

# Natural Families of Triangles II: A Locus of Symmedian Points

Julia Fisher, Adam Carr, Andrew Roberts, David Xu\*

March 29, 2008

## Abstract

We group triangles into families based on three parameters: the distance between the circumcenter  $O$  and the centroid  $G$ , the circumradius, and the measure of angle  $\angle GOA$  where  $A$  is one vertex. We focus on the family of triangles which allows  $\angle GOA$  to vary and fixes the other two parameters. By construction, this grouping produces triangles which share the same Euler line. Perhaps unexpectedly, if we examine the family's locus of a triangle center known as the symmedian point, we find that it always forms an arc of a circle centered at a specified point on the Euler line.

## 1 Introduction

An interesting question in the field of Euclidean geometry is how one might create natural families of triangles. Moreover, what are the properties of the triangle space produced by such a construction?

This abstract questioning becomes tangible and dynamic with the use of geometry software such as the Geometer's Sketchpad or GeoGebra. This type of software allows users to immediately observe the effects of discrete or continuous changes in figures and to formulate hypotheses based on those observations.

The results presented in this paper originate from the use of such software to address the above questions and were inspired by [Mueller]. We will not discuss the properties of the triangle space based on the families we describe below, however. Rather, we will examine an interesting and perhaps unexpected property that a triangle center known as the symmedian point exhibits in one of the families.

## 2 Natural Families of Triangles

Let us begin by returning to the previously posed question: how can we group triangles into natural families? Clearly, there are many ways to associate triangles with each other. For example, we could create families of similar triangles or of triangles with two side lengths related by a fixed ratio. However, in an effort to ensure that the triangles in a single family share certain intrinsic features, let us base that grouping on one special line of a triangle—its Euler line.

**Definition 1.** The Euler line of a triangle is the line that passes through its centroid and circumcenter.

---

\*We would like to thank our advisor Stephen Kennedy of the Carleton College Mathematics Department for all of his help and support.

The Euler line of a given triangle contains a great number of the triangle's special points. For example, the orthocenter, the nine-point center, and the DeLongchamps point all lie along it. Another perhaps surprising feature is that the ratio of distances between certain points on the Euler line remains constant. Three such points are the circumcenter, the centroid, and the orthocenter. If we call the circumcenter  $O$ , the centroid  $G$ , and the orthocenter  $H$ , then,  $OH = 3OG$  [Kimberling].

We use the above property of the Euler line to create our families of triangles. Specifically, we construct a triangle given its circumcenter  $O$ , its centroid  $G$ , and one vertex  $A$ . The relative positions of these points are defined by three pieces of numeric information: the distance between the circumcenter and the centroid ( $g$ ), the distance between the circumcenter and a vertex ( $r$ ), and  $m\angle GOA$  ( $\theta$ ). By fixing any two of these parameters and keeping the third constant, we create a family of triangles. We will focus our attention on the family formed by fixing  $g$  and  $r$  and allow  $\theta$  to vary. We call this the  $\theta$ -family.

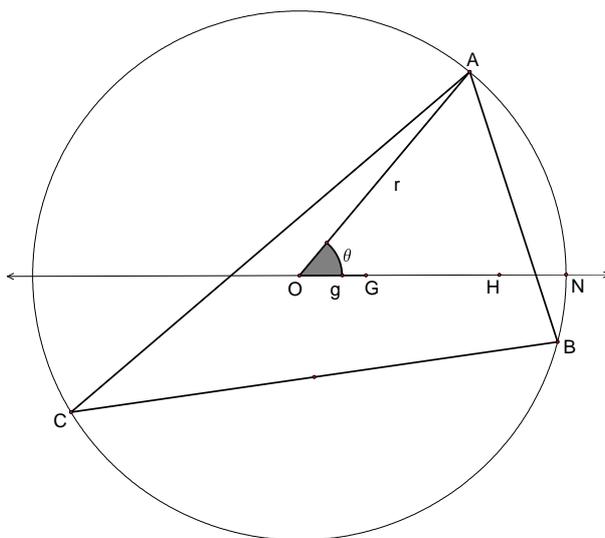


Figure 1: The basic triangle construction.

The curious reader can find a more in-depth discussion of this construction and its properties in Natural Families of Triangles I [Carr]. For our purposes, it is enough to note one fact and prove another. First, we find that when  $3OG > ON$ , where  $N$  is one intersection of the triangle's circumcircle and the Euler line, then certain angles  $\angle GOA$  will not produce a triangle. We will not prove this here. Second, all right triangles (up to scaling) occur in a single  $\theta$ -family—the family in which  $g = \frac{r}{3}$ . To show this, we need Proposition 1 and its corollary, Corollary 1. However, we will only prove Corollary 1 and Proposition 2 here. Proposition 1 follows from a proof by contradiction.

**Proposition 1.** *Let  $\triangle ABC$  with orthocenter  $H$  be given. If  $H$  lies on the circumcircle of  $\triangle ABC$ , then  $H$  is coincident with one vertex of  $\triangle ABC$ .*

**Corollary 1.** *Let  $\triangle ABC$  with orthocenter  $H$  be given. Then,  $H$  is coincident with vertex  $B$  if and only if  $m\angle ABC = \frac{\pi}{2}$ .*

*Proof.* Assume that  $H$  is coincident with vertex  $B$ . Since the altitude from vertex  $A$  to  $\overline{BC}$  passes through  $H$  and intersects  $\overline{BC}$  at a right angle, we see that  $m\angle ABC = \frac{\pi}{2}$ .

