

The Box Problem

Marissa Predmore
California Lutheran University
Advisor: Christina Soderlund

April 9, 2008

Abstract

The box problem is taken from Calculus 1, where a student is asked to maximize the volume of a box constructed from a rectangular piece of cardboard with squares removed at the corners. We are interested in what the width and length need to be in order to have at least a rational answer for the optimum height. In 2000, Cuoco used Eisenstein triples to find the dimensions. Hotchkiss expanded on Cuoco's work in 2002 and used an elliptical equation to find the dimensions needed for the box. This paper answers two open questions posed by Hotchkiss: proving that the smallest possible distinct dimensions that produce a rational solution are 3 and 8; and proving that the smallest possible distinct dimensions that produce an integral solution are 5 and 8. Also the minimum distinct dimensions are examined in general.

1 Introduction

Here is an example of a box problem ([5]):

A box with an open top is to be constructed from a rectangular piece of cardboard 3 ft. wide by 4 ft. long, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.

In order to solve this problem, a student first writes an equation for the volume of the box: $V(s) = (3 - 2s)(4 - 2s)s$. Then the student takes the derivative and finds the critical points of V and after solving, determines that when $s = \frac{7+\sqrt{13}}{6}$, the volume is the largest possible. Irrational answers such as this one are not very satisfying to the student, so Cuoco and Hotchkiss came up with methods to find the values of the dimensions (we call a and b) that ensure that s is either rational or an integer.

In [1], Cuoco used Eisenstein triples to solve this problem. Suppose the dimensions of the

rectangle are a and b . Then the volume of the box will be

$$V(s) = (a - 2s)(b - 2s)s = 4s^3 - 2(a + b)s^2 + abs,$$

where s is the height of the box that the student needs to find. Since we need to maximize the volume we take the derivative and get $V'(s) = 12s^2 - 4(a + b)s + ab = 0$ and then solve for s . By the quadratic formula, $s = \frac{4(a+b) \pm \sqrt{16(a+b)^2 - 48ab}}{24}$. This gives us two solutions, but we will always choose the solution that is positive and less than half the length of the piece of cardboard.

The goal is to find what a and b have to be, in order for the solution s to be rational or an integer. For this to occur, we need the discriminant $16(a + b)^2 - 48ab$ to be a perfect square. We know 16 is a perfect square, so factoring leaves us with the expression $(a + b)^2 - 3ab$, which simplifies to $a^2 - ab + b^2$. Since $\sqrt{a^2 - ab + b^2}$ must be rational, we know

$$a^2 - ab + b^2 = c^2, \tag{1}$$

where c is rational. Notice that this is the definition of an Eisenstein triple, (a, b, c) ,

In [2], Hotchkiss proves the following theorem.

Theorem 1.1 ([2]). *Let a and b be rational numbers. Then the solution to the Box Problem occurs at a rational value of s if and only if $a = r(1 - m^2)$ and $b = r(2m - m^2)$ where r and m are rational numbers with $0 < m < 1$.*

To illustrate this theorem, let us choose $r = 2$ and $m = \frac{1}{3}$. This gives us the dimensions $a = \frac{16}{9}$ and $b = \frac{10}{9}$. Notice that Hotchkiss' solution produces rational dimensions as well as rational answers for the height.

This paper will address the two open questions given in [2]:

Open Question 1. *Is $a = 3$ and $b = 8$ the smallest pair of distinct integers that give a rational solution to the Box Problem?*

Open Question 2. *Is $a = 5$ and $b = 8$ the smallest pair of distinct integers that give an integer solution to the Box Problem?*

We end with a conjecture for finding a formula for the minimum integer value of b given a .

The author would like to thank the referee for providing a thorough and helpful review of the paper, and the CLU mathematics department for the opportunity to research at this level as an undergraduate.

2 Smallest Dimensions for Rational Solutions

We can write Equation (1) as

$$a^2 - ab + b^2 = \left(\frac{p^2}{q^2}\right), \quad (2)$$

where p and q are integers. When we want a and b to be distinct we know more about q , which is demonstrated in the following lemma.

Lemma 2.1. *If $a^2 - ab + b^2 = \left(\frac{p^2}{q^2}\right)$, where a, b, p , and q are integers with p and q relatively prime, then $q = 1$ or 2 .*

Proof. Solving Equation (2) for b gives $b = \frac{1}{2} \cdot \left(a \pm \sqrt{-3a^2 + \frac{4p^2}{q^2}}\right)$.

