions in G up to conjugation are $(1, 4)$, $(3, 0)$, $(5, 0)$, $(7, 0)$, and $(7, 1)$. The index is $n = 8$ and the reader can easily compute that $\chi_{G/H}(5, 0) = 4$.

Consider a group G containing an involution $\tau \in G$ and a subgroup $H \leq G$ of index n . The action of $G \curvearrowright G/H$ gives us a permutation representation of τ .

As τ is an involution this permutation will be either the identity or a product of some number of disjoint 2-cycles and some number of 1-cycles. The number of fixed points of τ , that is the number of 1-cycles, will be the index n minus the number of moved points of those 2-cycles. Thus the possible values of $\chi_{G/H}(\tau)$ are $n, n-2, n-4, \dots$ and so on. If $\chi_{G/H}(\tau) = n$ then τ is in the kernel of the action thus (G, H, H') is a non-faithful triple. We have just seen in example 2 an involution τ in a non-trivial faithful triple satisfying $\chi_{G/H}(\tau) = n-4$. We will investigate the consequences when $\chi_{G/H}(\tau) = n-2$.

2. A STRUCTURE THEOREM FOR $\langle \tau^G \rangle$

In this section we will show that strict conditions are put on the structure of the group generated by the conjugacy class of an involution with $n-2$ fixed points. For the remainder of the section we make the following definitions. Let G be a group, $H \leq G$ a subgroup of index n , and let $\tau \in G$ be an involution such that $\chi_{G/H}(\tau) = n-2$. Define $N = \langle \tau^G \rangle$.

For the remainder of this paper there will only be one type of action. This action is G acting on the coset space G/H by left multiplication. We will however consider the orbits and stabilizers of this action when it is restricted to various subgroups of G . Given a subgroup $J \leq G$ we will refer to the orbits of the restricted action $J \curvearrowright G/H$ as J -orbits and will use $\text{Stab}_J(gH)$ to denote the stabilizer of a coset gH under the action of J .

We define k to be the number of distinct N -orbits and will denote these orbits $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_k$. Finally define K to be the kernel of the restricted action $N \curvearrowright G/H$. We begin with a lemma.

Lemma 1. *Let $J \leq G$ be any subgroup. Given any two distinct cosets $xH, yH \in G/H$, if there exists an element $a \in \langle \tau^G \cap J \rangle$ such that $axH = yH$ then there exists an element $a' \in \tau^G \cap J$ such that $a'xH = yH$.*

Proof. We will prove that if $w_1, w_2 \in \tau^G \cap J$ and $(w_1w_2)xH = zH$ where $zH \neq xH$ then there exists a $w_3 \in \tau^G \cap J$ such that $w_3xH = zH$. This suffices to prove the lemma because the element $a \in \langle \tau^G \cap J \rangle$ can be written as a word $a = w_1w_2 \cdots w_s$ with letters $w_i \in \tau^G \cap J$. We will have shown that we can reduce the length of this word without altering where the coset xH is sent. This reduction can be continued inductively until the word consists of a single letter $a' \in \tau^G \cap J$.

So let $w_1, w_2 \in \tau^G \cap J$ with $(w_1w_2)xH = zH$. Define $uH = w_2xH$ so that $w_1uH = zH$. If $uH = zH$ then set $w_3 = w_2$. Similarly if $uH = xH$ then set $w_3 = w_1$. All that is left is the case when uH, xH , and zH are 3 distinct cosets.

We have $w_1, w_2 \in \tau^G$ so these elements are involutions that move exactly two cosets. As $w_1uH = zH$ we know that w_1 moves uH and zH thus fixes xH . Set $w_3 = w_2^{w_1}$; this element is in the conjugacy class of τ and is a product of elements in J thus $w_3 \in \tau^G \cap J$. As desired we have $w_3xH = w_1w_2w_1xH = w_1w_2xH = zH$. \square

We now focus our attention on the restricted action $N \curvearrowright G/H$ with orbits $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_k$ in order to determine the structure of N .

