

# Reducibility of Second Order Differential Operators with Rational Coefficients

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## 1. INTRODUCTION

In this paper we discuss the reducibility of second order differential operators with rational coefficients. Second order linear homogeneous differential operators with rational coefficients are of the form:

$$D^2 + \alpha(x)D + \beta(x), \quad (1)$$

where  $\alpha(x), \beta(x)$  are rational functions and  $D^n = \frac{d^n}{dx^n}$  [4]. Hence,  $\alpha(x) = \frac{p(x)}{m(x)}$  and  $\beta(x) = \frac{q(x)}{n(x)}$ , where  $p(x), q(x), m(x), n(x)$  are polynomials with complex coefficients. The factorization of differential operators provides a method to find solutions to higher order differential equations by reducing the equation to a first order equation after factorization. First order linear equations are very easy to solve and are one of the equations studied extensively in an ordinary differential equations class.

We will begin Section 2 with a discussion of the background of differential operators and their reducibility. We will also focus on the Frobenius definition of reducibility and present a few theorems which provided ideas for the results in this paper, namely from Guerra, Shapiro, and Heilman. These results will lead to a discussion of the powerful relationship between the coefficients of our factored operator and the coefficients of our reduced operator.

Operators that factor into one first order operator with a rational function coefficient and one first order operator with a constant coefficient will be discussed in section 3. We will give specific tests for the reducibility of these operators. From there, we will present results for operators with polynomial coefficients. These results will not be as strong as other results in the paper; however, they do provide a condition for irreducibility. We will also briefly discuss a special case of second order operators with polynomial coefficients that reduce to two first order operators with rational function coefficients.

The remainder of our paper will concentrate on the reducibility of second order differential operators with rational coefficients. Our results pertain to operators with one or two regular singular points. Cauchy-Euler operators will be used as an example of such operators and to demonstrate that factorizations of some operators are not unique. Along with these results, we present modifications we made to the original indicial equation found by Frobenius. Further discussion will involve the relationship between the differential operator, the modified indicial equation, reducibility, factorizations of the operator, and solutions to homogeneous equations formed by the operator.

## 2. BACKGROUND

It is well-known that second order linear homogeneous differential equations with constant coefficients can be solved using the indicial (auxiliary) equation [4]. Differential equations with constant coefficients are of the form:

$$(c_1D^2 + c_2D + c_3)y = 0, \quad (2)$$

where  $c_1, c_2, c_3 \in \mathbb{C}$  and  $y$  is a function of  $x$ . The indicial equation for (2) is

$$c_1r^2 + c_2r + c_3 = 0. \quad (3)$$

The indicial equation either has two distinct roots or one repeated root. If there are two distinct roots to equation (3), call them  $r_1$  and  $r_2$ , then the solutions to (2) are  $y_1 = c_4e^{r_1x}$  and  $y_2 = c_5e^{r_2x}$  with  $c_4, c_5 \in \mathbb{C}$ . On the other hand, one repeated root,  $r_1$ , gives the two solutions  $y_1 = c_4e^{r_1x}$  and  $y_2 = c_5xe^{r_1x}$  [4]. In the following example, we will show how factoring is directly related to finding solutions.

Example 1:

$$(D^2 - \frac{5}{2}D - \frac{3}{2})y = 0$$

The solutions to our indicial equation are  $r_1 = -\frac{1}{2}$  and  $r_2 = 3$ . Hence our solutions are

$$y_1 = c_1e^{-\frac{x}{2}}; y_2 = c_2e^{3x}$$

The solution  $y_1$  yields,

$$\begin{aligned} \ln y_1 &= \ln c_1 - \frac{x}{2} \\ \frac{dy_1}{y_1} &= -\frac{1}{2}dx \\ \frac{dy_1}{dx} + \frac{1}{2}y_1 &= 0 \\ (D + \frac{1}{2})y_1 &= 0. \end{aligned}$$

Hence our first factor is  $(D + \frac{1}{2})$ . We can easily find the coefficient of our other factor by subtracting  $\frac{1}{2}$  from  $-\frac{5}{2}$ . Thus, our factored form is  $(D - 3)(D + \frac{1}{2})$ . Since our factors contain constant coefficients, we also see  $(D - 3)(D + \frac{1}{2}) = (D + \frac{1}{2})(D - 3)$ . Thus, the factors are commutative.

We see that factoring differential operators with constant coefficients is equivalent to factoring polynomials. Therefore, there will be no need to discuss second order operators with constant coefficients within this paper.

The study of the reducible differential operators began in 1873, when Frobenius presented the formal definition of reducible differential operators [3],[2]. Frobenius' definition of reducibility

is for operators of arbitrary order. Thus we present the general form of linear homogeneous operators:

$$L = D^n + \alpha_{n-1}(x)D^{n-1} + \alpha_{n-2}(x)D^{n-2} + \dots + \alpha_1(x)D + \alpha_0(x), \quad (4)$$

where each  $\alpha_i(x)$  are rational functions of  $x$  and  $D^k = \frac{d^k}{dx^k}$ .

**Definition 1 (Frobenius):** *L is reducible if there exists an operator M, also with rational coefficients, of order less than the order of L, such that  $Ly(x) = My(x) = 0$  for some function  $y(x) \neq 0$  [3],[2].*

**Example 2:**

Letting  $L_1 = D^2 - x^2 - 1$  and  $M_1 = D - x$  operate on  $y = e^{\frac{x^2}{2}}$ , we see

$$L_1(e^{\frac{x^2}{2}}) = M_1(e^{\frac{x^2}{2}}) = 0.$$

Thus,  $L_1$  is reducible. Furthermore  $L_1$  can be factored into two first order operators, namely  $(D + x)(D - x)$ .

Notice from the above example that there exists a relationship between factoring and reducing operators. We will further establish this relationship at the end of this section.

