UNIQUE PROPERTIES OF THE FIBONACCI AND LUCAS SEQUENCES

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Abstract. The algebraic structure of the set of all Fibonacci-like sequences, which includes the Fibonacci and Lucas sequences, is developed, utilizing an isomorphism between this set and a subset of the 2-by-2 integer matrices. Using this isomorphism, determinants of sequences, and Fibonacci-like matrices, can be defined. The following results are then obtained: (1) the Fibonacci sequence is the only such sequence with determinant equal to 1, (2) the set of all Fibonacci-like sequences forms an integral domain, (3) even powers of Lucas matrices are multiples of a Fibonacci matrix, (4) only powers of multiples of Fibonacci matrices or Lucas matrices are multiples of Fibonacci matrices.

1. INTRODUCTION

The Fibonacci numbers are a sequence of numbers that have captivated the world. Other sequences have a similar structure compared with the Fibonacci sequence, namely the Lucas sequence. These two sequences are a part of a family of recursive sequences. By establishing important algebraic concepts, we will be able to create a ring which includes these two sets. Yang [9], established an important isomorphsim between $\mathbb{Z}[A]$ and $\mathbb{Z}[\phi]$. We will take this isomorphism in consideration, in addition to the work of Horadam [5]. Although [1] studied the ring of all second-order recursive sequences under the rational numbers, we will only concern ourselves with a ring, $\Omega \in \mathscr{GL}(2,\mathbb{Z})$. Using the structure of the ring [3], we will prove specific relations between the Fibonacci sequence, the Lucas sequence, and other recursive sequences in the specific ring we will define.

2. BACKGROUND

A recursive sequence is any sequence that depends on the knowledge of previous numbers in the sequence to determine the next number. The types of recursive sequences that we will discuss in this paper are in the form $A_n = \alpha A_{n-1} + \beta A_{n-2}$, where $\alpha = 1$, $\beta = 1$. The Fibonacci numbers and the Lucas numbers are sets of numbers which belong to this family of recursive sequences.

Definition 1. We will define the Fibonacci numbers as

$$F_n = F_{n-1} + F_{n-2}, \quad F_1 = 1, \quad F_2 = 1.$$

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The Fibonacci numbers have been seen throughout nature and architecture, and the human body. The study of the arrangement of plants, phyllotaxis, lead to scientists and mathematicians alike to realize the importance of recursive sequences. The man who introduced the Fibonacci numbers to the world of number theory was Edouard Lucas.

Definition 2. We will define the Lucas numbers as

$$L_n = L_{n-1} + L_{n-2}, \quad L_1 = 1, \quad L_2 = 3$$

These two sequences have more in common then their recursive structure. There are many well-known and established relations between the Lucas and the Fibonacci sequences. These relations are not the only relations that exist, but this list will help us answer questions concerning the two sequences [7].

$$L_n = F_{n+1} + F_{n-1} \tag{1}$$

$$5F_n = L_{n-1} + L_{n+1} \tag{2}$$

3. FIBONACCI-LIKE SEQUENCES AND MATRICES

We will now discuss important properties of Fibonacci-like sequences. These properties will be discussed in terms of recursive sequences and in 2x2 matrices. This section will help us understand the commonalities between elements in the set of general Fibonacci-like sequences.

Definition 3. We will define a Fibonacci-like sequence as

$$A_n = A_{n-1} + A_{n+1}.$$

Definition 4. We will define a Fibonacci-like matrix to be a matrix in the form

$$\begin{bmatrix} A_n & A_{n-1} \\ A_{n-1} & A_{n-2} \end{bmatrix}.$$

Throughout this paper, we will think of Fibonacci-like matrices and Fibonacci-like sequences interchangeably. The set $\mathbb{F}$ will represent all 2x2 Fibonacci-like matrices whose entries are integer multiples of Fibonacci numbers. We will define $\mathbb{L}$ similarly for the Lucas numbers. The elements in $\mathbb{F}$ are called Fibonacci matrices, while elements in $\mathbb{L}$ are called Lucas matrices. The transformation from sequence representation and matrix representation provides another way to study recursive sequences.