Assume that  $m\angle ABC = \frac{\pi}{2}$ . Then,  $\overline{BC}$  is the altitude from vertex  $C$  to  $\overline{AB}$ , and  $\overline{AB}$  is the altitude from vertex  $A$  to  $\overline{BC}$ . Since the altitudes of a triangle coincide at  $H$ , it follows that  $B$  and  $H$  are coincident.  $\square$

Now we can show that all right triangles (up to scaling) occur in a single  $\theta$ -family.

**Proposition 2.** *Triangle  $\triangle ABC$  is a right triangle if and only if  $g = \frac{r}{3}$ .*

*Proof.* Assume  $\triangle ABC$  is a right triangle. Then, by Corollary 1,  $H$  is coincident with one vertex. Without loss of generality, let that vertex be vertex  $B$ . Thus,  $H$  is on the circumcircle. Consequently,  $OH = r$ . Since  $G$  is one-third of the way from  $O$  to  $H$ , it follows that  $g = \frac{r}{3}$ .

Assume  $g = \frac{r}{3}$ . Then,  $OH = r$ , and consequently  $H$  lies on the circumcircle of  $\triangle ABC$ . By Proposition 1 and Corollary 1,  $\triangle ABC$  is a right triangle. □

Now that we have a basic understanding of our families of triangles, let us shift our focus to the symmedian point and an interesting property the symmedian point exhibits in the  $\theta$ -family.

### 3 The Symmedian Point

In the expansive collection of triangle centers, few could be considered well-known. Occasionally, however, centers which should be a part of mathematicians' base knowledge disappear from the contemporary consciousness. The symmedian point is one such center. Well-explored many years ago, the symmedian or Lemoine point has a plethora of useful and fascinating properties. In his work *Episodes in 19th and 20th Century Euclidean Geometry*, Ross Honsberger calls it "one of the jewels of modern geometry" [Honsberger, 53]. In order to begin our brief study of this geometric gem, we first need to understand some established definitions and theorems. Thus, let us define the concept of isogonal conjugacy.

**Definition 2.** Let  $\angle A$  be given. The isogonal conjugate of  $\overline{AP}$ , where  $P$  is any point in the plane, is the reflection of  $\overline{AP}$  over the angle bisector of  $\angle A$ . Line  $\overline{AP}$  and its reflection are called isogonal conjugate lines or simply isogonal conjugates.

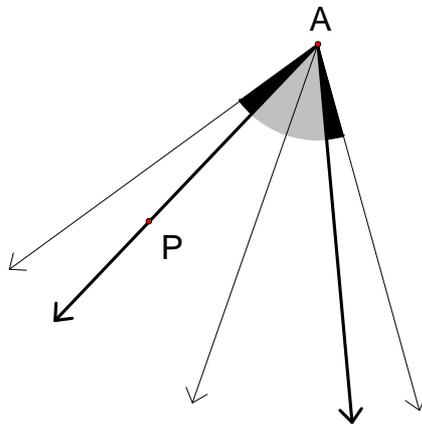


Figure 2: Isogonal conjugate lines.

In Figure 2, the middle line is the angle bisector of  $\angle A$ , and the two thick black lines are isogonal conjugates. One direct consequence of Definition 2 is that the angles formed by the isogonal conjugate lines and the angle bisector (the gray angles) are congruent. Also, the angles formed by the isogonal conjugate lines and the sides of the original angle (the black angles) are congruent.

Since isogonal conjugacy inherently involves angles, one question which arises is how isogonal conjugates relate to triangles. As Theorem 1 below states, they have at least one fascinating property.

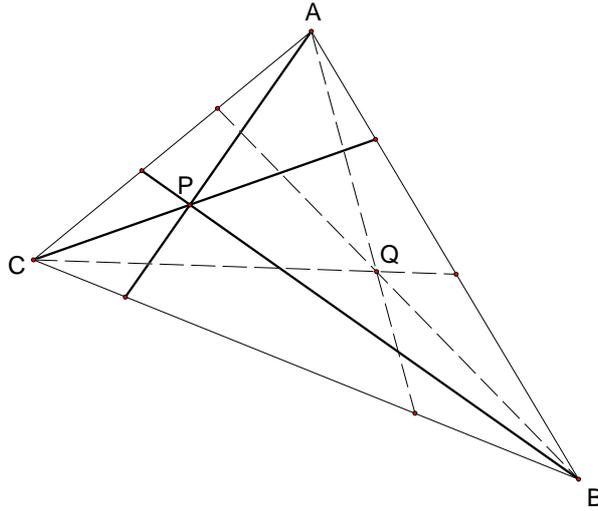


Figure 3: Isogonal conjugate points.

**Theorem 1.** Let  $\triangle ABC$  and a point  $P$  in the plane of  $\triangle ABC$  be given. The lines isogonal to  $\overline{AP}$ ,  $\overline{BP}$ , and  $\overline{CP}$ , meet at a point  $Q$ . Points  $P$  and  $Q$  are called isogonal conjugate points, or isogonal conjugates [Honsberger].

Although we omit it here, a proof of Theorem 1 can be found in Ross Honsberger's work, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*. Now that we have a basic understanding of isogonal conjugates and some of their properties, we are able to define the symmedian point.

**Definition 3.** The symmedian point  $K$  is the isogonal conjugate of the centroid  $G$ .

### 3.1 An Interesting Property of the Symmedian Point

As one might expect, the loci of certain special points in our  $\theta$ -family of triangles form sections of curves. The locus of symmedian points lie on a particularly nice curve—a circle. Let us formally state this result.