Since the Frobenius definition of reducibility was introduced, we have found no published research on the reducibility of differential operators with variable coefficients until 2003. In 2003, Guerra and Shapiro used the Frobenius notion of reducibility of differential operators to prove the following results:

**Theorem 1 (Guerra-Shapiro):** *Bessel's equation of order m is irreducible.*

and

**Theorem 2 (Guerra-Shapiro):** *Bessel functions of integer order cannot satisfy an equation of the form  $u^{(m)} + r_{m-1}u^{(m-1)} + \dots + r_0u = 0$ , where the  $r_m$  are polynomial coefficients.*

Guerra and Shapiro's paper not only proved the above results for Bessel's equation, but extended the result to other equations within the same class as Bessel's equation. This opened the discussion of the reducibility of differential operators once again. We mention it here because several ideas in this paper, mainly the dependence upon solutions to the indicial equation for reducibility, came from Guerra and Shapiro's paper. Guerra began studying the reducibility of differential operators with undergraduates during a research project with Dennison Heilman, funded by the Arkansas Space Grant Consortium. Heilman proved the following result in his research:

**Theorem 3 (Heilman):** *If  $D^n + a_{n-1}D^{n-1} + \dots + a_1D + f(x)$ , where each  $a_k$  is a constant coefficient,  $f(x)$  is a polynomial function, and  $n$  does not divide  $(f(x))$ , then the operator is irreducible.*

Some of the same ideas in Heilman's paper are used in this paper as well. Heilman discovered that reducibility had some dependence upon the degree of the polynomial coefficients. The use of his idea will be prevalent in our proof of the reducibility of second order operators with

polynomial coefficients to two first order differential operators with polynomial coefficients.

Since the reducibility of differential operators has not been widely studied, our work will begin with an introduction to the basic principles of factoring operators. The study of reducible differential operators requires establishing relationships between the coefficients of our operators in their expanded form and the coefficients of the factored form. In order to establish this relationship, we used test functions. Frobenius introduced the idea of test functions to study the reducibility of differential operators and also to develop his method for finding series solutions to certain equations [2]. A test function can be any arbitrary function of  $x$  on which we allow both representations of our operator to act [2]. These test functions provide a relationship between  $\alpha(x)$  and  $\beta(x)$ , the coefficients of our expanded operator, to  $w(x)$  and  $f(x)$ , the coefficients of our factored operator. We will use the functions  $y = 1$  and  $y = x$ , as these two functions give the simplest relationship between our coefficients. Many of our results depend upon the relationships gained by working with test functions. Hence, we will derive these relationships now for arbitrary operators  $D^2 + \alpha(x)D + \beta(x)$  and  $(D + w(x))(D + f(x))$ , where  $\alpha(x), \beta(x), w(x), f(x)$  are rational functions with complex coefficients. Notice that  $\beta(x)$  operating on  $y$  is the operation of multiplying two rational functions, but  $D^n = \frac{d^n}{dx^n}$ , as defined before. Then, we see

$$\begin{aligned}
 (D^2 + \alpha(x)D + \beta(x))(1) &= (D + w(x))(D + f(x))(1) \\
 (0 + \alpha(x) \cdot 0 + \beta(x)) &= (D + w(x))(0 + f(x)) \\
 \beta(x) &= (D + w(x))(f(x)) \\
 \\ 
 \beta(x) &= f'(x) + w(x)f(x)
 \end{aligned} \tag{5}$$

and

$$\begin{aligned}
 (D^2 + \alpha(x)D + \beta(x))(x) &= (D + w(x))(D + f(x))(x) \\
 (0 + \alpha(x) \cdot 1 + x \cdot \beta(x)) &= (D + w(x))(1 + x \cdot f(x)) \\
 \alpha(x) + x \cdot \beta(x) &= 0 + 1 \cdot f(x) + x \cdot f'(x) + w(x) + x \cdot w(x)f(x) \\
 \alpha(x) + x \cdot \beta(x) &= f(x) + w(x) + x \cdot (f'(x) + w(x)f(x)) \\
 \alpha(x) + x \cdot \beta(x) &= f(x) + w(x) + x \cdot \beta(x) \\
 \\ 
 \alpha(x) &= f(x) + w(x)
 \end{aligned} \tag{6}$$

We note that the product rule was used in the third line of the derivation of (6), when  $D$  operated on  $x \cdot f(x)$ . The presence of the product rule is the unique quality which separates the idea of factoring differential operators from other forms of factoring, including that of differential operators with constant coefficients. Now that we have established the above relationships for  $\alpha(x), \beta(x)$ , we must show they hold for all  $y = g(x)$ . Thus,

$$\begin{aligned}
 (D^2 + \alpha(x)D + \beta(x))g(x) &= (D^2 + (f(x) + w(x))D + f'(x) + w(x)f(x))(g(x)) \\
 &= g''(x) + (f(x) + w(x))g'(x) + (f'(x) + w(x)f(x))g(x).
 \end{aligned}$$

Also,

$$\begin{aligned}
(D + w(x))(D + f(x))g(x) &= (D + w(x))(g'(x) + f(x)g(x)) \\
&= g''(x) + f'(x)g(x) + g'(x)f(x) + w(x)g'(x) + w(x)f(x)g(x) \\
&= g''(x) + (f(x) + w(x))g'(x) + (f'(x) + w(x)f(x))g(x)
\end{aligned}$$

Therefore, our factored and expanded forms are equivalent when acting on all functions of  $x$ . Consider the following example, where  $y$  is a function of  $x$ :

Example 3:

$$\begin{aligned}
(D + x)(D - x)y &= (D + x)(y' - xy) \\
&= y'' - y - xy' + xy' - x^2y \\
&= (D^2 - 1 - x^2)y \\
(D + x)(D - x) &= (D^2 - 1 - x^2).
\end{aligned}$$

However,

$$\begin{aligned}
(D - x)(D + x)y &= (D - x)(y' + xy) \\
&= y'' + y + xy' - xy' - x^2y \\
&= (D^2 + 1 - x^2)y \\
(D - x)(D + x) &= (D^2 + 1 - x^2).
\end{aligned}$$

In Example 3, we see our factors are not commutative. More generally, equations (5) and (6) give  $(D + f(x))(D + w(x)) = D^2 + (f(x) + w(x))D + w'(x) + w(x)f(x)$ . Hence, factors are not commutative if at least one of the functions  $f(x)$  or  $w(x)$  is not constant.

As mentioned earlier, there exists a relationship between reducibility and factorizations. We will develop this relationship now with Theorem 4.