Since b is an integer, $a \pm \sqrt{-3a^2 + \frac{4p^2}{q^2}}$ must be equal to an integer that is divisible by 2 and $\sqrt{-3a^2 + \frac{4p^2}{q^2}}$ is an integer. Let $\sqrt{-3a^2 + \frac{4p^2}{q^2}} = w$ where w is an integer. Then $\frac{4p^2}{q^2} = w^2 + 3a^2$. This means that $\frac{4p^2}{q^2}$ must be an integer so the denominator is a factor of 4. Therefore q must equal 1 or 2. \square

A search for examples suggested that a cannot be 4; indeed, we have the following result.

Lemma 2.2. *If $a = 4$ and the solution s is rational, there are no distinct integer solutions for b .*

Proof [3]. From Equation (2) we let $\sqrt{a^2 - ab + b^2} = \frac{p}{q}$, where $p, q \in \mathbf{Z}$ and p and q are relatively prime. When $a = 4$, we get $\sqrt{16 - 4b + b^2} = \frac{p}{q}$ and $b = \frac{1}{2} \cdot \left(4 \pm \sqrt{-48 + \frac{4p^2}{q^2}}\right)$. In order for b to be an integer, the expression $\sqrt{-48 + \frac{4p^2}{q^2}}$, must be an even integer.

For the sake of contradiction, assume that $\sqrt{-48 + \frac{4p^2}{q^2}} = 2n$, where n is a positive integer, but $n \neq 2$ because $b \neq 4$. One can check that $n^2 + 12 = \frac{p^2}{q^2}$. Since n and 12 are integers, $n^2 + 12$ is an integer. Therefore $\frac{p^2}{q^2}$ is an integer so we can assume $q = 1$ and let $n^2 + 12 = p^2$. To find our contradiction, we need to show that the only possible solution to $n^2 + 12 = p^2$ is when $n = 2$. We know that

$$12 = (p - n)(p + n) \quad (3)$$

and p and n are integers so there is a finite set of integer factors of 12 which are 1,2,3,4,6,12. Let us find out what n equals when $(p - n)$ equals one of the possible factors.

Let $(p - n) = 1$, so $p = 1 + n$. Plugging this into Equation (3) gives $12 = 1 + 2n$ so $n = \frac{11}{2}$. However n must be an integer so we have reached a contradiction. When $(p - n)$ is any of the other factors, similar calculations lead to a contradiction demonstrated in the following table.

$(p - n)$	n	<i>Contradiction</i>
2	2	$n \neq 2$
3	$\frac{1}{2}$	n is an integer
4	$-\frac{1}{2}$	n is positive
6	-2	n is positive
12	$-\frac{11}{2}$	n is positive

Therefore there are no solutions to the equation $n^2 + 12 = p^2$, except for when $n = \pm 2$.

In order for b to be a distinct integer, we assumed that $\sqrt{-48 + \frac{4p^2}{q^2}} = 2n$, where n is a positive integer and $n \neq 2$. We have just shown that n is never an integer other than 2. Therefore our assumption was false, and b is never an integer other than 4. \square

Given these lemmas, now we can now address Hotchkiss' Open Question (1): Is $a = 3$ and $b = 8$ the smallest pair of distinct integers that give a rational solution to the Box Problem?

Theorem 2.1. *If a and b are distinct integers that give a rational solution to the Box Problem, then $(a > 3)$ or $(a = 3$ and $b \geq 8)$.*

Proof. First we will show that a cannot be less than 3. We do so by considering the cases when $a = 1$ and when $a = 2$.

When $a = 1$, suppose for the sake of contradiction, that b is an integer other than 1. From Equation (2) we let $\sqrt{a^2 - ab + b^2} = \frac{p}{q}$, where $p, q \in \mathbf{Z}$ and p and q are relatively prime. Since $a = 1$, $\sqrt{1 - b + b^2} = \frac{p}{q}$. One can check that $b = \frac{1}{2} \left(\sqrt{\frac{4p^2}{q^2} - 3} + 1 \right)$.