**Theorem 2.** Let  $\Omega$  be a  $\theta$ -family of triangles. Let  $E \subseteq [0, 2\pi]$  such that for all  $\theta \in E$ , there exists a triangle  $\triangle ABC_\theta \in \Omega$  with  $m\angle GOA = \theta$ . Let  $K_\theta$  be the symmedian point of  $\triangle ABC_\theta$ . Let  $K_n$  be the point on the ray  $\overrightarrow{OG}$  such that  $OK_n = \frac{2gr^2}{r^2 - g^2}$ . Then, for all  $\theta \in E$ ,  $K_\theta$  lies on the circle with radius  $\frac{2g^2r}{r^2 - g^2}$  centered at  $K_n$ . We call this circle the Carleton circle and its center  $K_n$  the Knights' point.

Moreover, if  $g < \frac{r}{3}$ , every point on the Carleton circle is the symmedian point of a triangle in  $\Omega$ . If  $g = \frac{r}{3}$ , then every point except the intersection of the Carleton circle with the circumcircle is the symmedian point of a triangle in  $\Omega$ . If  $g > \frac{r}{3}$ , then every point  $P$  on the Carleton circle such that  $P$  is strictly contained in the interior of the disc enclosed by the circumcircle of  $\Omega$  is the symmedian point of a triangle in  $\Omega$ .

While in the specific case in which  $g = \frac{r}{3}$  a geometric proof of Theorem 2 is readily apparent, a synthetic argument for the general case is much more difficult. Thus, we will approach the general situation from an analytic perspective. In order to more clearly understand precisely what Theorem 2 states, however, let us begin with the geometric proof of the case in which  $g = \frac{r}{3}$ .

**Lemma 1.** Let  $\Omega$  be a  $\theta$ -family of triangles in which  $g = \frac{r}{3}$ . Then, for every  $\triangle ABC_\theta \in \Omega$ , the symmedian point of  $\triangle ABC_\theta$  lies on the Carleton circle. Moreover, every point except  $H$  on the Carleton circle is the symmedian point of a triangle in  $\Omega$ .

In order to prove Lemma 1, we will use the following commonly-known proposition. We omit a proof of it here; the interested reader can find one in Honsberger.

**Proposition 3.** If  $\triangle ABC$  is a right triangle with the right angle at vertex  $B$ , then its symmedian point  $K$  is the midpoint of the symmedian line, i.e., the isogonal conjugate of the median, from vertex  $B$  [Honsberger].

Now, let us prove Lemma 1.

*Proof.* We need to show that for all  $\theta \in E$ ,  $K_\theta$  lies on the circle with radius  $\frac{2g^2r}{r^2-g^2}$  centered at the point  $K_n$  which lies on the ray  $\overrightarrow{OG}$  such that  $OK_n = \frac{2gr^2}{r^2-g^2}$ . Substituting  $\frac{r}{3}$  in for  $g$ , we see that the radius of the circle simplifies to  $\frac{1}{4}r$  and the distance between  $O$  and  $K_n$  simplifies to  $\frac{3}{4}r$ . Moreover, by Proposition 2 and Corollary 1, when  $g = \frac{r}{3}$ ,  $H$  coincides with one vertex of  $\triangle ABC_\theta$ , making  $\overline{OH}$  the radius of the circumcircle. Thus,  $OK_n = \frac{3}{4}OH$ . By similar logic, the radius of the Carleton circle is  $\frac{1}{4}OH$ .

Now, also by Proposition 2 and Corollary 1,  $\triangle ABC_\theta$  is a right triangle, and the vertex with which  $H$  is coincident is the vertex at the right angle. Without loss of generality, let this vertex be vertex  $B$ . Also of note, the circumcenter  $O$  of  $\triangle ABC_\theta$  will be the midpoint of  $\overline{AC}$ . Construct the symmedian line from  $B$ , and let  $K_b$  be the intersection of that symmedian line with  $\overline{AC}$ . We will first show that  $K_b$  lies on the circle centered at  $M$ , the midpoint of segment  $\overline{OH}$  (or  $\overline{OB}$ ).

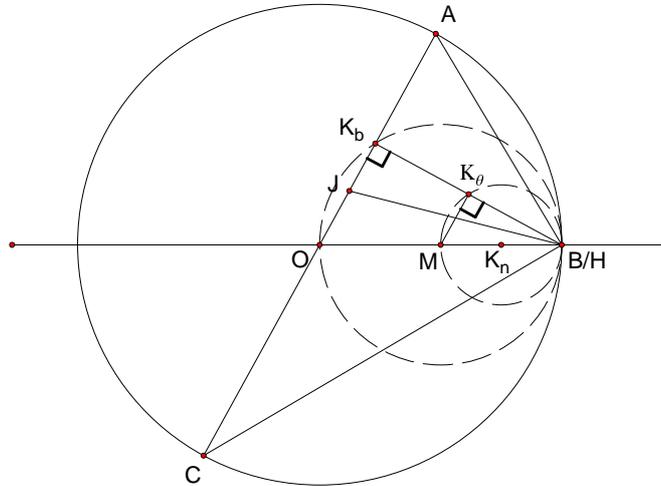


Figure 4: Proof that the symmedian point lies on a circle when  $\triangle ABC$  is a right triangle.