**Theorem 4:** *If  $D^2 + \alpha(x)D + \beta(x)$  is factorable, then it is reducible.*

**Proof:** Let  $D^2 + \alpha(x)D + \beta(x)$  factor into  $(D + w(x))(D + f(x))$  and let  $y_1(x)$  be the solution to  $(D + f(x))(y(x)) = 0$ . We know  $y_1$  exists since  $(D + f(x))(y(x)) = 0$  is a first order linear equation [4, Sect. 2.3]. Then,  $(D + w(x))(D + f(x))(y_1(x)) = (D + w(x))(0) = 0$ . Since,  $(D^2 + \alpha(x)D + \beta(x))(y(x)) = (D + w(x))(D + f(x))(y(x))$  for all  $y$ ,  $(D^2 + \alpha(x)D + \beta(x))(y_1(x)) = 0$ . Thus,  $(D + f(x))$  and  $D^2 + \alpha(x)D + \beta(x)$  share a solution. Therefore,  $D^2 + \alpha(x)D + \beta(x)$  is reducible.  $\square$

Notice in the proof of Theorem 4, reducibility ensures that our factored form is equivalent to the expanded form when acting on  $y_1(x)$ , the solution to  $(D + f(x))y(x) = 0$ . In order to establish the factored form, we find the second factor,  $(D + w(x))$ , so that the expanded operator is equivalent to the factored operator for all  $y(x)$ . In this paper, we will only use the implication that factorization ensures reducibility.

### 3. RATIONAL AND CONSTANT COEFFICIENTS:

This section will provide tests for factoring differential operators with one factor containing a

constant coefficient and the other containing a rational function coefficient. Since factors are not commutative, we have two different cases to consider. Letting  $c \in \mathbb{C}$  and  $h(x)$  be a rational function of  $x$ , the two possible forms for reducible operators are

$$(D + h(x))(D + c) \tag{7}$$

$$(D + c)(D + h(x)). \tag{8}$$

Before we begin discussing the tests for these operators, it is important to note the following special case. If we let  $c = 0$ , then operators (7) and (8) become  $(D + h(x))D \equiv D^2 + h(x)D$  and  $D(D + h(x)) \equiv D^2 + h(x)D + h'(x)$ , respectively. We will return to this special case in Theorem 5 of Section 4.

*Test 1: Let  $p(x)$  and  $q(x)$  be polynomial coefficients, and let  $c = \frac{q}{p}$ , where  $q$  and  $p$  are the leading coefficients of  $q(x)$  and  $p(x)$  respectively. Also, define  $h(x) := p(x) - c$ . Then,  $D^2 + p(x)D + q(x)$  factors as a product of first order operators, one with a constant coefficient  $c$  and one with a polynomial coefficient  $h(x)$  if the two conditions below are met:*

*i.  $\deg(p(x)) = \deg(q(x))$ ,*

*ii. and one of:*

*(a)  $q(x) = c(h(x))$*

*or*

*(b)  $q(x) = c(h(x)) + h'(x)$ .*

*Furthermore, operators corresponding to Case iia factor as  $(D + h(x))(D + c)$ , and operators corresponding to Case iib factor as  $(D + c)(D + h(x))$ .*

The following example will demonstrate the above test.

**Example 4:** Consider the operator  $D^2 + (3x^2 + 6)D + 12x^2 + 8$ . Then,  $c = \frac{12}{3} = 4$  and  $h(x) = 3x^2 + 6 - 4 = 3x^2 + 2$ . Further notice that

$$12x^2 + 8 = 4(3x^2 + 2).$$

Therefore,  $D^2 + (3x^2 + 6)D + 12x^2 + 8 = (D + 3x^2 + 2)(D + 4)$ .

The next test can be used for differential operators with rational coefficients.

*Test 2: Let  $c = \frac{p}{q}$ , where  $p$  and  $q$  are the leading coefficients of  $p(x)$  and  $q(x)$  respectively, and define  $h(x) := p(x) - c$ . Then,  $D^2 + \frac{p(x)}{s(x)}D + \frac{q(x)}{s^2(x)}$  factors as a product of first order operators, one with a constant coefficient  $c$  and one with a rational coefficient  $\frac{h(x)}{s(x)}$  if the conditions in one of the two cases below are met:*

*Case I:*

*i.  $\deg(q(x)) \geq \deg(p(x)) \geq \deg(s(x))$  and  $\deg(q(x)) \geq 2(\deg(s(x)))$ ,*

*ii. and one of:*

*(a)  $\frac{q(x)}{s^2(x)} = c \left( \frac{h(x)}{s(x)} \right)$ ,*

*or*

*(b)  $\frac{q(x)}{s^2(x)} = \frac{d}{dx} \left( \frac{h(x)}{s(x)} \right) + c \left( \frac{h(x)}{s(x)} \right)$ .*

*Furthermore, operators corresponding to Case iia factor as  $\left( D + \frac{h(x)}{s(x)} \right) (D + c)$ , and operators corresponding*

to Case iib factor as  $(D + c)\left(D + \frac{h(x)}{s(x)}\right)$ .

Case II:

- i.  $\deg(q(x)) \geq \deg(p(x)) = \deg(s(x))$  and  $\deg(q(x)) < 2(\deg(s(x)))$ ,
- ii. dividing  $p(x)$  by  $s(x)$  yields  $\frac{r(x)}{s(x)} + k$ , where  $k \in \mathbb{C}$  and  $r(x) \in \mathbb{C}[x]$ ,
- iii. and one of:

$$(a) \frac{q(x)}{s^2(x)} = k \left( \frac{r(x)}{s(x)} \right),$$

or

$$(b) \frac{q(x)}{s^2(x)} = \frac{d}{dx} \left( \frac{r(x)}{s(x)} \right) + k \left( \frac{r(x)}{s(x)} \right).$$

Furthermore, operators corresponding to Case iia factor as  $\left(D + \frac{r(x)}{s(x)}\right)(D + k)$ , and operators corresponding to Case iib factor as  $(D + k)\left(D + \frac{r(x)}{s(x)}\right)$ .

Notice Case I of Test II is very similar to Test 1. Also, note the different requirements needed for Case I and Case II. Case I requires  $\deg(q(x)) \geq 2\deg(p(x))$ , and Case II requires  $\deg(q(x)) < 2\deg(p(x))$ . The following example will demonstrate the use of Test 2 Case I.

