UNIQUE PROPERTIES OF THE FIBONACCI AND LUCAS SEQUENCES

STEPHEN A. PARRY

Abstract. The algebraic structure of the set of all Fibonacci-like sequences, which includes the Fibonacci and Lucas sequences, is developed, utilizing an isomorphism between this set and a subset of the 2-by-2 integer matrices. We will then proceed to define the determinant of a sequence and Fibonacci-like matrices. The following results are then obtained: (1) the Fibonacci sequence is the only such sequence with determinant equal to 1; (2) the set of all Fibonacci-like sequences forms an integral domain; (3) even powers of Lucas matrices are multiples of a Fibonacci matrix; and (4) only powers of multiples of Fibonacci matrices or Lucas matrices are multiples of Fibonacci matrices.

1. INTRODUCTION

The Fibonacci and Lucas sequences are subsets of a family of recursive sequences. By establishing important algebraic concepts, we will be able to create a ring that includes these two sets. Yang [9] established an important isomorphism between $\mathbb{Z}[A]$ and $\mathbb{Z}[\phi]$. We will take this isomorphism in addition to the work of Horadam [5] into consideration. Although Dannan [1] studied the ring of all second-order recursive sequences under the rational numbers, we will only concern ourselves with a ring, $\Omega \in \mathscr{GL}(2,\mathbb{Z})$. Using the structure of the ring [3], we will prove specific relations among the Fibonacci sequence, the Lucas sequence, and other recursive sequences.

2. BACKGROUND

A recursive sequence is any sequence of numbers indexed by $n \in \mathbb{Z}$, which can be generated by solving the recurrence equation. The types of recursive sequences that we will discuss in this paper are in the form $A_n = \alpha A_{n-1} + \beta A_{n-2}$, where$\alpha = 1,\ \beta = 1$. The Fibonacci sequence and the Lucas sequence are sequences that belong to this particular family of recursive sequences.

Definition 1. We will define the Fibonacci numbers as

$$F_n = F_{n-1} + F_{n-2}, \quad F_1 = 1, \quad F_2 = 1. \tag{1}$$

Definition 2. We will define the Lucas numbers as

$$L_n = L_{n-1} + L_{n-2}, \quad L_1 = 1, \quad L_2 = 3. \tag{2}$$

These two sequences have more in common then their recursive structure. There are many well-known and established relations between the Lucas and the Fibonacci sequences. We will find the following relations to be the most helpful [7].

$$L_n = F_{n+1} + F_{n-1} \tag{3}$$

$$5F_n = L_{n-1} + L_{n+1} \tag{4}$$

3. FIBONACCI-LIKE SEQUENCES AND MATRICES

We will now define and discuss important properties of Fibonacci-like sequences in terms of recursive sequences and 2x2 matrices. This section will help us understand the commonalities between elements in the set of general Fibonacci-like sequences.

Definition 3. We will define a Fibonacci-like sequence as

$$A_n = A_{n-1} + A_{n+1}.$$

Theorem 1. *Any Fibonacci-like sequence can be written as*

$$A_n = A_1 F_n + (A_2 - A_1)F_{n-1}.$$

Proof. Let n=1

$$A_1 = A_1 F_1 + (A_2 - A_1)F_0.$$

Let n=2

$$A_2 = A_1 F_2 + (A_2 - A_1)F_1.$$

By adding the two expressions, we obtain

$$\begin{array}{rcc} A_1 = & A_1F_1 & +(A_2-A_1)F_0 \\ A_2 = & A_1F_2 & +(A_2-A_1)F_1 \\ \hline A_1 + A_2 = & A_1F_3 & +(A_2-A_1)F_2 \end{array}$$

Since A_1 and A_2 are constants, we can use the recursive definition (1) to conclude the sum is equal to A_3.

$$\begin{array}{rcc} A_{k-1} = & A_1F_{k-1} & +(A_2-A_1)F_{k-2} \\ A_{k-2} = & A_1F_{k-2} & +(A_2-A_1)F_{k-3} \\ \hline A_k = & A_1F_k & +(A_2-A_1)F_{k-1} \end{array}$$

□

Example. If we have a sequence $B_n = \{..., 1, 9, 10, 19, ...\}$, where $B_1 = 1$, we can write B_n as $F_n + 8F_{n-1}$.