From Lemma 2.1 we know $q = 1$ or 2. If $q = 1$, then $b = \frac{1}{2} \left(\sqrt{4p^2 - 3} + 1 \right)$. Since we assumed b must be an integer, then $\sqrt{4p^2 - 3}$ must be odd. If we write $\sqrt{4p^2 - 3} = 2k + 1$, where k is a positive integer, then $p^2 = k^2 + k + 1$. Notice that $k^2 < k^2 + k + 1 < (k + 1)^2$. There does not exist a perfect square between k^2 and $(k + 1)^2$, so p cannot be an integer, which contradicts our definition of p . If $q = 2$, then $b = \frac{1}{2} \left(\sqrt{p^2 - 3} + 1 \right)$. Since we assumed that b is an integer, then $\sqrt{p^2 - 3}$ must be odd. If we write $\sqrt{p^2 - 3} = 2k + 1$, where k is a positive integer, then $p^2 = 4k^2 + 4k + 4$. Notice that $4k^2 + 4k + 4 = 2^2(k^2 + k + 1)$ and we have shown that $k^2 + k + 1$ cannot be a perfect square, which contradicts our definition of p . Therefore when $a = 1$, there is no distinct integer b satisfying equation (1).

When $a = 2$, suppose for the sake of contradiction that b is an integer other than 2. Substituting $a = 2$ into Equation (2) and solving for b gives $b = \sqrt{\frac{p^2}{q^2} - 3} + 1$. Since we are assuming that b is an integer, $\sqrt{\frac{p^2}{q^2} - 3}$ is also an integer. Let us call that integer y , so that $\frac{p^2}{q^2} - 3 = y^2$. This

means $\frac{p^2}{q^2} = y^2 + 3$, which is an integer. Since p and q are relatively prime, $\frac{p^2}{q^2}$ cannot be equal to an integer, thereby giving a contradiction. Therefore, when $a = 2$, there is no distinct integer b satisfying Equation (1).

We have already shown that $a \geq 3$ and now we will show that if $a = 3$, b cannot be less than 8. Since we are only considering integer dimensions, we only need to show that the following pairs (a, b) are not possible dimensions for the box problem: (3,1), (3,2), (3,4), (3,5), (3,6), (3,7).

From Theorem 1.1, we let $a = r(1 - m^2)$ and $b = r(2m - m^2)$. By solving the first equation for r , we get $r = \frac{a}{1-m^2}$ and by substitution, $b = \frac{a(2m-m^2)}{(1-m^2)}$. Thus,

$$(1 - m^2)b = a(2m - m^2) \tag{4}$$

where m is rational and $0 < m < 1$.

By plugging 3 into Equation (4) for a , and solving for b , we get the expression $b = \frac{3(2m-m^2)}{(1-m^2)}$. For all values of b , m is irrational, which is illustrated in the following table:

b	m
2	$3 - \sqrt{7}$
4	$-3 + \sqrt{13}$
5	$\frac{-3+\sqrt{19}}{2}$
6	$-1 + \sqrt{3}$
7	$\frac{-3+\sqrt{37}}{4}$

Thus, when $a = 3$, b cannot be an integer less than 8.

□

3 Smallest Dimensions for Integer Solutions

Thus far we have considered rational values for the solution to the Box Problem. Hotchkiss was able to limit the dimensions of the box even further to ensure that the solution to the Box Problem gives an integer solution with the following theorem:

Theorem 3.1 ([2]). *Let a and b be rational numbers. Then the maximum volume of the box constructed in the Box Problem occurs at an integral value of s if and only if $a = r(1 - m^2)$, $b = r(2m - m^2)$, and $r(m - m^2)$ is an even integer, where r and m are rational numbers with $0 < m < 1$.*

Now consider the Open Question (2): Is $a = 5$ and $b = 8$ the smallest pair of distinct integers that give an integral solution to the Box Problem?

Theorem 3.2. *If a and b are distinct integers that give an integral solution to the Box Problem, then $(a > 5)$ or $(a = 5$ and $b \geq 8)$.*

Proof. First we will show that a cannot be less than 5. We have already shown that $a \neq 1$ and $a \neq 2$ for rational solutions to the box problem in the proof of Theorem 2.1. It remains to show that $a \neq 3$ or 4.

When $a = 3$, suppose for the sake of contradiction that b is an integer other than 3. From Theorem 3.1, we let $3 = r(1 - m^2)$ where r and m are rational and $0 < m < 1$. Thus,

$$r = \frac{3}{1 - m^2}. \quad (5)$$

We know from [2], Equation 8, that the solution to the box problem is $s = \frac{r(m-m^2)}{2}$, so in order for s to be an integer, $r(m - m^2)$ must be an even integer. Since we are assuming that $a = 3$ provides an integral solution to the box problem when b is an integer other than 3, we can also assume that $r(m - m^2)$ is an even integer. After substituting equation (5) into this expression, we get $r(m - m^2) = \frac{3m}{1+m}$.