Proposition 3. *Given any two N -orbits \mathcal{O}_i and \mathcal{O}_j we have $|\mathcal{O}_i| = |\mathcal{O}_j|$.*

Proof. Choose two elements $x, y \in G$ such that $xH \in \mathcal{O}_i$ and $yH \in \mathcal{O}_j$. Every coset in \mathcal{O}_i can be written in the form axH for some $a \in N$. Define the map $\phi: \mathcal{O}_i \rightarrow \mathcal{O}_j$ in terms of the chosen representatives x and y by $axH \xrightarrow{\phi} a^{yx^{-1}}yH$.

Note that ϕ maps into \mathcal{O}_j because N is normal so $a^{yx^{-1}} \in N$. Also ϕ is well defined; it does not depend on the choice of element $a \in N$. For assume $axH = bxH$ for some $a, b \in N$; then $axy^{-1}yH = bxy^{-1}yH \Rightarrow yx^{-1}axy^{-1}yH = yx^{-1}bxy^{-1}yH \Rightarrow a^{yx^{-1}}yH = b^{yx^{-1}}yH \Rightarrow \phi(axH) = \phi(bxH)$.

I claim that ϕ is onto. Every coset in \mathcal{O}_j can be written in the form ayH for some $a \in N$. Then $a^{xy^{-1}}xH \in \mathcal{O}_i$ and $\phi(a^{xy^{-1}}xH) = (a^{xy^{-1}})^{yx^{-1}}yH = ayH$. Thus ϕ is onto giving $|\mathcal{O}_i| \geq |\mathcal{O}_j|$. By symmetry we have $|\mathcal{O}_j| \geq |\mathcal{O}_i|$ thus $|\mathcal{O}_i| = |\mathcal{O}_j|$. \square

We now define $m = |\mathcal{O}_1|$. By the previous proposition m is the order of any N -orbit. There are k such orbits thus $n = km$. Note that τ is an element in N that moves two cosets in some N -orbit so $m > 1$.

We've just seen that G/H is the union of k orbits under N . For $i \in \{1, 2, \dots, k\}$ the action of N on the N -orbit \mathcal{O}_i is transitive on m points. By associating the cosets in \mathcal{O}_i to the integers $\{1, 2, \dots, m\}$ we obtain a homomorphism $\pi_i: N \rightarrow S_m$ called the *permutation representation of N acting on the i^{th} N -orbit*.

Note that each element of τ^G moves exactly 2 cosets in G/H . Thus if $t \in \tau^G$ transposes two cosets in \mathcal{O}_i then it fixes the cosets in every other N -orbit giving $t \in \ker \pi_j$ for all $j \neq i$. That is, each element of τ^G is in the support of π_i for exactly one $i \in \{1, 2, \dots, k\}$ (the *support* of a homomorphism is the set of elements in the domain that are not in the kernel).

Proposition 4. *Let $I = \{1, 2, \dots, k\}$, then we have the following.*

- (i) *Each $t \in \tau^G$ is in the support of π_i for exactly one $i \in I$.*
- (ii) *For each $t \in \tau^G$ and each $i \in I$, the permutation $\pi_i(t)$ is either the identity or a 2-cycle.*
- (iii) *For each $i \in I$ every 2-cycle in S_m is contained in the set $\pi_i(\tau^G)$.*
- (iv) *π_i maps N onto S_m .*

Proof.

- (i) This is given in the paragraph above.
- (ii) Let $t \in \tau^G$ and $i \in I$. If $\pi_i(t)$ is not the identity then t moves cosets in the i^{th} N -orbit. As $t \in \tau^G$ we know that t is an involution that moves exactly 2 cosets in G/H . Thus $\pi_i(t)$ is a permutation that moves exactly 2 points so $\pi_i(t)$ is a 2-cycle.
- (iii) Let $i \in I$ and let $\sigma \in S_m$ be any 2-cycle, $\sigma = (a b)$. The points a and b correspond to two cosets in the i^{th} N -orbit, call them xH and yH . Because they are in the same N -orbit there exists an element $g \in N$ such that $gxH = yH$. Lemma 1 with $J = N$ states that there exists a $t \in \tau^G$ such that $txH = yH$. Then $\pi_i(t)$ is a two cycle that moves the points associated to xH and yH , that is $\pi_i(t) = \sigma$.
- (iv) A standard generating set for S_m is the set of all 2-cycles that move the point 1. By (iii) these 2 cycles are all contained in $\pi_i(\tau^G) \subset \pi_i(N) \leq S_m$ thus $\pi_i(N) = S_m$. \square

Now we are in a position to say something about the structure of N .