Definition 5. We define the set, Ω, to contain all 2x2 Fibonacci-like matrices.

$$\Omega = \begin{bmatrix} a+b & b \\ b & a \end{bmatrix} \in \mathscr{GL}(2, \mathbb{Z})$$

Theorem 1. *Any Fibonacci-like sequence can be written as*

$$A_n = A_1 F_n + (A_2 - A_1) F_{n-1}.$$

Proof. Let n=1

$$A_1 = A_1 F_1 + (A_2 + A_1) F_0$$

Let n=2

$$A_2 = A_1 F_2 + (A_2 + A_1) F_1$$

By adding the two expressions, we obtain

$$A_1 + A_2 = A_1F_1 + (A_2 - A_1)F_0 + A_1F_2 + (A_2 - A_1)F_1 = A_3$$

$$\begin{array}{rll} A_{k-1} = & A_1F_{k-1} & +(A_2 - A_1)F_{k-2} \\ A_{k-2} = & A_1F_{k-2} & +(A_2 - A_1)F_{k-3} \\ \hline A_k = & A_1F_k & +(A_2 - A_1)F_{k-1} \end{array}$$

□

Example 1. If we have a sequence $B_n = \{..., 1, 9, 10, 19, ...\}$, we can write it as $B_n = F_n + 8F_{n-1}$.

$$\begin{array}{rcrrrr} B_n & = & 1 & 9 & 10 & 19 \\ -1F_n & = & -1 & -1 & -2 & -3 \\ \hline C_n & = & 0 & 8 & 8 & 16 \end{array}$$

Then,

$$\begin{array}{rcrrrr} C_n & = & 0 & 8 & 8 & 16 \\ -8F_n & = & 0 & -8 & -8 & -16 \\ \hline D_n & = & 0 & 0 & 0 & 0 \end{array}$$

Definition 6. We will express the determinant of a Fibonacci-like matrix

$$\begin{vmatrix} A_n & A_{n-1} \\ A_{n-1} & A_{n-2} \end{vmatrix} = |A_nA_{n-2} - A_{n-1}^2|$$

By converting Fibonacci-like sequences into Fibonacci-like matrices, we can take the determinant of the matrix. Having the determinant of the matrix will provide us with a way to classify the type of matrix, or sequence, it is in the set of sequences.

Theorem 2. *The Fibonacci sequence is the only Fibonacci-like sequence with determinant equal to 1.*

Proof. Given the characteristic polynomial of the Fibonacci sequence, $x^2 = x + 1$, we can write x as a continued fraction [6].

$$x = \cfrac{\ddots}{\ddots + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}$$

We can also express any Fibonacci ratio as a continued fraction:

$$\frac{F_{n+1}}{F_n} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots + \cfrac{\ddots}{1 + \cfrac{F_2}{F_1}}}}}.$$

If we consider the determinant of $\begin{bmatrix} c+d & d \\ d & c \end{bmatrix}$, we obtain $|c^2 + dc - d^2|$. For simplicity, we will let $c^2 + dc - d^2 = 1$. Then,

$$1 + \frac{d}{c} - \left(\frac{d}{c}\right)^2 = \frac{1}{c^2}$$

$$\vdots$$

$$\frac{d}{c} = 1 + \cfrac{1}{\cfrac{c}{\cfrac{c^2-1}{d}}}.$$

We know that $d - c = \frac{c^2-1}{d}$ because $c^2 + dc - d^2 = 1 \Rightarrow cd - d^2 = 1 - c^2$. Therefore, since this is true, then we can in the sequence of numbers, $\{..., d-c, c, d, ...\}$, which is Fibonacci, since Fibonacci numbers can be expressed in that continued fraction form. □

Remark 1. When we have any continued fraction whose numerators all equal one, we can condense our notion by writing a list of the denominators: $x = [d_1, d_2, ..., d_{n-2}, d_{n-1}, d_n]$.

Definition 7. We will define the shift map, σ, to be equal to $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in \mathbb{F}$ [4].

Theorem 3. *Let $B \in E \subseteq \Omega$, then $B\sigma^n \in E \; \forall \; n \in \mathbb{Z}$.*

Proof.

$$B = \begin{bmatrix} B_n & B_{n-1} \\ B_{n-1} & B_{n-2} \end{bmatrix}$$

$$B\sigma = \begin{bmatrix} B_{n-+1} & B_n \\ B_n & B_{n-1} \end{bmatrix} \in E.$$

$$\sigma^n = \begin{bmatrix} F_k & F_{k-1} \\ F_{k-1} & F_{k-2} \end{bmatrix}$$

$$B\sigma^n = \begin{bmatrix} B_{n+k-1} & B_{n+k-2} \\ B_{n+k-2} & B_{n+k-3} \end{bmatrix}$$

$$B\sigma^n \in E.$$

□

Remark 2. By expressing a sequence in matrix, we can obtain every elements in the sequence by multiplying the recursive matrix by powers of σ.