Construct the angle bisector of  $\angle ABC$  and label it  $\overline{JB}$ , where  $J$  is its intersection with  $\overline{AC}$ . Now, since the symmedian line is the isogonal conjugate of the median,  $\angle OJB \cong \angle JBK_b$  and  $\angle K_bBA \cong \angle OBC$ . Since  $O$  is the circumcenter of  $\triangle ABC$ , it follows that  $\triangle OBC$  is isosceles with  $\overline{OB} \cong \overline{OC}$ . Thus,  $\angle ACB \cong \angle OBC$ . Now,  $m\angle ABC = \frac{\pi}{2}$ . This implies that  $m\angle ACB + m\angle CAB = \frac{\pi}{2}$  which in turn implies that  $m\angle K_bBA + m\angle CAB = \frac{\pi}{2}$ . Thus,  $m\angle AK_bB = \frac{\pi}{2}$ , and  $K_b$  lies on the circle centered at point  $M$  with radius  $MO$ . Note that this shows that the altitude and the symmedian line from vertex  $B$  are coincident.

Now, by Proposition 3,  $K_\theta$  is the midpoint of  $\overline{K_bB}$ . Construct  $\overline{K_\theta M}$ . By SAS similarity,  $\triangle K_\theta BM \sim \triangle K_b BO$ . Thus,  $m\angle MK_\theta B = m\angle OK_b B = \frac{\pi}{2}$ . Therefore,  $K_\theta$  lies on the circle centered at the midpoint  $K_n$  of  $\overline{MB}$ ,  $\frac{3}{4}$  of the way from  $O$  to  $H$ . The radius of the circle will be  $\frac{1}{2}MB$ , which is  $\frac{1}{4}OH$ .

To show that every point on the Carleton circle except  $H$  is a symmedian point of a triangle in the given  $\theta$ -family, let  $K$  be a point on the Carleton circle such that  $K$  is not coincident with  $H$ . Since  $H$  is on the circumcircle, by Proposition 1, it must coincide with one vertex of every triangle in the given  $\theta$ -family. Without loss of generality, let  $B$  be that vertex. Construct  $\overline{BK}$ . Then, extend  $\overline{BK}$  past  $K$  to a point  $K_b$  such that  $\overline{KK_b} \cong \overline{BK}$ . By the argument above,  $K_b$  is the foot of the altitude from  $B$ . Construct the line perpendicular to  $\overline{K_bB}$  through  $K_b$ . This line will intersect the circumcircle at points  $A$  and  $C$ . By construction,  $\triangle ABC_\theta$  has symmedian point  $K$ . Thus, every point except  $H$  on the Carleton circle is the symmedian point of a triangle in the given  $\theta$ -family.<sup>1</sup>  $\square$

To show the general case of Theorem 2, we will first place the  $\theta$ -family of triangles in a coordinate system. We will find the  $x$ - and  $y$ -coordinates of the Knights' point  $K_n$  and of the symmedian point  $K_\theta$  of any triangle in that family. Then, we will use the distance formula to find the distance between these two points. If that distance is not dependent on  $\theta$ , then, since  $K_n$  is fixed in a  $\theta$ -family,  $K_\theta$  will always lie on a circle centered at  $K_n$ . The following proof makes use of two propositions, both of which are stated below. We will prove neither here. The interested reader will find a proof of the second in [Honsberger].

**Proposition 4.** *Let vectors  $\vec{a}$  and  $\vec{b}$  be given. Define vector  $\vec{c}$  thus:  $\vec{c} = |\vec{b}|\vec{a} + |\vec{a}|\vec{b}$ . Vector  $\vec{c}$  bisects the angle created by vectors  $\vec{a}$  and  $\vec{b}$  [Stewart].*

**Proposition 5.** *Let triangle  $\triangle ABC$  with altitude  $\overline{AH_a}$  be given. The symmedian point  $K$  of  $\triangle ABC$  lies on the line connecting the midpoint  $M_h$  of  $\overline{AH_a}$  to the midpoint  $M_a$  of  $\overline{BC}$  [Honsberger].*

Now, let us prove Theorem 2.

*Proof.* Let  $\Omega$  be the  $\theta$ -family of triangles. Choose the circumcenter  $O$  to be the origin of the coordinate system and the Euler line  $\overline{OG}$  to be the  $x$ -axis. We will first find expressions for the coordinates of the vertices  $A$ ,  $B$ , and  $C$  of any triangle in the  $\theta$ -family. The method we will use to find them algebraically mimics the geometric construction described earlier.

By construction,  $O$  has coordinates  $(0,0)$ ,  $G$  has coordinates  $(g,0)$ , and  $H$  has coordinates  $(3g,0)$ . Since  $A$  lies on the circle centered at the origin with radius  $r$ , and since angle  $\theta$  is the angle between  $\overline{OA}$  and the  $x$ -axis, it follows that  $A$  has coordinates  $(r \cos \theta, r \sin \theta)$ . Using this information and the fact that the centroid of a triangle lies  $\frac{2}{3}$  of the way from any vertex to the opposite side, we can find the equation of line  $\overline{AG}$ , the coordinates of the midpoint  $M_a$  of side  $\overline{BC}$ , and the equation of line  $\overline{AH}$ . Since  $\overline{AH}$  is perpendicular to  $\overline{BC}$ , we can use the coordinates of  $M_a$  and the opposite reciprocal of the slope of  $\overline{AH}$  to find the equation of  $\overline{BC}$ . Finally, we can easily intersect the equation of the circumcircle ( $x^2 + y^2 = r^2$ ) with the equation of  $\overline{BC}$  to find