Theorem 1. *There exists an epimorphism $\psi: N \rightarrow (S_m)^k$ and the kernel of this epimorphism is K , the kernel of $N \curvearrowright G/H$.*

Proof. Define $\psi: N \rightarrow (S_m)^k$ by $g \mapsto (\pi_1(g), \pi_2(g), \dots, \pi_k(g))$. Each π_i is a homomorphism thus ψ is a homomorphism. By (iv) in proposition 4 every $\sigma_i \in S_m$ has a π_i pre-image, $g_i \in N$. As $N = \langle \tau^G \rangle$ express g_i as a word in the letters τ^G . Then redefine g_i by throwing out any letters in the kernel of π_i . We still have $\pi_i(g_i) = \sigma_i$ but by (i) in proposition 4 we have chosen a g_i such that $\pi_j(g_i) = 1$ for all $j \neq i$.

Now given any element $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k) \in (S_m)^k$ we choose a pre-image $g_i \in N$ for each $\sigma_i \in S_m$ as described above and form the product $g = \prod_i g_i$. Then we have

$$\begin{aligned} \psi(g) &= (\pi_1(g), \pi_2(g), \dots, \pi_k(g)) \\ &= \left(\pi_1 \left(\prod_i g_i \right), \pi_2 \left(\prod_i g_i \right), \dots, \pi_k \left(\prod_i g_i \right) \right) \\ &= \left(\prod_i \pi_1(g_i), \prod_i \pi_2(g_i), \dots, \prod_i \pi_k(g_i) \right) \\ &= (\pi_1(g_1), \pi_2(g_2), \dots, \pi_k(g_k)) \\ &= (\sigma_1, \sigma_2, \dots, \sigma_k) \\ &= \sigma \end{aligned}$$

Thus ψ is onto. Finally K , the kernel of $N \curvearrowright G/H$, consists of exactly those elements that are in the kernel of the action $N \curvearrowright \mathcal{O}_i$ for every i . Thus $K = \bigcap_i \ker \pi_i = \ker \psi$. \square

The previous theorem along with the first isomorphism theorem gives that $N/K \cong (S_m)^k$. If $G \curvearrowright G/H$ is faithful we have that $N \cong (S_m)^k$.

3. THE STRUCTURE OF GASSMANN TRIPLES

In this section we will continue the analysis of the previous section and so will retain the definitions of the group G , subgroup H , and involution τ . We add the condition that there is a subgroup $H' \leq G$ and the triple (G, H, H') is faithful Gassmann.

We also wish to define the subgroup $N_{H'} = \langle \tau^G \cap H' \rangle$. The $N_{H'}$ -orbits of this action will be essential in investigating the relationship between H and H' . We begin this investigation by determining the cardinality of some of the sets that interest us.

Proposition 5. *The number of elements in G conjugate to τ is $|\tau^G| = \frac{1}{2}n(m-1)$. The number of conjugates of τ in H' is $|\tau^G \cap H'| = \frac{1}{2}(m-1)(n-2)$.*

Proof. As the action is faithful, ψ of theorem 1 is an isomorphism. So to determine $|\tau^G|$ we count its image under ψ . By (i)-(iii) of proposition 4, $\psi(\tau^G)$ is the set of all 2-cycles in all factors of $(S_m)^k$. There are $\frac{1}{2}m(m-1)$ 2-cycles in S_m and k factors of S_m , thus $|\tau^G| = k \cdot \frac{1}{2}m(m-1) = \frac{1}{2}n(m-1)$.

To determine $|\tau^G \cap H'|$ note that $\chi_{G/H'}(\tau) = n-2$. Proposition 1 then gives $|\tau^G \cap H'| = \frac{(n-2)|H'|}{|C_G(\tau)|}$. Using $|\tau^G \cdot |C_G(\tau)|| = |G|$ we have $|\tau^G \cap H'| = \frac{(n-2)|H'| \cdot |\tau^G|}{|G|} = \frac{(n-2)}{n} |\tau^G| = \frac{1}{2}(m-1)(n-2)$. \square

What will be of particular interest to us are the number of elements in τ^G that are not in H' . This can easily be computed from proposition 5. We have $|\tau^G - H'| = |\tau^G| - |\tau^G \cap H'| = \frac{1}{2}n(m-1) - \frac{1}{2}(m-1)(n-2) = m-1$.