Theorem 4. *The set Ω forms an integral domain.*

Proof. In order for this to be true, we must first prove that Ω is an abelian group under addition. Afterwards, we will then show multiplication is associative and that the left and right distributive laws hold. Then, we can prove that $I \in \Omega$ and $0 \in \Omega$. In order to have an integral domain, we must also prove Ω is commutative under multiplication, and $\nexists$ zero divisors.

We will prove that $\nexists$ zero divisors in Ω.

$$\begin{bmatrix} a+b & b \\ b & a \end{bmatrix} \begin{bmatrix} c+d & d \\ d & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} ac+ad+bc+2bd & ad+bd+bc \\ bc+db+ad & db+ac \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

By solving these four linear equations, we find the general solutions are

$$a = 1/2(-b-\sqrt{5}b), \ , c = 1/2(-d+\sqrt{5}d) \ \ni \ b, d \in \mathbb{Z}$$

Since b,a $\in \mathbb{R}$ then $\nexists$ zero divisors in Ω. We leave the remainder of the proof to the reader. □

4. POWERS OF LUCAS 2x2 MATRICES

The Fibonacci and Lucas sequences provide an interesting pattern when we multiply their respective recursive matrices together. When we multiply two elements in $\mathbb{F}$ we obtain another element in $\mathbb{F}$. The reason why this happens is because the Fibonacci matrices are the units in the ring Ω. When we multiply an element in $\mathbb{L}$ by another element in $\mathbb{L}$, we obtain an element in $\mathbb{F}$. In this section, we will discuss this phenomenon. For simplification, we will consider our primitive Lucas matrix to be $A = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}$.

Theorem 5. *If* $A = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}$, *then*

$$A^{2k} = 5^k \begin{bmatrix} F_{4k+1} & F_{4k} \\ F_{4k} & F_{4k-1} \end{bmatrix}$$

$$A^{2k+1} = 5^k \begin{bmatrix} L_{4k+3} & L_{4k+2} \\ L_{4k+2} & L_{4k+1} \end{bmatrix}$$

Proof.

$$FL(k) = \left\{ \begin{array}{c} A^{2k+1} \\ A^{2k} \end{array} \right\} = \left\{ \begin{array}{c} 5^k \begin{bmatrix} L_{4k+3} & L_{4k+2} \\ L_{4k+2} & L_{4k+1} \end{bmatrix} \\ 5^k \begin{bmatrix} F_{4k+1} & F_{4k} \\ F_{4k} & F_{4k-1} \end{bmatrix} \end{array} \right\}$$

$$FL(1) = \left\{ \begin{array}{c} A^3 \\ A^2 \end{array} \right\} = \left\{ \begin{array}{c} \begin{bmatrix} 145 & 90 \\ 90 & 55 \end{bmatrix} \\ \begin{bmatrix} 25 & 15 \\ 15 & 10 \end{bmatrix} \end{array} \right\} = \left\{ \begin{array}{c} 5 \begin{bmatrix} L_7 & L_6 \\ L_6 & L_5 \end{bmatrix} \\ 5 \begin{bmatrix} F_5 & F_4 \\ F_4 & F_3 \end{bmatrix} \end{array} \right\}$$

By letting k=n-1, we obtain the following expression:

$$FL(n-1) = \left\{ \begin{array}{c} A^{2n-1} \\ A^{2n-2} \end{array} \right\} = \left\{ \begin{array}{c} 5^{n-1} \begin{bmatrix} L_{4n-1} & L_{4n-2} \\ L_{4n-2} & L_{4n-3} \end{bmatrix} \\ 5^{n-1} \begin{bmatrix} F_{4n-3} & F_{4n-4} \\ F_{4n-4} & F_{4n-5} \end{bmatrix} \end{array} \right\}$$

$$A^{2n} = A^{2n-1}A$$

$$5 \begin{bmatrix} F_{4n+1} & F_{4n} \\ F_{4n} & F_{4n-1} \end{bmatrix} = \begin{bmatrix} L_{4n-1} & L_{4n-2} \\ L_{4n-2} & L_{4n-3} \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}.$$