Example 5: Consider the operator  $D^2 + \frac{x+2}{x-2}D + \frac{3x^2-9x+5}{(x-2)^2}$ . From Test 2 Case I, we see that  $c = \frac{3}{1} = 3$ ,  $h(x) = x + 2 - 3 = x - 1$ , and

$$\frac{d}{dx} \left( \frac{x-1}{x-2} \right) + \frac{c(h(x))}{s(x)} = \frac{-1}{(x-2)^2} + \frac{3x-3}{x-2} = \frac{3x^2-9x+5}{(x-2)^2}.$$

Therefore,  $D^2 + \frac{x+2}{x-2}D + \frac{3x^2-9x+5}{(x-2)^2} = (D + 3)\left(D + \frac{x-1}{x-2}\right)$ .

Example 6 will show how Test II Case II can be used.

Example 6: Consider the operator  $D^2 + \frac{6x^2+6x+7}{3x^2+2x+1}D + \frac{12x^3+32x^2-6x+2}{(3x^2+2x+1)^2}$ . Then,

$$r = \frac{6x^2 + 6x + 7}{3x^2 + 2x + 1} = \frac{2x + 5}{3x^2 + 2x + 1} + 2.$$

Hence  $k = 2$  and  $r(x) = 2x + 5$ . Now, we can verify that property iib. is satisfied:

$$\frac{d}{dx} \left( \frac{2x + 5}{3x^2 + 2x + 1} \right) + 2 \left( \frac{2x + 5}{3x^2 + 2x + 1} \right) = \frac{-6x^2 - 30 - 8}{(3x^2 + 2x + 1)^2} + \frac{4x + 10}{3x^2 + 2x + 1} = \frac{12x^3 + 32x^2 - 6x + 2}{(3x^2 + 2x + 1)^2}.$$

Therefore,  $D^2 + \frac{6x^2+6x+7}{3x^2+2x+1}D + \frac{12x^3+32x^2-6x+2}{(3x^2+2x+1)^2} = (D + 2)\left(D + \frac{2x+5}{3x^2+2x+1}\right)$ .

The intricacy of the conditions in the above tests demonstrate the difficulty in obtaining factorizations for these operators. For this reason, our results are limited to very specific operators for the rest of the paper, with the exception of Theorem 5.

#### 4. OPERATORS WITH POLYNOMIAL COEFFICIENTS

We will now consider the reducibility of second order operators with polynomial coefficients.

This section will deal with second order operators of the form in (1) with  $\alpha(x), \beta(x) \in \mathbb{C}[x]$ . Theorem 5 provides a sufficient condition for irreducibility of second order differential operators with polynomial coefficients to first order operators with polynomial coefficients.

From Example 3, we see two operators that reduce to polynomial coefficients and  $\deg(\alpha(x)) < \deg(\beta(x))$ . But from Heilman's result, we also see that  $D^2 - x^3$  is irreducible, even though  $\deg(\alpha(x)) < \deg(\beta(x))$ . Therefore, the condition in Theorem 5 is not necessary for irreducibility.

**Theorem 5:** *Let  $\alpha(x), \beta(x) \in \mathbb{C}[x]$ , with  $\beta(x) \neq 0$ , and  $\beta(x) \neq \alpha'(x)$ . If  $\deg(\alpha(x)) > \deg(\beta(x))$ , then  $D^2 + \alpha(x)D + \beta(x)$  cannot be factored as the product of two first order operators with polynomial coefficients,  $(D + w(x))(D + f(x))$ .*

We note here that the conditions  $\beta(x) \neq 0$ , and  $\beta(x) \neq \alpha'(x)$  eliminate the trivial case in Section 3 where one of the factors is  $D$ .

**Proof:** Let  $\alpha(x), \beta(x) \in \mathbb{C}[x]$  with  $\beta(x) \neq 0$ ,  $\beta(x) \neq \alpha'(x)$  and  $\deg(\alpha(x)) > \deg(\beta(x))$ . Assume  $D^2 + \alpha(x)D + \beta(x)$  can be factored into  $(D + w(x))(D + f(x))$ . The relationships found in (5) and (6) hold for all rational functions. Thus,

$$\begin{aligned}\alpha(x) &= f(x) + w(x) \\ \beta(x) &= f'(x) + w(x)f(x)\end{aligned}$$

The conditions  $\beta(x) \neq 0$  and  $\beta(x) \neq \alpha'(x)$  imply  $w(x) \neq 0$  and  $f(x) \neq 0$ . Since the  $\deg(f(x)) \geq \deg(f'(x))$ ,  $w(x) \neq 0$ , and  $f(x) \neq 0$ , we see

$$\deg(\beta(x)) = \deg(w(x)) + \deg(f(x))$$

Furthermore, the degree of  $\alpha(x)$  is equal to at most  $\deg(w(x))$  or  $\deg(f(x))$ . However,  $\deg(w(x)) + \deg(f(x)) \geq \deg(w(x))$  and  $\deg(w(x)) + \deg(f(x)) \geq \deg(f(x))$ . In either of the cases,

$$\deg(\alpha(x)) \leq \deg(\beta(x)).$$

This is a contradiction to our assumption  $\deg(\alpha(x)) > \deg(\beta(x))$ . Therefore,  $D^2 + \alpha(x)D + \beta(x)$  cannot be factored as the product of two first order operators with polynomial coefficients, if  $\deg(\alpha(x)) > \deg(\beta(x))$ , with  $\beta(x) \neq 0$ , and  $\beta(x) \neq \alpha'(x)$ .

The above theorem does not mention the irreducibility of second order operators with polynomial coefficients to first order differentiable operators with rational coefficients. For quite some time, we believed the result in Theorem 5 could be extended to include this conjecture; however, there do exist operators with polynomial coefficients reducible to two first order operators with rational coefficients. Theorem 6 will provide us with a general form for one such operator.