Now consider the subgroup $N_{H'} = \langle \tau^G \cap H' \rangle$. The generating set $\tau^G \cap H'$ of $N_{H'}$ is contained in the generating set τ^G of N thus $N_{H'} \leq N$. This implies that N -orbits are disjoint unions of one or more $N_{H'}$ -orbits. The question we now ask is when moving from the action $N \curvearrowright G/H$ to the action $N_{H'} \curvearrowright G/H$ how do the N -orbits decompose into the $N_{H'}$ -orbits?

It cannot be the case that the $N_{H'}$ -orbits are exactly the N -orbits. As $m > 1$ we have shown above that there are elements in τ^G that are not in H' . If $t \in \tau^G - H'$ moves xH and yH then those two cosets must be in different $N_{H'}$ -orbits. For suppose there exists an $a \in N_{H'}$ such that $axH = yH$. Then lemma 1, with $J = H'$, states that there exists an $a' \in \tau^G \cap H'$ such that $a'xH = yH$. The action on the coset space G/H is assumed to be faithful thus we must have $a' = t$. But $t \notin H'$ so this is a contradiction.

Thus at least one N -orbit must decompose into two or more $N_{H'}$ -orbits. We will see that exactly one of the N -orbits decomposes into exactly two $N_{H'}$ -orbits of length 1 and $m-1$.

Theorem 2. *The $N_{H'}$ -orbits consist of $k-1$ orbits of length m , 1 orbit of length $m-1$, and 1 orbit of length 1.*

Proof. For $A = N$ or $N_{H'}$ define the undirected graph $\Gamma(A)$ as follows. The vertex set of $\Gamma(A)$ is the coset space G/H . We associate $\tau^G \cap A$ with the edge set, that is we join two vertices if there is a conjugate of τ in A that transposes the cosets associated with those vertices. Note that because the action on G/H is faithful no two distinct elements of τ^G transpose the same two cosets, thus the association of $\tau^G \cap A$ with the edge set is a one-to-one correspondence.

Lemma 1 with $J = N$ shows that if two cosets are in the same N -orbit then they are connected by an edge in $\Gamma(N)$. Similarly lemma 1 with $J = H'$ shows that if two cosets are in the same $N_{H'}$ -orbit then they are connected by an edge in $\Gamma(N_{H'})$. Thus the graph $\Gamma(A)$ represents the A -orbits of the action on the coset space G/H . An edge connects two vertices in $\Gamma(N)$ iff the cosets that those vertices represent are in the same N -orbit and an edge connects two vertices in $\Gamma(N_{H'})$ iff the cosets that those vertices represent are in the same $N_{H'}$ -orbit.

So $\Gamma(N)$ is a disconnected graph with k components. Each component has m vertices and is complete (every vertex in the component is connected to every other vertex in that component). As N -orbits decompose into $N_{H'}$ -orbits the graph $\Gamma(N_{H'})$ is obtained from the graph $\Gamma(N)$ by removing a number of edges. As discussed above, elements in $\tau^G - H'$ transpose cosets that are in the same N -orbit but not in the same $N_{H'}$ -orbit. Thus the edges removed from $\Gamma(N)$ to obtain $\Gamma(N_{H'})$ are precisely the $m-1$ edges associated with elements in $\tau^G - H'$.

As stated above at least one of the N -orbits $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_k$ decomposes into two or more $N_{H'}$ -orbits. Let \mathcal{O}_i be such an N -orbit and let \mathcal{A} be one of the $N_{H'}$ -orbits that \mathcal{O}_i decomposes into. We have $\mathcal{O}_i \neq \mathcal{A}$ because \mathcal{O}_i decomposes thus define $\mathcal{B} = \mathcal{O}_i - \mathcal{A}$ to be the complement of \mathcal{A} in \mathcal{O}_i .