$$\begin{array}{ccrrrr} B_n & = & 1 & 9 & 10 & 19 \\ -1F_n & = & -1 & -1 & -2 & -3 \\ \hline C_n & = & 0 & 8 & 8 & 16 \end{array}$$

Then,

$$\begin{array}{ccrrrr} C_n & = & 0 & 8 & 8 & 16 \\ -8F_n & = & 0 & -8 & -8 & -16 \\ \hline D_n & = & 0 & 0 & 0 & 0 \end{array}$$

Definition 4. We will define a Fibonacci-like matrix to be a matrix in the form

$$\begin{bmatrix} A_n & A_{n-1} \\ A_{n-1} & A_{n-2} \end{bmatrix}.$$

Throughout this paper, we will think of Fibonacci-like matrices and Fibonacci-like sequences interchangeably. The set $\mathbb{F}$ will represent all 2x2 Fibonacci-like matrices whose entries are integer multiples of Fibonacci numbers. We will define $\mathbb{L}$ similarly for the Lucas numbers. The elements in $\mathbb{F}$ are called Fibonacci matrices, while elements in $\mathbb{L}$ are called Lucas matrices.

Definition 5. We define the set, Ω, which contains all 2x2 Fibonacci-like matrices.

$$\Omega = \begin{bmatrix} a+b & b \\ b & a \end{bmatrix} \subset \mathscr{GL}(2,\mathbb{Z}).$$

Definition 6. We will express the determinant of a Fibonacci-like matrix

$$\begin{vmatrix} A_n & A_{n-1} \\ A_{n-1} & A_{n-2} \end{vmatrix} = |A_n A_{n-2} - A_{n-1}^2|.$$

Remark 1. The determinant of a Fibonacci-like sequence is alternating. Therefore, if we neglected to include the absolute value of the determinant in our definition, then the values for the determinant would either be $-\lambda$ or λ; in order to simplify this behavior, we include the absolute value.

After converting Fibonacci-like sequences into Fibonacci-like matrices, we take the determinant of each matrix, which provides us with a way to classify every Fibonacci-like sequence.

Theorem 2. *The Fibonacci sequence is the only Fibonacci-like sequence with determinant equal to 1.*

Proof. Given the characteristic polynomial of the Fibonacci sequence, $x^2 = x + 1$, we can write x as a continued fraction [6].

$$x = \cfrac{\ddots}{\ddots + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}$$

We can also express any Fibonacci ratio as a continued fraction:

$$\frac{F_{n+1}}{F_n} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots + \cfrac{\ddots}{1 + \cfrac{F_2}{F_1}}}}}.$$

The determinant of $\begin{bmatrix} c+d & d \\ d & c \end{bmatrix}$ is $|c^2 + dc - d^2|$. For simplicity, we will let $c^2 + dc - d^2 = 1$. Then,

$$1 + \frac{d}{c} - \left(\frac{d}{c}\right)^2 = \frac{1}{c^2}$$

$$\vdots$$

$$\frac{d}{c} = 1 + \cfrac{1}{\cfrac{c}{\cfrac{c^2-1}{d}}}.$$

We know that $d - c = \frac{c^2-1}{d}$ since $c^2 + dc - d^2 = 1$, which implies $cd - d^2 = 1 - c^2$. Therefore, we can conclude the sequence of numbers, $\{..., d-c, c, d, ...\}$ is Fibonacci, since Fibonacci numbers can be expressed in that specific continued fraction form. □

Remark 2. When we have any continued fraction whose numerators all equal 1, we can condense our notation by writing the number as a list of the denominators: $x = [d_1, d_2, ..., d_{n-2}, d_{n-1}, d_n]$.

Definition 7. We will define the shift map, σ, to be equal to $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in \mathbb{F}$ [4].