**Theorem 6:** *If  $y = c_1x + c_2$  is a solution to  $D^2 + \alpha(x)D + \beta(x)$ , where  $\alpha(x), \beta(x) \in \mathbb{C}[x]$  then there exists  $p(x) \in \mathbb{C}[x]$  such that  $D^2 + \alpha(x)D + \beta(x)$  factors into  $(D + p(x)(c_1x + c_2) - \frac{c_1}{c_1x+c_2})(D - \frac{c_1}{c_1x+c_2})$ .*

**Example 7:**

Notice  $y = 2x + 1$  is a solution to  $D^2 + (2x + 1)^2D - 2(2x + 1)$ , and



$$D^2 + (2x + 1)^2 D - 2(2x + 1) = (D + (2x + 1)^2 + \frac{2}{2x+1})(D - \frac{2}{2x+1}).$$

In order to find operators of this form, we need to know the possible solutions to  $D^2 + \alpha(x)D + \beta(x)$ , where  $\alpha(x), \beta(x) \in \mathbb{C}[x]$ . Since factoring ensures reducibility, we know  $(D^2 + \alpha(x)D + \beta(x))y = 0$  must have a common solution with our first factor;  $(D + f(x))y = 0$ . Since  $f(x)$  is a rational function, the first order equation above can be written as  $(D - \frac{j(x)}{k(x)})y = 0$  with  $j(x), k(x) \in \mathbb{C}[x]$ . From this representation we find a solution to the original operator:

$$\left(D - \frac{j(x)}{k(x)}\right)y = 0$$

$$Dy - \frac{j(x)}{k(x)}y = 0$$

$$Dy = \frac{j(x)}{k(x)}y$$

$$\frac{dy}{dx} = \frac{j(x)}{k(x)}y$$

$$\frac{dy}{y} = \frac{j(x)}{k(x)}dx$$

$$\ln(y) = \int \frac{j(x)}{k(x)}dx$$

$$y = e^{\int \frac{j(x)}{k(x)}dx} \quad (9)$$

Let  $D^2 + \alpha(x)D + \beta(x)$  with  $\alpha(x), \beta(x) \in \mathbb{C}[x]$ . Then, it is well known that all solutions to  $D^2 + \alpha(x)D + \beta(x)$  are entire functions [2](i.e. a function analytic everywhere in the complex plane [1]). Notice  $y = e^{\frac{2x}{x^2+1}}$  is of the form in (9), since  $\frac{2x}{x^2+1} = \int \frac{-2(x^2-1)}{x^2+1}dx$  and  $\frac{-2(x^2-1)}{x^2+1}$  is a rational function. From the previous statement,  $y = e^{\frac{2x}{x^2+1}}$  is not a solution to  $D^2 + \alpha(x)D + \beta(x)$  with  $\alpha(x), \beta(x) \in \mathbb{C}[x]$  because it is not analytic at  $x = \pm i$ . But, there are some analytic solutions of the form in (9) that lead to first order operators with rational coefficients. Notice that  $y_1 = c_1x + c_2$  is an example of such a function. From example 7, we know the solution  $y = c_1x + c_2$  yields a factorization of the second order operator that it solves. Now, we will prove the reducibility of the general form in Theorem 6.

Proof: Let  $y = c_1x + c_2$  be a solution to  $D^2 + \alpha(x)D + \beta(x)$  with  $\alpha(x), \beta(x) \in \mathbb{C}[x]$ . So,

$$\begin{aligned} y &= c_1x + c_2 \\ \ln(y) &= \ln(c_1x + c_2) \\ \frac{dy}{y} &= \frac{c_1 dx}{c_1x + c_2} \\ \frac{dy}{dx} &= \frac{c_1 y}{c_1x + c_2} \\ \frac{dy}{dx} - \frac{c_1 y}{c_1x + c_2} &= 0 \\ \left(D - \frac{c_1}{c_1x + c_2}\right)y &= 0 \end{aligned}$$

Thus, our first factor is  $D - \frac{c_1}{c_1x + c_2}$ . Our second factor is found to be  $D + \alpha(x) + \frac{c_1}{c_1x + c_2}$  by substituting our first factor into (6). Hence, we see

$$\begin{aligned} \left(D + \alpha(x) + \frac{c_1}{c_1x + c_2}\right)\left(D - \frac{c_1}{c_1x + c_2}\right) &= D^2 + \frac{c_1^2}{(c_1x + c_2)^2} + \alpha(x)D + \frac{c_1\alpha(x)}{c_1x + c_2} - \frac{c_1^2}{(c_1x + c_2)^2} \\ &= D^2 + \alpha(x)D + \frac{c_1\alpha(x)}{c_1x + c_2} \end{aligned}$$

Furthermore,  $c_1x + c_2$  cannot divide  $c_1$ . Therefore,  $c_1x + c_2$  divides  $\alpha(x)$  since  $\beta(x) = \frac{c_1\alpha(x)}{c_1x + c_2} \in \mathbb{C}[x]$ . Letting  $p(x)(c_1x + c_2) = \alpha(x)$ , our second order operator then becomes

$$D^2 + p(x)(c_1x + c_2)D + c_1p(x) \tag{10}$$

and we are done.  $\square$

Notice the expanded operator in example 7 is of the form in (10). We have not been able to determine if this is the only factorization with rational coefficients for second order operators with polynomial coefficients, or if others exist. There are many differential equations with solutions of the form in (9), and sorting through these cases would take some time.

## 5. OPERATORS WITH ONE REGULAR SINGULARITY:

The remainder of this paper will focus on second order linear differential operators with singular points. In this section, we will consider operators with one singularity, while in Section 6 we will present a result for operators with two singularities. Frobenius has provided a framework for solutions to differential operators with one singular point. If our operator is  $D^2 + \frac{p(x)}{x-x_0}D + \frac{q(x)}{(x-x_0)^2}$  with  $p(x), q(x) \in \mathbb{C}[x]$ , then  $x = x_0$  is a regular singular point and the following theorem holds for this operator.

**Theorem 7 (Frobenius):** *If  $x = x_0$  is a regular singular point of the differential equation formed from  $(D^2 + \alpha(x)D + \beta(x))y = 0$ , then there exists at least one solution of the form*

$$y = \sum_{n=0}^{\infty} c_n(x - x_0)^{n+r}, \tag{11}$$

where the number  $r$  is a constant to be determined by the indicial equation. The series will converge at least on some interval  $0 < (x - x_0) < R$ . [4, Sect. 6.2]

The indicial equation mentioned in Theorem 7 is of the form

$$r(r - 1) + p(x_0)r + q(x_0) = 0. \quad (12)$$

Frobenius discovered the indicial equation when using his method of assuming a solution of the form (11) and substituting this function into a second order differential equation with one singularity. However, finding the coefficient  $c_n$  requires a recursive formula, which can take some time. Furthermore, simplification of the solution requires us to find the limiting function of the series of functions in (11). Since Cauchy-Euler Operators are the simplest operators of this form, we will begin this section by discussing these operators and show they have more than one factorization. Then, we will present a result that allows us to find a solution to certain types of second order equations without using the recursive formula or finding a limiting function.