---

<sup>1</sup>The reason that  $H$  cannot be a symmedian point is a result of the fact that the symmedian point  $K$  of a  $\triangle ABC$  can never fall on the circumcircle of  $\triangle ABC$ . To see this, suppose that  $K$  does lie on the circumcircle. Then, either  $K$  is coincident with one vertex, say vertex  $A$ , or  $K$  is not coincident with any vertex. Suppose  $K$  is coincident with vertex  $A$ . Then,  $\overline{AB}$  and  $\overline{AC}$  are the symmedian lines from vertices  $B$  and  $C$ , respectively. Consequently,  $\overline{BC}$  must be the median from both vertices  $B$  and  $C$ . However, that would force  $C$  to be the midpoint of  $\overline{AC}$  and  $B$  to be the midpoint of  $\overline{AB}$ . This cannot occur. Now, suppose that  $K$  is not coincident with any vertex of  $\triangle ABC$ . Then,  $K$  lies outside  $\triangle ABC$ . However, that would force the median from at least one vertex to lie outside of  $\triangle ABC$ . This is a contradiction. Thus,  $K$  must always be strictly inside the disk enclosed by the circumcircle of  $\triangle ABC$ .

the coordinates of vertices  $B$  and  $C$ . Much of this computation was completed with the help of *Mathematica*. Thus, we simply list the results below:

$$\overline{AG} : y = \frac{r \sin \theta (x - g)}{r \cos \theta - g}$$

$$M_a : \left( \frac{3g - r \cos \theta}{2}, \frac{-r \sin \theta}{2} \right)$$

$$\overline{AH} : y = \frac{r \sin \theta (x - 3g)}{r \cos \theta - 3g}$$

$$\overline{BC} : y = \frac{3g - r \cos \theta}{r \sin \theta} x + \frac{-r^2 \sin^2 \theta - (3g - r \cos \theta)^2}{2r \sin \theta}$$

$$B : \left( \frac{54g^3 + 12gr^2 - 2r(27g^2 + r^2) \cos \theta + 6gr^2 \cos 2\theta}{4(9g^2 + r^2 - 6gr \cos \theta)} + \frac{r\sqrt{3} \csc \theta (\cos 2\theta - 1) \sqrt{-27g^4 + r^4 + 36g^3r \cos \theta - 4gr^3 \cos \theta - 6g^2r^2 \cos 2\theta}}{4(9g^2 + r^2 - 6gr \cos \theta)}, \right. \\ \left. - \sin \theta \left( \frac{r}{2} + \frac{\sqrt{3} \csc \theta (3g - r \cos \theta) \sqrt{-27g^4 + r^4 + 36g^3r \cos \theta - 4gr^3 \cos \theta - 6g^2r^2 \cos 2\theta}}{2(9g^2 + r^2 - 6gr \cos \theta)} \right) \right)$$

$$C : \left( \frac{54g^3 + 12gr^2 - 2r(27g^2 + r^2) \cos \theta + 6gr^2 \cos 2\theta}{4(9g^2 + r^2 - 6gr \cos \theta)} - \frac{r\sqrt{3} \csc \theta (\cos 2\theta - 1) \sqrt{-27g^4 + r^4 + 36g^3r \cos \theta - 4gr^3 \cos \theta - 6g^2r^2 \cos 2\theta}}{4(9g^2 + r^2 - 6gr \cos \theta)}, \right. \\ \left. - \sin \theta \left( \frac{r}{2} - \frac{\sqrt{3} \csc \theta (3g - r \cos \theta) \sqrt{-27g^4 + r^4 + 36g^3r \cos \theta - 4gr^3 \cos \theta - 6g^2r^2 \cos 2\theta}}{2(9g^2 + r^2 - 6gr \cos \theta)} \right) \right)$$

Knowing the coordinates of the vertices of a generic triangle in  $\Omega$ , we can now find the coordinates of the symmedian point. We will begin by finding the equation of the symmedian line from vertex  $A$ . In order to do this, we need to bisect  $\angle BAC$ . Thus, we consider segments  $\overline{AB}$  and  $\overline{AC}$  to be vectors with their heads at  $B$  and  $C$ , respectively. Using Proposition 4, we find the vector which bisects them. This vector gives us the slope of the angle bisector. Since vertex  $A$  lies on the angle bisector, we can find its equation. We will not list this equation or the steps leading to it as the process was algebraically intensive and involved the use of *Mathematica*.

To find a point on the symmedian line from vertex  $A$ , we first find the equation of the line through  $G$  perpendicular to the angle bisector of  $\angle BAC$ . Again, since the equation of this line is quite complicated, we will not include it here. Next, we find the  $x$ -coordinate of the intersection  $J$  of this new perpendicular line (which we will now refer to as  $\overline{GJ}$ ) with the angle bisector. Finally, we find the coordinates of the point  $L$  on  $\overline{GJ}$  such that  $LJ = GJ$ . This point will be a point on the symmedian line from vertex  $A$  by the following geometric argument:



$$\left( \frac{2gr((9g^3 + 6gr^2) \cos \theta - r(9g^2 + r^2 + 6g^2 \cos 2\theta - gr \cos 3\theta))}{(g^2 - r^2)(9g^2 + r^2 - 6gr \cos \theta)}, \frac{2g^2r(9g^2 + r^2 - 12gr \cos \theta + 2r^2 \cos 2\theta) \sin \theta}{(g^2 - r^2)(9g^2 + r^2 - 6gr \cos \theta)} \right).$$

In order to prove that the symmedian point is always a fixed distance from the point  $K_n$  with coordinates  $(\frac{2gr^2}{r^2-g^2}, 0)$ , we use the distance formula to find the distance between these two points. Using *Mathematica*, we arrive at the following:

$$\begin{aligned} & \left[ \left( \frac{2gr((9g^3 + 6gr^2) \cos \theta - r(9g^2 + r^2 + 6g^2 \cos 2\theta - gr \cos 3\theta))}{(g^2 - r^2)(9g^2 + r^2 - 6gr \cos \theta)} - \frac{2gr^2}{r^2 - g^2} \right)^2 \right. \\ & \left. + \left( \frac{2g^2r(9g^2 + r^2 - 12gr \cos \theta + 2r^2 \cos 2\theta) \sin \theta}{(g^2 - r^2)(9g^2 + r^2 - 6gr \cos \theta)} - 0 \right)^2 \right]^{\frac{1}{2}} \\ & = \frac{2g^2r}{r^2 - g^2}. \end{aligned}$$