We have that the component of $\Gamma(N)$ corresponding to \mathcal{O}_i is the complete graph on m vertices. Let $a = |\mathcal{A}|$ and $b = |\mathcal{B}|$ so that the component of $\Gamma(N_{H'})$ corresponding to \mathcal{A} is the complete graph on a vertices and $a+b = m$. Now when passing from $\Gamma(N)$ to $\Gamma(N_{H'})$ we must disconnect the component of $\Gamma(N)$ corresponding

to \mathcal{A} from the vertices in \mathcal{B} . Each of the a vertices in \mathcal{A} is connected to each of the b vertices in \mathcal{B} , thus we must remove $e = ab$ edges. Given $e = ab$, $a + b = m$, and $1 \leq a, b \leq m - 1$ it is a simple calculus problem to find the minimum possible value of e when considered as a function of a and b . This value is $e = m - 1$ and occurs when a and b take the values 1 and $m - 1$.

Thus the minimal case of disconnecting only a single component of $\Gamma(N_{H'})$ requires the removal of $m - 1$ edges. As only $m - 1$ edges are to be removed it cannot be the case that the component of $\Gamma(N)$ that corresponds to \mathcal{O}_i decomposes into more than two $\Gamma(N_{H'})$ components for that would require the removal of additional edges. It must decompose into exactly 2 components of sizes 1 and $m - 1$. It also cannot be the case that any component of $\Gamma(N)$ other than the one that corresponds to \mathcal{O}_i decomposes because that would again involve removing more than $m - 1$ edges. Thus exactly one of the N -orbits decomposes into two $N_{H'}$ -orbits of length 1 and $m - 1$. The other $k - 1$ N -orbits do not decompose. Hence the $N_{H'}$ -orbits consist of $k - 1$ orbits of length m , 1 orbit of length $m - 1$, and 1 orbit of length 1. \square

Now that we know something about the $N_{H'}$ -orbits we ask how these orbits behave under the action of H' . It turns out that H' takes $N_{H'}$ -orbits to $N_{H'}$ -orbits.

Proposition 6. *If xH and yH are in the same $N_{H'}$ -orbit, then for every $h \in H'$ the cosets hxH and hyH are also in the same $N_{H'}$ -orbit.*

Proof. Let xH and yH be two cosets in the same $N_{H'}$ -orbit and let h be an arbitrary element of H' . There exists an $a \in N_{H'}$ such that $axH = yH$. Note that $N_{H'} \trianglelefteq H'$ thus we may define $b = a^h$ and we have $b \in N_{H'}$. Then $bhxH = hah^{-1}hxH = haxH = hyH$ thus hxH and hyH are in the same $N_{H'}$ -orbit. \square

We now must consider 2 cases, when $m = 2$ and when $m > 2$. When $m > 2$ we have $m - 1 > 1$ so theorem 2 gives us that there is a unique $N_{H'}$ -orbit of length 1. As H' sends $N_{H'}$ -orbits to $N_{H'}$ -orbits it must send this orbit to itself, that is it must stabilize the coset in this orbit. On the other hand when $m = 2$ we have $m - 1 = 1$ so theorem 2 gives us that there are 2 $N_{H'}$ -orbits of length 1. Thus in this case it is not immediately obvious that some coset of H is stabilized.

We wish to show that it must always be the case that H' stabilizes some coset of H and that this implies that the Gassmann triple (G, H, H') is trivial. This is a key result of this paper, that a non-trivial faithful Gassmann triple of index n cannot contain in involution with $n - 2$ fixed points. The main theorem given in the introduction will follow easily from this.

Theorem 3. *If (G, H, H') is a faithful Gassmann triple of index n and if there exists an involution $\tau \in G$ such that $\chi_{G/H}(\tau) = n - 2$ then the triple (G, H, H') is trivial.*

Proof. We begin by showing that if H' stabilizes any coset of H , then the triple is trivial. As stabilizers of the action $G \curvearrowright G/H$ are well known this is easy to see. Assume xH is a coset stabilized by H' . Then we have $H' \leq \text{Stab}_G(xH) = xHx^{-1}$. But $|H'| = |H| = |xHx^{-1}|$ thus $H' = xHx^{-1}$. This proves that the triple (G, H, H') is trivial.

Now we wish to show that H' always stabilizes some coset in G/H . The case when $m > 2$ is done; as discussed above $m > 2$ implies that H' stabilizes the unique $N_{H'}$ -orbit of length 1. What remains is only the case when $m = 2$.