*Theorem 8: Cauchy-Euler Operators are reducible to first order operators with rational coefficients. Furthermore, there are at least two factorizations with rational coefficients if the operator has two distinct roots to its indicial equation, and there exists a unique factorization with rational coefficients when the indicial equation has one real repeated root.*

Cauchy-Euler Operators are of the form  $D^2 + \frac{c_1}{x}D + \frac{c_2}{x^2}$  with  $c_1, c_2 \in \mathbb{C}$ . Since  $p, q$  are constant functions in such operators, our indicial equation (12) becomes

$$r(r - 1) + c_1r + c_2 = 0. \quad (13)$$

In the following example, we will demonstrate the use of the indicial equation to find solutions and factorizations for Cauchy-Euler Operators.

Example 8: Consider  $D^2 + \frac{6}{x}D + \frac{6}{x^2}$ . Solving

$$r(r - 1) + 6r + 6 = 0,$$

we see our roots are  $r_1 = -2$  and  $r_2 = -3$ . From Zill [4, Sect. 4.7], we know that these two roots ensure two solutions of the form  $y_1 = c_1x^{-2}$  and  $y_2 = c_2x^{-3}$ . Thus,

$$\begin{aligned} y_1 &= c_1x^{-2} \\ \ln(y_1) &= \ln(c_1) + \ln(x^{-2}) \\ \frac{dy_1}{y_1} &= \frac{-2}{x}dx \\ \frac{dy_1}{dx} &= \frac{-2}{x}y_1 \\ \frac{dy_1}{dx} + \frac{2}{x}y_1 &= 0 \\ (D + \frac{2}{x})(y_1) &= 0 \end{aligned}$$

Similarly, we could show that  $y_2 = c_2x^{-3}$  leads to the factor  $D + \frac{3}{x}$ . Hence our first factors are  $D + \frac{2}{x}$  and  $D + \frac{3}{x}$ . Our second factors can now be found by subtracting the coefficients of our first order operators from  $\alpha(x)$ .

$$\begin{aligned}\frac{6}{x} - \frac{2}{x} &= \frac{4}{x} \\ \frac{6}{x} - \frac{3}{x} &= \frac{3}{x}\end{aligned}$$

Therefore, our two factored forms are

$$\begin{aligned}(D + \frac{4}{x})(D + \frac{2}{x}) \\ (D + \frac{3}{x})(D + \frac{3}{x})\end{aligned}$$

Theorem 8 is the first theorem to provide necessary and sufficient conditions for the reducibility of operators. It is important to notice Theorem 8 relates the idea of reducibility to the roots of the indicial equation. This relationship will also be seen later in Theorem 9. The following is the proof of Theorem 8.

Proof: First, let our Cauchy-Euler operator,  $D^2 + \frac{c_1}{x}D + \frac{c_2}{x^2}$ , have two distinct roots to the indicial equation. Then, our solutions are of the form

$$y_1 = c_3x^{r_1}, y_2 = c_4x^{r_2}. [4, Sect. 4.7]$$

By manipulating the first solution, we obtain

$$\begin{aligned}\ln(y_1) &= \ln(c_3x^{r_1}) \\ \frac{dy_1}{y_1} &= \frac{r_1 dx}{x} \\ \frac{dy_1}{dx} &= \frac{r_1 y_1}{x}\end{aligned}$$

$$\left(D - \frac{r_1}{x}\right)y_1 = 0. \tag{14}$$

Similarly  $y_2$  gives

$$\left(D - \frac{r_2}{x}\right)y_2 = 0. \tag{15}$$

Hence, there are two linear operators with rational coefficients which share solutions with our second order operator. By equation (5), there are two factorizations with rational coefficients for our second order operator:

$$\left(D + \frac{c_1+r_1}{x}\right)\left(D - \frac{r_1}{x}\right); \left(D + \frac{c_1+r_2}{x}\right)\left(D - \frac{r_2}{x}\right).$$

Now assume our Cauchy-Euler operator has one real repeated root to its indicial equation. Then, its solutions are

$$y_1 = c_3x^{r_1}, y_2 = c_4\ln(x)x^{r_1}. [4, Sect. 4.7]$$

From the previous case, we know our first solution gives

$$\left(D + \frac{c_1 + r_1}{x}\right)\left(D - \frac{r_1}{x}\right). \quad (16)$$

Once again, manipulating our second solution gives the following:

$$\begin{aligned} \ln(y_2) &= \ln(c_4\ln(x)x^{r_1}) \\ \frac{dy_2}{y_2} &= \frac{r_1\ln(x) + 1}{x\ln(x)}dx \\ \frac{dy_2}{dx} &= \frac{r_1\ln(x) + 1}{x\ln(x)}y_2. \end{aligned}$$

This would provide a factor of the form:

$$\left(D - \frac{r_1\ln(x) + 1}{x\ln(x)}\right)y_2 = 0, \quad (17)$$

which does not have rational coefficients. Now, we must consider linear combinations of our two solutions. Thus, consider  $y = c_3x^{r_1} + c_4\ln(x)x^{r_1}$ . Using the same manipulations as above, we see

$$\begin{aligned} y &= c_3x^{r_1} + c_4\ln(x)x^{r_1} \\ \ln(y) &= \ln(c_3x^{r_1} + c_4\ln(x)x^{r_1}) \\ \frac{dy}{y} &= \frac{c_3r_1 + c_4 + c_4\ln(x)r_1}{(c_3 + c_4\ln(x))x^{r_1}}dx \\ \frac{dy}{dx} &= \frac{c_3r_1 + c_4 + c_4\ln(x)r_1}{(c_3 + c_4\ln(x))x^{r_1}}y \\ \frac{dy}{dx} - \frac{c_3r_1 + c_4 + c_4\ln(x)r_1}{(c_3 + c_4\ln(x))x^{r_1}}y &= 0 \\ \left(D - \frac{c_3r_1 + c_4 + c_4\ln(x)r_1}{(c_3 + c_4\ln(x))x^{r_1}}\right)y &= 0. \end{aligned}$$

Once again, our operator's coefficient is not a rational function of  $x$ . Since we take the natural logarithm of our solution, the coefficient of our factor does not depend upon the constant in our first solution. Therefore, we have a unique factorization with rational coefficients given by (16).□

From the above result, we see that the reducibility of Cauchy-Euler operators depends upon the indicial equations. Therefore, we investigated the application of the indicial equation to other second order operators with regular singularities. We found that a shared root between the original indicial equation (12) and a modified form of the indicial equation provides sufficient

means for reducibility. We can also find the solutions to homogeneous equations formed by these operators without having to apply the Frobenius method.