Theorem 3. *Let $B \in E \subset \Omega$, then $B\sigma^n \in E$ for all $n \in \mathbb{Z}$.*

Proof.

$$B = \begin{bmatrix} B_n & B_{n-1} \\ B_{n-1} & B_{n-2} \end{bmatrix}.$$

$$B\sigma = \begin{bmatrix} B_{n+1} & B_n \\ B_n & B_{n-1} \end{bmatrix} \in E.$$

$$\sigma^n = \begin{bmatrix} F_k & F_{k-1} \\ F_{k-1} & F_{k-2} \end{bmatrix}.$$

$$B\sigma^n = \begin{bmatrix} B_{n+k-1} & B_{n+k-2} \\ B_{n+k-2} & B_{n+k-3} \end{bmatrix}.$$

$$B\sigma^n \in E.$$

□

Remark 3. By expressing three consecutive elements of a Fibonacci-like sequence in matrix form, we can obtain every element in the sequence by multiplying the matrix by powers of σ.

Theorem 4. *The set Ω forms an integral domain.*

Proof. We must first prove that Ω is an abelian group under addition. We will then show multiplication is associative and the left and right distributive laws hold. Then, we can prove that $I \in \Omega$ and $0 \in \Omega$. In order to have an integral domain, we must also prove Ω is commutative under multiplication, and there are no zero divisors.

Here we prove that there are no zero divisors, leaving the remainder of the proof to the reader.

$$\begin{bmatrix} a+b & b \\ b & a \end{bmatrix} \begin{bmatrix} c+d & d \\ d & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} ac+ad+bc+2bd & ad+bd+bc \\ bc+db+ad & db+ac \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

By solving these four linear equations, we find the general solution is

$$a = 1/2(-b - \sqrt{5}b),\ c = 1/2(-d + \sqrt{5}d) \ni b,\ d \in \mathbb{Z}.$$

Since $b,\ a \in \mathbb{R}$, there exists no zero divisors in Ω. We leave the remainder of the proof to the reader. □

4. POWERS OF LUCAS 2x2 MATRICES

The Fibonacci and Lucas sequences provide an interesting pattern when we multiply their respective recursive matrices together. When we multiply two elements in $\mathbb{F}$ we obtain another element in $\mathbb{F}$, which happens because the Fibonacci matrices are the units in the ring Ω. When we multiply an element in $\mathbb{L}$ by another element in $\mathbb{L}$, we obtain an element in $\mathbb{F}$. In this section, we will discuss this phenomenon. For simplification, we will consider our primitive Lucas matrix to be $A = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}$.

Theorem 5. *If* $A = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}$, *then*

$$A^{2k} = 5^k \begin{bmatrix} F_{4k+1} & F_{4k} \\ F_{4k} & F_{4k-1} \end{bmatrix}, \textit{and}$$

$$A^{2k+1} = 5^k \begin{bmatrix} L_{4k+3} & L_{4k+2} \\ L_{4k+2} & L_{4k+1} \end{bmatrix}.$$

Proof.

$$FL(k) = \left\{ \begin{matrix} A^{2k+1} \\ A^{2k} \end{matrix} \right\} = \left\{ \begin{matrix} 5^k \begin{bmatrix} L_{4k+3} & L_{4k+2} \\ L_{4k+2} & L_{4k+1} \end{bmatrix} \\ 5^k \begin{bmatrix} F_{4k+1} & F_{4k} \\ F_{4k} & F_{4k-1} \end{bmatrix} \end{matrix} \right\}.$$

$$FL(1) = \left\{ \begin{matrix} A^3 \\ A^2 \end{matrix} \right\} = \left\{ \begin{matrix} \begin{bmatrix} 145 & 90 \\ 90 & 55 \end{bmatrix} \\ \begin{bmatrix} 25 & 15 \\ 15 & 10 \end{bmatrix} \end{matrix} \right\} = \left\{ \begin{matrix} 5 \begin{bmatrix} L_7 & L_6 \\ L_6 & L_5 \end{bmatrix} \\ 5 \begin{bmatrix} F_5 & F_4 \\ F_4 & F_3 \end{bmatrix} \end{matrix} \right\}.$$