So we assume that $m = 2$ and no coset in G/H is stabilized by H' . We intend to derive a contradiction from this assumption by constructing an element $h' \in H'$ such that $\chi_{G/H}(h') = 0$. By condition (iii) in proposition 2 the element h' is conjugate to some element $h \in H$. As $\chi_{G/H}$ is a character of G it is constant on conjugacy classes. This gives $\chi_{G/H}(h) = 0$, but h fixes the coset H thus this is a contradiction.

Recall that $n = mk = 2k$; proposition 5 then gives $|\tau^G| = k$. Each N -orbit contains cosets that are transposed by some conjugate of τ and there are k N -orbits thus there is exactly one such conjugate of τ for each N -orbit. Let $\tau^G = \{t_1, t_2, \dots, t_k\}$ where t_j is the conjugate of τ that transposes the two cosets in the j^{th} N -orbit \mathcal{O}_j .

By theorem 2 exactly 1 N -orbit decomposes into 2 $N_{H'}$ -orbits of length 1. Let \mathcal{O}_i be this N -orbit. The element t_i transposes the two cosets in \mathcal{O}_i ; as these cosets are not in the same $N_{H'}$ -orbit we have $t_i \notin N_{H'}$. The set $\tau^G \cap H'$ generates $N_{H'}$ and $t_i \in \tau^G$ so $t_i \notin H'$. Proposition 5 gives $|\tau^G \cap H'| = k - 1$ thus t_i is the only element of τ^G that does not lie in H' .

Let $xH, yH \in \mathcal{O}_i$ be the two cosets that are transposed by t_i . These two cosets form the two length 1 $N_{H'}$ -orbits. As H' does not stabilize any coset there exists some $g \in H'$ such that g moves xH . We must have $gxH = yH$ because H' takes $N_{H'}$ -orbits to $N_{H'}$ -orbits and yH is the only other orbit of length 1. So the coset yH is also moved by g .

For $j \in \{1, 2, \dots, k\}$ note that g either fixes both cosets in \mathcal{O}_j or moves both cosets. This is because g moves both cosets in \mathcal{O}_i and for $j \neq i$ if g moves the orbit \mathcal{O}_j then it obviously must move both cosets in the orbit. If g does not move the orbit \mathcal{O}_j then it can only move one coset in the orbit by sending it to the other coset in that same orbit, thus g either transposes the two cosets or fixes the two cosets.

Define the set $F = \{j \in \{1, 2, \dots, k\} \mid g \text{ fixes both cosets in } \mathcal{O}_j\}$ and let

$$\eta = \prod_{j \in F} t_j.$$

Note that $i \notin F$ so η is a product of conjugates of τ , none of which is equal to t_i , thus $\eta \in H'$. Finally we define $h' = \eta g$.

Observe that a coset is moved by η iff the coset belongs to an orbit \mathcal{O}_j whose index j is in F . Also a coset is moved by g iff the coset belongs to an orbit \mathcal{O}_j whose index j is not in F . Thus every coset is moved by $h' = \eta g \in H'$ as desired. \square

We end the theoretical section of this paper by using two lemmas of Beaulieu to prove our final result, that the previous theorem can be lifted to the case when (G, H, H') is not faithful.

Main Theorem. *If (G, H, H') is a Gassmann triple of index n and if there exists an involution $\tau \in G$ such that $\chi_{G/H}(\tau) = n - 2$ then the triple (G, H, H') is trivial.*

Lemma 2 (Lemma 1.13 in [Bea91]). *Let $H \leq G$ be a subgroup and let $K \leq H$ be normal in G . Then $\chi_{G/H}(g) = \chi_{(G/K)/(H/K)}(gK)$ for all $g \in G$.*

Lemma 3 (Lemma 1.14 in [Bea91]). *Let (G, H, H') be a non-trivial Gassmann triple. Let K be the common kernel of the actions $G \curvearrowright G/H$ and $G \curvearrowright G/H'$. Then $(G/K, H/K, H'/K)$ is a non-trivial faithful Gassmann triple.*