**Theorem 9:** Let  $D^2 + \frac{c_1x+c_2}{x-x_0}D + \frac{c_3x+c_4}{(x-x_0)^2}$ , with  $c_1, c_2, c_3, c_4 \in \mathbb{C}$ , have a solution,  $r_1$ , to the indicial equation

$$r(r-1) + (c_1x_0 + c_2)r + (c_3x_0 + c_4) = 0. \quad (18)$$

The following two implications hold:

i. If  $r_1$  is a solution to

$$r(r-1) + c_2r + c_4 = 0, \quad (19)$$

then  $D^2 + \frac{c_1x+c_2}{x-x_0}D + \frac{c_3x+c_4}{(x-x_0)^2}$  factors into  $\left(D + \frac{c_1x+c_2+r}{x-x_0}\right)\left(D - \frac{r}{x-x_0}\right)$ , and there is a solution of the form  $y = (x-x_0)^r$  to the homogeneous equation  $(D^2 + \frac{c_1x+c_2}{x-x_0}D + \frac{c_3x+c_4}{(x-x_0)^2})(y) = 0$ .

ii. If  $r_1$  is a solution to

$$r(r-1) + c_2r + c_1x_0 + c_4 = 0, \quad (20)$$

then  $D^2 + \frac{c_1x+c_2}{x-x_0}D + \frac{c_3x+c_4}{(x-x_0)^2}$  factors into  $\left(D + \frac{1-r_1}{x-x_0}\right)\left(D + \frac{c_1x+c_2+r_1-1}{x-x_0}\right)$ , and there is a solution  $y = (x-x_0)^{-(c_1x_0+c_2+r_1-1)}e^{-c_1x}$  to the homogeneous equation  $(D^2 + \frac{c_1x+c_2}{x-x_0}D + \frac{c_3x+c_4}{(x-x_0)^2})(y) = 0$ .

Since  $p(x) = c_1x + c_2$  and  $q(x) = c_3x + c_4$ , we see that equation (19) in Case i is equivalent to  $r(r-1) + p(0)r + q(0) = 0$ , the indicial equation if  $x = 0$  was our singular point. Equation (20) in Case ii is equivalent to the modified indicial equation of  $r(r-1) + p(0)r + p(x_0) - p(0) + q(0) = 0$ . The addition of  $p(x_0) - p(0)$  to (19) in order to get (20) is believed to occur since the derivative of our first factor's coefficient is not identically zero.

Theorem 9 only provides sufficient conditions for the reducibility of  $D^2 + \frac{c_1x+c_2}{x-x_0}D + \frac{c_3x+c_4}{(x-x_0)^2}$ . Examples 9 and 10 will further explain the tests in Case i and Case ii, respectively, and demonstrate their usefulness.

**Example 9:** Consider the following operator:

$$D^2 + \frac{2x+1}{x-3}D - \frac{8x+16}{(x-3)^2}$$

We will begin testing with Frobenius's original indicial equation. Since  $p(x) = 2x + 1$ ,  $q(x) = -(8x + 16)$ , and our singular point is  $x = 3$ , our indicial equation becomes:

$$r(r-1) + 7r - 40 = 0.$$

The roots of the original indicial equation are  $r_1 = 4$  and  $r_2 = -10$ . Next, we see the indicial equation we would obtain if  $x = 0$  were a singular point,  $r(r-1) + c_2r + c_4 = 0$ , takes on the form

$$r(r-1) + 1r + 16 = 0.$$

Notice,  $r_1 = 4$  is a solution to our second indicial equation. By Theorem 9,  $D^2 + \frac{2x+1}{x-3}D - \frac{8x+16}{(x-3)^2}$  factors into  $(D + \frac{2x+5}{x-3})(D - \frac{4}{x-3})$  and  $D^2 + \frac{2x+1}{x-3}D - \frac{8x+16}{(x-3)^2}(y) = 0$  has a solution  $y = (x-3)^4$ . For verification,

$$\begin{aligned} & \left(D + \frac{2x+5}{x-3}\right)\left(D - \frac{4}{x-3}\right) \\ D^2 + \frac{4}{(x-3)^2} + \frac{2x+5}{x-3}D - \frac{4}{x-3}D - \frac{8x+20}{(x-3)^2} \\ & D^2 + \frac{2x+1}{x-3}D - \frac{8x+16}{(x-3)^2} \end{aligned}$$

Also,

$$\begin{aligned} \left(D^2 + \frac{2x+1}{x-3}D - \frac{8x+16}{(x-3)^2}\right)((x-3)^4) &= 12(x-3)^2 + \frac{2x+1}{x-3}(4(x-3)^3) - \frac{8x+16}{(x-3)^2}(x-3)^4 \\ &= 12(x-3)^2 + 4(2x+1)(x-3)^2 - (8x+16)(x-3)^2 \\ &= 8x(x-3)^2 - 8x(x-3)^2 + 16(x-3)^2 - 16(x-3)^2 \\ &= 0 \end{aligned}$$

So, we see that  $y = (x-3)^4$  is a solution to  $(D^2 + \frac{2x+1}{x-3}D - \frac{8x+16}{(x-3)^2})(y) = 0$ .

Example 10: Let us consider the operator

$$D^2 + \frac{4x+10}{x-2}D + \frac{12x+6}{(x-2)^2}.$$

Since  $x = 2$  is the singular point of our equation,  $p(x_0) = p(2) = 18$  and  $q(x_0) = q(2) = 30$ . So, our original indicial equation is

$$r(r-1) + 18r + 30 = 0,$$

which has roots  $r_1 = -2$  and  $r_2 = -15$ . Neither of these roots satisfy  $r(r-1) + 10r + 6 = 0$ , which would require it to take the form in Case i. Thus, we try our modified indicial equation in Case ii. Given our differential operator, our modified indicial equation becomes

$$r(r-1) + 10r + 14 = 0.$$

Since  $r_1 = -2$  is a root to this equation, we meet the criteria for Case ii. This implies  $D^2 + \frac{4x+10}{x-2}D + \frac{12x+6}{(x-2)^2}$  factors into  $(D + \frac{3}{x-2})(D + \frac{4x+7}{x-2})$  and  $D^2 + \frac{4x+10}{x-2}D + \frac{12x+6}{(x-2)^2}(y) = 0$  has a solution of the form

$$y = (x-2)^{-(4(2)+10-2-1)}e^{-4x} = (x-2)^{-15}e^{-4x}.$$

We now verify Case ii by expanding the factored form and testing our solution.