Define $m = \frac{u}{v}$ where u and v are positive integers, relatively prime, with $u < v$. Then,

$$r(m - m^2) = \frac{3m}{1 + m} = \frac{3\frac{u}{v}}{1 + \frac{u}{v}} = \frac{3u}{v + u}.$$

Since we assumed that $r(m - m^2)$ is an even integer, let $\frac{3u}{v+u} = 2n$ where n is a positive integer. Then one can check that $u = \frac{2nv}{3-2n}$. Since u , v , and n are all positive, this means that $\frac{3}{2} > n$. Since n is also an integer, we get $n = 1$. Consequently, $u = 2v$ so $m = \frac{u}{v} = 2$. We have reached a contradiction because Theorem 3.1 requires that $0 < m < 1$. Therefore when $a = 3$ there is no distinct pair of integer dimensions that gives an integral solution.

When $a = 4$, we know by Lemma 2.2, there are no distinct integer solutions for b .

We have already shown that $a \geq 5$ and now we will show that if $a = 5$, b cannot be less than 8. Since we are only considering integer dimensions, we only need to show that the following pairs (a, b) are not possible dimensions: $(5,1)$, $(5,2)$, $(5,3)$, $(5,4)$, $(5,6)$, $(5,7)$.

From Theorem 3.1, we let $a = r(1 - m^2)$ and $b = r(2m - m^2)$. By solving the first equation for

r , we get $r = \frac{a}{1-m^2}$ and by substitution, $b = \frac{a(2m-m^2)}{(1-m^2)}$. Thus,

$$(1 - m^2)b = a(2m - m^2) \tag{6}$$

where m is rational and $0 < m < 1$.

By plugging 5 into equation (6) and solving for b we get the expression $b = \frac{5(2m-m^2)}{(1-m^2)}$. For all values of b , m is irrational, which is illustrated in the following table:

b	m
2	$\frac{5-\sqrt{19}}{3}$
3	$\frac{5-\sqrt{19}}{2}$
4	$5 - \sqrt{21}$
6	$-5 + \sqrt{31}$
7	$\frac{-5+\sqrt{39}}{2}$

Thus, we have show that when $a = 5$, b cannot be an integer less than 8.

□

4 Minimizing b

Since it has been proven that $a = 3$, $b = 8$, are the minimum values for distinct dimensions in order for there to be a rational answer, it is natural to ask the following question: Given any value of a , what is the distinct minimum value of b that ensures a rational solution to the Box Problem?

By solving Equation (2) for b , we get the expression $b = \frac{1}{2} \left(a \pm \sqrt{-3a^2 + \frac{4p^2}{q^2}} \right)$. From Lemma 2.1, we know $q = 1$ or 2 . In order for b to be an integer, the expression $\sqrt{-3a^2 + \frac{4p^2}{q^2}}$ must be rational. For each prime value of a , we used MapleTM to evaluate iterates of the two functions $f(x) = \sqrt{-3a^2 + 4x^2}$, for when $q = 1$, and $f(x) = \sqrt{-3a^2 + x^2}$, for when $q = 2$.

The outcome of this computer search (see table) suggests when a is prime, the minimum value of b occurs when r is a perfect square. The result is Conjecture 1.

a	b	r
5	8	9
7	15	16
11	35	36
13	48	49
17	80	81
19	99	100
23	143	169
29	224	289
31	255	324
37	360	441

Conjecture 1. *When a is an odd prime of the form $2k + 1$, the minimum distinct value for b is $k^2 + 2k$. Also, when $a > 9$ and odd but not prime, $b = k^2 + 2k$ is the third minimum distinct value of b .*

To illustrate the second statement in Conjecture 1, when $a = 15$, the smallest value for b is 27. The next integer value for b is 48 and the third minimum value for b is 63, which is of the form $k^2 + 2k$ where $k = 7$.

References

- [1] Cuoco, Al. "Meta-Problems in Mathematics." The College Mathematics Journal 31.5 (2000): 373-378.
- [2] Hotchkiss, Philip K. "It's Perfectly Rational." The College Mathematics Journal 33.2 (2002): 113-117.
- [3] Lee, Catherine. private communication, 2007.
- [4] P. Wade and W. Wade, "Recursions that produce Pythagorean triples." The College Mathematics Journal 2 (2000): 98-101.
- [5] Stewart, James. Calculus. 5th edition. 2003: 337.