Proof of Main Theorem. Let (G, H, H') be a Gassmann triple of index n and let $\tau \in G$ be an involution satisfying $\chi_{G/H}(\tau) = n - 2$. Let K be the common kernel of the coset actions. Assume that (G, H, H') is non-trivial. By lemma 3 $(G/K, H/K, H'/K)$ is a non-trivial faithful Gassmann triple and clearly the index remains n . As τ moves two cosets $\tau \notin K$ but $\tau^2 = 1 \in K$. We conclude that $\tau K \in G/K$ is an involution and by lemma 2 we have $\chi_{(G/K)/(H/K)}(\tau K) = n - 2$. This contradicts theorem 3 thus the original assumption that the triple (G, H, H') is non-trivial is false. \square

4. SEARCHING FOR GASSMANN TRIPLES

In order to further study Gassmann triples we wish to generate a large number of non-trivial faithful examples. The computer algebra system GAP was chosen to perform a brute force search for Gassmann triples. As some of the computations involved in this search are time consuming we have written a collection of GAP functions that perform the search and save the resulting triples to file for future study.

These functions look for non-trivial faithful Gassmann triples up to isomorphism. Given two Gassmann triples (G_1, H_1, H'_1) and (G_2, H_2, H'_2) we say the two triples are isomorphic when there exists an isomorphism $\phi: G_1 \rightarrow G_2$ that satisfies either $\phi(H_1) = H_2$ and $\phi(H'_1) = H'_2$ or $\phi(H_1) = H'_2$ and $\phi(H'_1) = H_2$. We allow the second condition because given two Gassmann equivalent subgroup H and H' of a group G we wish to consider (G, H, H') and (G, H', H) to be equivalent triples.

There are two methods implemented to search for Gassmann triples. The first method uses GAP's library of transitive subgroups of the symmetric groups. This method was intended to be able to find all Gassmann triples of a given index n but memory overflow problems occurred when attempting a complete search of S_{10} . The symmetric groups of degree 9 and less have 90 transitive subgroups [GAP06]. A total of 143 Gassmann triples were found.

The second method uses GAP's small groups library. This method searches for all Gassmann triples (G, H, H') in which the group G has a given order. Due to time constraints we have only searched all groups of order 200 and less. We have encountered no memory problems and believe that searching through larger orders is quite feasible. There are 6065 groups of order less than or equal to 200 [GAP06]. A total of 531 Gassmann triples were found.

Accounting for the overlap between the two methods 6088 groups were searched. In total 657 non-trivial faithful non-isomorphic Gassmann triples were found. The source code for the GAP functions is fully commented and is available online at <http://www.math.lsu.edu/~jstark/gassmann.g>. The data file containing the results of the search described above is at <http://www.math.lsu.edu/~jstark/gassmann.dat>.

5. FURTHER QUESTIONS

Here we give two questions that may motivate further research on the topic of Gassmann triples and involutions in Gassmann triples.

5.1. What is the minimum number of prime factors in the order of a group containing a non-trivial faithful Gassmann triple? Let G be a group whose order has the prime factorization $|G| = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$. Define $P(G) = \sum_{i=1}^k a_i$. That is $P(G)$ is the number of primes in $|G|$ counting duplicates. In each of the 657 examples of non-trivial faithful Gassmann triples that we have found it is always the case that $P(G) \geq 5$. Does a non-trivial faithful Gassmann triple (G, H, H') with $P(G) < 5$ exist?

5.2. Can we search for Gassmann triples by index efficiently? Currently the program can only search for Gassmann triples within a given group. Systematic searches for triples rely on a systematic ordering of groups, in this case by searching through transitive permutation groups. If (G, H, H') is a faithful Gassmann triple of index n then using the action $G \curvearrowright G/H$ we can see that G is a transitive subgroup of S_n .

Theoretically an algorithm such as the one executed by `FindGassmannInSn()` could search through S_n to find all such triples. Unfortunately our implementation fails due to memory problems at $n = 10$. With more resources we may be able to achieve a full search of S_{10} ; but, given any finite amount of memory we can easily imagine a large enough n for the current method to exhaust it. Additionally this method is slow to execute and the complexity is very high. Larger values of n will quickly run into problems with processing time.

Thus the current algorithm is simply not practical. Given an arbitrary index n it is not known if there is a practical method of constructing a list of all non-trivial faithful Gassmann triples of index n .

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