As we see above, the distance between  $K_n$  and  $K_\theta$  is not dependent on  $\theta$ . This implies that  $K_\theta$  will always lie on the circle of radius  $\frac{2g^2r}{r^2-g^2}$  centered at the Knights' point  $K_n$ . Additionally, it is nice to note that since  $r$  and  $g$  are distances, they are always strictly greater than zero. Moreover, since  $r > g$ , it follows that  $\frac{2g^2r}{r^2-g^2} > 0$  and is always defined.<sup>2</sup>

Let us now turn our attention to the second half of Theorem 2. We must show that every point on the arc of the Carleton circle strictly contained within the disk enclosed by the circumcircle is the symmedian point of a triangle in  $\Omega$ . We have three cases:  $g < \frac{r}{3}$ ,  $g = \frac{r}{3}$ , and  $g > \frac{r}{3}$ .

**Case 1.** Let  $g < \frac{r}{3}$ . To show that every point on the Carleton circle is the symmedian point of a triangle in  $\Omega$ , we will consider the  $x$ - and  $y$ -coordinates of the symmedian point to be functions of  $\theta$ . Then, we will carefully construct a closed interval whose endpoints, when acted upon by our functions, produce the intersections of the Carleton circle with the Euler line. Finally, we will use a connectedness argument and symmetry to show that the entire Carleton circle is composed of symmedian points of triangles in  $\Omega$ .

To begin, let

$$A(\theta) = \frac{2gr((9g^3 + 6gr^2) \cos \theta - r(9g^2 + r^2 + 6g^2 \cos 2\theta - gr \cos 3\theta))}{(g^2 - r^2)(9g^2 + r^2 - 6gr \cos \theta)},$$

and

$$B(\theta) = \frac{2g^2r(9g^2 + r^2 - 12gr \cos \theta + 2r^2 \cos 2\theta) \sin \theta}{(g^2 - r^2)(9g^2 + r^2 - 6gr \cos \theta)}.$$

Thus,  $K_\theta$  is a function of  $\theta$ , and we write  $K_\theta = K(\theta) = (A(\theta), B(\theta))$ . We will show that  $K(\theta)$  is continuous. As stated above,  $g^2 - r^2 \neq 0$  since  $g \neq r$ . Moreover,  $9g^2 + r^2 - 6gr \cos \theta \geq 9g^2 + r^2 - 6gr = (3g - r)^2 > 0$  since  $g \neq \frac{r}{3}$ . Thus,  $(g^2 - r^2)(9g^2 + r^2 - 6gr \cos \theta) \neq 0$ . Since the numerators of both  $A(\theta)$  and  $B(\theta)$  are combinations of continuous functions, it follows that both  $A(\theta)$  and  $B(\theta)$  are continuous, making  $K(\theta) = (A(\theta), B(\theta))$  continuous.

<sup>2</sup>If  $g$  were equal to  $r$ , then this would force all three of the vertices of a  $\triangle ABC$  to be coincident with  $G$ ; a point is not a triangle. If  $g$  were greater than  $r$ , then we would not be able to construct a triangle. Thus,  $r > g$ .

Now, let us construct our closed interval. Consider the expressions  $\frac{3g+r}{2r}$  and  $\frac{3g-r}{2r}$ . We want the arc cosines of these expressions to be the endpoints of our interval. However, before we can take their arc cosines, we must be certain that it is valid to do so. Since  $3g+r > 0$  and  $g < \frac{r}{3}$ , it follows that  $0 < \frac{3g+r}{2r} < 1$ . Also,  $3g+r > 0 \iff -1 < \frac{3g-r}{2r}$ ; and  $g < \frac{r}{3}$  implies that  $\frac{3g-r}{2r} < 0$ . Combining the last two inequalities, we see that  $-1 < \frac{3g-r}{2r} < 0$ . Thus, we can legitimately examine the angles  $\cos^{-1}(\frac{3g+r}{2r})$  and  $\cos^{-1}(\frac{3g-r}{2r})$ .

We now observe that  $0 < \cos^{-1}(\frac{3g+r}{2r}) < \frac{\pi}{2}$  since  $0 < \frac{3g+r}{2r} < 1$ , and  $\frac{\pi}{2} < \cos^{-1}(\frac{3g-r}{2r}) < \pi$  since  $-1 < \frac{3g-r}{2r} < 0$ . Thus,  $\cos^{-1}(\frac{3g+r}{2r}) < \cos^{-1}(\frac{3g-r}{2r})$ , and we can examine the closed interval from  $\cos^{-1}(\frac{3g+r}{2r})$  to  $\cos^{-1}(\frac{3g-r}{2r})$ . Since the interval  $[\cos^{-1}(\frac{3g+r}{2r}), \cos^{-1}(\frac{3g-r}{2r})]$  is a connected set and  $K(\theta)$  is continuous, it follows from the fact that continuous functions map connected sets to connected sets that the set  $K([\cos^{-1}(\frac{3g+r}{2r}), \cos^{-1}(\frac{3g-r}{2r})])$  is connected. Moreover, as we can see below, the function  $K$  evaluated at the endpoints of the interval gives the two points of intersection of the Carleton circle with the Euler line.