We will now solve the first relation of due to this matrix multiplication.

$$\begin{aligned} 5F_{4n+1} &= 4L_{4n-1} + 3L_{4n-2} \\ &= 3L_{4n-1} + 3L_{4n-2} + L_{4n-1} \\ &= 3L_{4n} + L_{4n-1} \\ &\vdots \\ 5F_{4n+1} &= L_{4n+2} + L_{4n}. \end{aligned}$$

We know this is true by (2). The reader can prove the other three relations using a similar technique. We must also show that this equality holds true for A^{2n+1}.

$$A^{2n+1} = A^{2n}A$$

$$\begin{bmatrix} L_{4n+3} & L_{4n+2} \\ L_{4n+2} & L_{4n+1} \end{bmatrix} = \begin{bmatrix} F_{4n+1} & F_{4n} \\ F_{4n} & F_{4n-1} \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}.$$

The reader can simplify these relations using (1) and (2). □

Remark 3. The behavior between $\mathbb{L}$ and $\mathbb{F}$ is similar to that of $\mathbb{R}$ and $\mathbb{C}$, where these sets represent the real numbers and complex numbers respectively. When we consider elements in $\mathbb{R}$ under multiplication, they stay in $\mathbb{R}$. Elements in $\mathbb{C}$ alternate by having imaginary numbers under multiplication. In addition, the reader may conceptualize this behavior as considering Lucas matrices to be the square root matrices of the Fibonacci matrices.

Corollary 1. The multiplication of any two Lucas matrices will yield a Fibonacci matrix.

Proof. If we have $D,\ E \in \mathbb{L}$, then

$$\begin{aligned} D = A\sigma^n&,\ E = A\sigma^m \\ DE &= A\sigma^n A\sigma^m \\ DE &= AA\sigma^n\sigma^m \\ DE &= 5\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\sigma^{n+m}. \end{aligned}$$

□

Theorem 6. *If* $C \in \Omega$ *and* $C^2 = \begin{bmatrix} w+y & y \\ y & w \end{bmatrix} \in \mathbb{F}$, *then* $C \in \mathbb{F}$ *or* $C \in \mathbb{L}$.

Proof. Let $C = \begin{bmatrix} a+b & b \\ b & a \end{bmatrix}$, then

$$C^2 = \begin{bmatrix} (a+b)^2 + b^2 & (a+b)^2 - a^2 \\ b(a+b) + ab & a^2 + b^2 \end{bmatrix}$$

$$X = (a+b)^2 + b^2.$$

$$Y = (a+b)^2 - a^2.$$

$$Z = a^2 + b^2.$$

X and Z will always be positive since they are the sum of squares. Since recursive sequences are bi-infinite, it is impossible to have a Fibonacci-like sequence containing all nonnegative numbers. At some point, every sequence will have an entry that is zero or negative. Therefore, we study when $Y = 0$.

$$(a+b)^2 - a^2 = 0$$

$$2ab + b^2 = 0$$
$$(b)(2a + b) = 0.$$

If $b = 0$, then $C = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in \mathbb{F}$. If $b = 2a$, then $C = \begin{bmatrix} -a & -2a \\ -2a & a \end{bmatrix} \in \mathbb{L}$. Therefore, C can only be a multiple of a Fibonacci matrix or a multiple of a Lucas matrix. □

Remark 4. In [2] and [8], the authors are concerned about matrices which have Fibonacci numbers, and when raised to any power, produce a matrix with Fibonacci numbers. This result states that there exists matrices that are not Fibonacci matrices, but when raised to an even power, will produce a Fibonacci matrix.

5. CONCLUSIONS

This discovery of a square root of a matrix can lead into several different directions. We can attempt to generalize this phenomenon for $A_n = \alpha A_{n-1} + \beta A_{n-2}$. In addition, there may exist an isomorphism map from sequences and characteristic polynomials to their continued fractions. For example, we can express ratios of Fibonacci numbers as continued fractions; each of these ratios will be in the form [...,1,1,1]. Similarly, we can express ratios of Lucas numbers, and we obtain [...1,1,3]. In addition, there may be a connection between determinants and continued fraction expansion.

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