By letting $k = n - 1$, we obtain the following expression:

$$FL(n-1) = \left\{ \begin{matrix} A^{2n-1} \\ A^{2n-2} \end{matrix} \right\} = \left\{ \begin{matrix} 5^{n-1} \begin{bmatrix} L_{4n-1} & L_{4n-2} \\ L_{4n-2} & L_{4n-3} \end{bmatrix} \\ 5^{n-1} \begin{bmatrix} F_{4n-3} & F_{4n-4} \\ F_{4n-4} & F_{4n-5} \end{bmatrix} \end{matrix} \right\}.$$

$$A^{2n} = A^{2n-1}A$$

$$5^n \begin{bmatrix} F_{4n+1} & F_{4n} \\ F_{4n} & F_{4n-1} \end{bmatrix} = 5^{n-1} \begin{bmatrix} L_{4n-1} & L_{4n-2} \\ L_{4n-2} & L_{4n-3} \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}.$$

We will now solve the first relation from the above equation.

$$5F_{4n+1} = 4L_{4n-1} + 3L_{4n-2}.$$
$$= 3L_{4n-1} + 3L_{4n-2} + L_{4n-1}.$$
$$= 3L_{4n} + L_{4n-1}.$$
$$\vdots$$
$$5F_{4n+1} = L_{4n+2} + L_{4n}.$$

We know this is true by (4). The reader can prove the other three relations using a similar technique. We must also show that this equality holds true for A^{2n+1}.

$$A^{2n+1} = A^{2n}A.$$

$$\begin{bmatrix} L_{4n+3} & L_{4n+2} \\ L_{4n+2} & L_{4n+1} \end{bmatrix} = \begin{bmatrix} F_{4n+1} & F_{4n} \\ F_{4n} & F_{4n-1} \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}.$$

The reader can simplify these relations using (3) and (4). □

Corollary. The multiplication of any two Lucas matrices will yield a Fibonacci matrix.

Proof. If we have $D,\ E \in \mathbb{L}$, then

$$D = A\sigma^n,\ E = A\sigma^m.$$

$$\begin{aligned} DE = &\quad A\sigma^n A\sigma^m. \\ = &\quad AA\sigma^n\sigma^m. \\ = &\quad 5\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\sigma^{n+m}. \end{aligned}$$

□

Theorem 6. *Let $C \in \Omega$ and $C^2 = \begin{bmatrix} w+y & y \\ y & w \end{bmatrix} \in \mathbb{F}$. Then $C \in \mathbb{F}$ or $C \in \mathbb{L}$.*

Proof. Let $C = \begin{bmatrix} a+b & b \\ b & a \end{bmatrix}$, then

$$C^2 = \begin{bmatrix} (a+b)^2 + b^2 & (a+b)^2 - a^2 \\ b(a+b) + ab & a^2 + b^2 \end{bmatrix}.$$

Let

$$X = (a+b)^2 + b^2.$$

$$Y = (a+b)^2 - a^2.$$

$$Z = a^2 + b^2.$$

Our goal is to consider the possible values for each entry of C^2 in terms of a and b; we can then transfer this information to conclude the possibilities of matrix C. X and Z will always be positive since they are each a sum of squares. Since recursive sequences are bi-infinite, it is impossible to have a Fibonacci-like sequence containing all nonnegative numbers. At some point, every sequence will have an entry that is 0 or negative. Y is the only expression that can equal 0.

$$(a+b)^2 - a^2 = .$$

$$2ab + b^2 = .$$

$$b(2a+b) := 0.$$

If $b = 0$, then $C = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in \mathbb{F}$. If $b = -2a$, then $C = \begin{bmatrix} -a & -2a \\ -2a & a \end{bmatrix} \in \mathbb{L}$. Therefore, C can only be a multiple of a Fibonacci matrix or a multiple of a Lucas matrix. □

Remark 4. In [2] and [8], the authors are concerned about matrices that have Fibonacci numbers, and when raised to any power, produce a matrix with Fibonacci numbers. This result states that there exists matrices that are not Fibonacci matrices, but when raised to an even power, will produce a Fibonacci matrix.

5. CONCLUSIONS

This discovery of a square root of a matrix can lead in several different directions. We can attempt to generalize this phenomenon for $A_n = \alpha A_{n-1} + \beta A_{n-2}$. In addition, there may exist an isomorphism map from sequences and characteristic polynomials to their continued fractions. For example, we can express ratios of Fibonacci numbers as continued fractions; each of these ratios will be in the form $[..., 1, 1, 1]$. Similarly, we can express ratios of Lucas numbers, and we obtain $[...1, 1, 3]$. In addition, there may be a connection between determinants and continued fraction expansion.

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