$$K(\cos^{-1}(\frac{3g+r}{2r})) = (\frac{2gr}{r-g}, 0).$$

$$K(\cos^{-1}(\frac{3g-r}{2r})) = (\frac{2gr}{r+g}, 0).$$

Since we showed that for all  $\theta$ ,  $K(\theta)$  lies on the Carleton circle, it follows that the only way  $K([\cos^{-1}(\frac{3g+r}{2r}), \cos^{-1}(\frac{3g-r}{2r})])$  can be a connected set is if it minimally encompasses either the upper or lower half of the Carleton circle (including the intersections of the Carleton circle with the Euler line). In other words,  $K([\cos^{-1}(\frac{3g+r}{2r}), \cos^{-1}(\frac{3g-r}{2r})])$  must be at least

$$\{(x, y) \mid (x - \frac{2gr^2}{r^2 - g^2})^2 + y^2 = \frac{4g^4 r^2}{(r^2 - g^2)^2} \wedge \text{either } y \geq 0 \text{ or } y \leq 0 \text{ but not both}\}.$$

Now, in the discussion above, we saw that  $[\cos^{-1}(\frac{3g+r}{2r}), \cos^{-1}(\frac{3g-r}{2r})] \subseteq (0, \pi)$ . Note that by symmetry, if  $\triangle ABC_\theta$  is a triangle produced by an angle  $\theta$ , then  $\triangle ABC_{2\pi-\theta}$  produced by angle  $2\pi-\theta$  will simply be the reflection of  $\triangle ABC_\theta$  over the Euler line. This implies that no matter which half of the Carleton circle is produced by  $[\cos^{-1}(\frac{3g+r}{2r}), \cos^{-1}(\frac{3g-r}{2r})]$ , the other half will be produced by  $[2\pi - \cos^{-1}(\frac{3g-r}{2r}), 2\pi - \cos^{-1}(\frac{3g+r}{2r})]$ .

Therefore, when  $g < \frac{r}{3}$ , every point on the Carleton circle is the symmedian point of a triangle in  $\Omega$ .

**Case 2.** Let  $g = \frac{r}{3}$ . This case was covered in Lemma 1.

**Case 3.** Let  $g > \frac{r}{3}$ . To show that for all  $(x, y) \in \{(x, y) \mid (x - \frac{2gr^2}{r^2 - g^2})^2 + y^2 = \frac{4g^4 r^2}{(r^2 - g^2)^2} \wedge x^2 + y^2 < r^2\}$  there exists a triangle in  $\Omega$  with symmedian point  $(x, y)$ , we follow a method similar to that of Case 1. Here, the angles we use are  $\cos^{-1}(\frac{3g-r}{2r})$  and  $\cos^{-1}(\frac{3g^2-r^2}{2rg})$ . As in Case 1, we must verify that we can always take the arc cosine of  $\frac{3g-r}{2r}$  and  $\frac{3g^2-r^2}{2rg}$ . Let us first examine  $\frac{3g-r}{2r}$ . Since  $g > \frac{r}{3}$ ,  $0 = \frac{3g-3g}{2r} < \frac{3g-r}{2r}$ . Also,  $g < r \iff \frac{3g-r}{2r} < 1$ . Thus,  $0 < \frac{3g-r}{2r} < 1$ . Turning to the second expression, we see that  $\frac{3g^2-r^2}{2rg} > \frac{3(\frac{r^2}{9})-r^2}{2rg} = \frac{-r}{3g} > -1$  since  $g > \frac{r}{3}$ . Moreover,  $0 > (3g+r)(g-r) \iff 1 > \frac{3g^2-r^2}{2rg}$ . Thus,  $-1 < \frac{3g^2-r^2}{2rg} < 1$ , and we can always take the arc cosine of both  $\frac{3g-r}{2r}$  and  $\frac{3g^2-r^2}{2rg}$ .

Now, as in Case 1, we need to determine which of the two angles discussed above is larger within the interval from 0 to  $\pi$ . We easily see, however, that  $g < r \iff -g > -r \iff 3g^2 - rg > 3g^2 - r^2 \iff \frac{3g-r}{2r} > \frac{3g^2-r^2}{2rg}$ . Thus,  $\cos^{-1}(\frac{3g-r}{2r}) < \cos^{-1}(\frac{3g^2-r^2}{2rg})$ .

The natural next step is to determine an interval upon which the function  $K$  can act. Before we do this, however, we need to consider one issue. In Case 1, the Carleton circle was entirely contained within the circumcircle of  $\Omega$  since  $\frac{2gr}{r-g}$  and  $\frac{2gr}{r+g}$  are both less than  $r$  for  $g < \frac{r}{3}$ . When  $g > \frac{r}{3}$ ,  $\frac{2gr}{r+g}$  remains less than  $r$ , but  $\frac{2gr}{r-g}$  does not. We can see this by using *Mathematica* to find the intersections of the circumcircle and the Carleton circle —  $\left(\frac{3g^2+r^2}{4g}, \frac{1}{4}r\sqrt{10 - \frac{9g^2}{r^2} - \frac{r^2}{g^2}}\right)$  and  $\left(\frac{3g^2+r^2}{4g}, -\frac{1}{4}r\sqrt{10 - \frac{9g^2}{r^2} - \frac{r^2}{g^2}}\right)$ . For all  $x \geq \frac{3g^2+r^2}{4g}$ , if  $(x, y)$  is a point on the Carleton circle, then  $(x, y)$  will lie on or outside the circumcircle of  $\Omega$ . As discussed in the footnote in the proof of Lemma 1, the symmedian point must lie strictly inside the circumcircle. Thus,  $\{x \mid x \in [\frac{3g^2+r^2}{4g}, \frac{2gr}{r-g}]\}$  are not valid  $x$ -values for symmedian points.

Keeping this in mind, let us examine  $K(\cos^{-1}(\frac{3g-r}{2r}))$  and  $K(\cos^{-1}(\frac{3g^2-r^2}{2rg}))$ . As we saw earlier,  $K(\cos^{-1}(\frac{3g-r}{2r})) = (\frac{2gr}{r+g}, 0)$ . Using *Mathematica*, we find that

$$K(\cos^{-1}(\frac{3g^2-r^2}{2rg})) = \left(\frac{3g^2+r^2}{4g}, -\frac{1}{4}r\sqrt{10 - \frac{9g^2}{r^2} - \frac{r^2}{g^2}}\right).$$

Thus,  $K(\cos^{-1}(\frac{3g-r}{2r}))$  is a valid symmedian point, but  $K(\cos^{-1}(\frac{3g^2-r^2}{2rg}))$  is not.

With this knowledge in hand, let us choose our interval to be  $[\cos^{-1}(\frac{3g-r}{2r}), \cos^{-1}(\frac{3g^2-r^2}{2rg})]$ . By the preservation of connectedness under continuous functions, since  $[\cos^{-1}(\frac{3g-r}{2r}), \cos^{-1}(\frac{3g^2-r^2}{2rg})]$  is connected,  $K([\cos^{-1}(\frac{3g-r}{2r}), \cos^{-1}(\frac{3g^2-r^2}{2rg})])$  is also connected. Since as shown above,  $K(\cos^{-1}(\frac{3g^2-r^2}{2rg}))$  is one of the intersections of the circumcircle of  $\Omega$  with the Carleton circle, and since  $K(\cos^{-1}(\frac{3g-r}{2r}))$  is the intersection contained within the circumcircle of the Carleton circle with the  $x$ -axis, it follows that  $K([\cos^{-1}(\frac{3g-r}{2r}), \cos^{-1}(\frac{3g^2-r^2}{2rg})])$  contains at least

$$\{(x, y) \mid (x - \frac{2gr^2}{r^2 - g^2})^2 + y^2 = \frac{4g^4r^2}{(r^2 - g^2)^2} \wedge x^2 + y^2 < r^2 \wedge \text{either } y \leq 0 \text{ or } y \geq 0\}.$$

By the previous discussion,  $[\cos^{-1}(\frac{3g-r}{2r}), \cos^{-1}(\frac{3g^2-r^2}{2rg})] \subseteq (0, \pi)$ . Because of this, by the same argument as in Case 1, the reflection across the Euler line of the above arc will also be composed of symmedian points. Therefore, every point on the arc of the Carleton circle strictly contained within the disk enclosed by the circumcircle of  $\Omega$  is the symmedian point of a triangle in  $\Omega$ .  $\square$

## 4 Conclusion

We have seen that if we group triangles into families based on the distance between the circumcenter  $O$  and the centroid  $G$ , the length of the circumradius, and the measure of the angle  $\angle GOA$ , where  $A$  is one vertex, then the symmedian point has a fascinating property—it always lies on a circle centered at a point on the Euler line. Moreover, the locus of symmedian points forms an arc of that circle.

After completing the work above, the authors became aware that the result which we see in Case 1 when  $g < \frac{r}{3}$  is posed as a question in Nathan Altshiller Court's well-known geometry text *College Geometry: An Introduction to the Modern Geometry of the Triangle and the Circle*. The problem states, “[a] variable triangle has a fixed circumcenter and a fixed centroid. Show that the locus of the . . . [symmedian]. . . point is a circle” [Altshiller Court, 292]. As we have seen, there are two more cases to consider. All symmedian points fall on a specified circle; however, the locus of all possible symmedian points is not always a complete circle.

The property of the symmedian point that we examined here is only one of many interesting facts that arise while exploring the  $\theta gr$ -families of triangles. A more thorough treatment of this subject can be found in the authors' senior thesis, "A New Way to Think About Triangles" at [http://apps.carleton.edu/curricular/math/Math\\_Comps/Math\\_Comps\\_0607/](http://apps.carleton.edu/curricular/math/Math_Comps/Math_Comps_0607/) and in [Carr]. Even that work, however, only begins to touch upon the properties of the triangle space created by these familial groupings. It would be fascinating to continue the exploration of this topic to see if new or more general results can be obtained.

## References

- [Altshiller Court] Altshiller Court, Nathan. *College Geometry: An Introduction to the Modern Geometry of the Triangle and the Circle*. New York: Barnes & Noble, Inc., 1952.
- [Carr] Carr, Adam, Julia Fisher, Andrew Roberts, David Xu. "Natural Families of Triangles I: Parametrizing Triangle Space," *Rose-Hulman Undergraduate Math Journal*, Spring 2008.
- [Honsberger] Honsberger, Ross. *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*. Washington: Mathematical Association of America, 1995.
- [Kimberling] Kimberling, Clark. "Triangle Centers." *Encyclopedia of Triangle Centers - ETC*. University of Evansville. 12 Mar. 2007. <http://faculty.evansville.edu/ck6/tcenters/index.html>.
- [Mueller] Mueller, William. "Centers of Triangles of Fixed Center: Adventures in Undergraduate Research." *Mathematics Magazine*. Vol. 70, No. 4 (Oct., 1997), pp. 252-262.
- [Munkres] Munkres, James R. *Topology*. 2nd ed. Upper Saddle River, NJ: Prentice Hall, Inc., 2000.
- [Stewart] Stewart, James. *Calculus*. 5th ed. Australia: Brooks/Cole, 2003.