$$\begin{aligned} \left(D + \frac{3}{x-2}\right)\left(D + \frac{4x+7}{x-2}\right) &= D^2 - \frac{15}{(x-2)^2} + \frac{3}{x-2}D + \frac{4x+7}{x-2}D + \frac{12x+21}{(x-2)^2} \\ &= D^2 + \frac{4x+10}{x-2}D + \frac{12x+6}{(x-2)^2}. \end{aligned}$$

Then,

$$\begin{aligned}
\left(D + \frac{3}{x-2}\right)\left(D + \frac{4x+7}{x-2}\right)(y) &= \left(D + \frac{3}{x-2}\right)\left(D + \frac{4x+7}{x-2}\right)((x-2)^{-15}e^{4x}) \\
&= \frac{8(2x^2+7x+8)e^{-4x}}{(x-2)^{17}} - \frac{4x+10}{x-2} \cdot \frac{(4x+7)e^{-4x}}{(x-2)^{16}} + \frac{12x+6}{(x-2)^2} \cdot \frac{e^{-4x}}{(x-2)^{15}} \\
&= [8(2x^2+7x+8) - (4x+10)(4x+7) + 12x+6] \left(\frac{e^{-4x}}{(x-2)^{17}}\right) \\
&= (16x^2+56x+64 - 16x^2 - 68x - 70 + 12x+6) \left(\frac{e^{-4x}}{(x-2)^{17}}\right) \\
&= 0
\end{aligned}$$

Thus,  $y = (x-2)^{-15}e^{-4x}$  is a solution to  $\left(D^2 + \frac{4x+10}{x-2}D + \frac{12x+6}{(x-2)^2}\right)(y) = 0$ .

Proof: Let  $D^2 + \frac{c_1x+c_2}{x-x_0}D + \frac{c_3x+c_4}{(x-x_0)^2}$ , with  $c_1, c_2, c_3, c_4 \in \mathbb{C}$ , and suppose the indicial equation has a solution,  $r_1$ . For case i, assume  $r_1$  is also a root of  $r(r-1) + c_2r + c_4 = 0$ . Thus,

$$\begin{aligned}
r_1(r_1-1) + (c_1x_0 + c_2)r_1 + (c_3x_0 + c_4) - (r_1(r_1-1) + c_2r_1 + c_4) &= 0 \\
c_1x_0r_1 + c_3x_0 &= 0 \\
c_1x_0r_1 &= -c_3x_0 \\
c_3 &= -c_1r_1.
\end{aligned}$$

Solving (19) for  $c_4$ , we see  $-r_1(r_1 + c_2 - 1) = c_4$ . Now, we can substitute these two values into our operator for  $c_3$  and  $c_4$  and obtain  $D^2 + \frac{c_1x+c_2}{x-x_0}D + \frac{-c_1r_1x-r_1(r_1+c_2-1)}{(x-x_0)^2}$ . From here we separate our coefficients into two fractions and perform the following factoring to achieve our factorization:

$$\begin{aligned}
D^2 + \left(\frac{-r_1}{x-x_0} + \frac{c_1x+c_2+r_1}{x-x_0}\right)D + \frac{-r_1(c_1x+c_2+r_1)}{(x-x_0)^2} + \frac{r_1}{(x-x_0)^2} \\
D^2 + \frac{r_1}{(x-x_0)^2} - \frac{r_1}{x-x_0}D + \frac{c_1x+c_2}{x-x_0}D - \frac{r_1(c_1+c_2+r_1)}{(x-x_0)^2}.
\end{aligned}$$

Since  $\frac{d}{dx}\left(\frac{-r_1}{x-x_0}\right)(y) = \frac{r_1}{(x-x_0)^2}y - \frac{r_1}{x-x_0}\left(\frac{dy}{dx}\right)$ , we can factor our above operator into

$$\begin{aligned}
D\left(D - \frac{r_1}{x-x_0}\right) + \frac{c_1x+c_2+r_1}{x-x_0}\left(D - \frac{r_1}{x-x_0}\right) \\
\left(D + \frac{c_1x+c_2+r_1}{x-x_0}\right)\left(D - \frac{r_1}{x-x_0}\right).
\end{aligned}$$

Thus, we have our factorization. Furthermore, our first factor always provides a solution to our second order operator. Hence, we will consider the equation  $\left(D - \frac{r_1}{x-x_0}\right)(y) = 0$ . Solving this first



order equation, we obtain:

$$\begin{aligned}\frac{dy}{dx} - \frac{r_1}{x - x_0}y &= 0 \\ \frac{dy}{dx} &= \frac{r_1}{x - x_0}y \\ \frac{dy}{y} &= \frac{r_1}{x - x_0}dx \\ \ln(y) &= r_1 \ln(x - x_0) + c_5 \\ \ln(y) &= \ln((x - x_0)_1^{r_1}) + c_5 \\ y &= c_6(x - x_0)_1^{r_1}.\end{aligned}$$

Then,  $y = c_6(x - x_0)_1^{r_1}$  is a solution to  $(D^2 + \frac{c_1x+c_2}{x-x_0}D + \frac{c_3x+c_4}{(x-x_0)^2})y = 0$ .

Now, we must consider our second case. Let  $r_1$  also be a solution to  $r(r - 1) + c_2r + c_1x_0 + c_4 = 0$ . Once again subtracting our modified indicial equation from the original indicial equation (18), we see

$$\begin{aligned}r_1(r_1 - 1) + (c_1x_0 + c_2)r_1 + (c_3x_0 + c_4) - (r_1(r_1 - 1) + c_2r_1 + c_1x_0 + c_4) &= 0 \\ c_1x_0r_1 + c_3x_0 - c_1x_0 &= 0 \\ c_3x_0 &= (c_1 - c_1r_1)x_0 \\ c_3 &= c_1 - c_1r_1.\end{aligned}$$

Also solving equation (20) for  $c_4$ , yields  $c_4 = -(r_1^2 + (c_2 - 1)r_1 + c_1x_0)$ . In the next two equations, we will substitute our values for  $c_3$  and  $c_4$  into our differential operator:

$$\begin{aligned}D^2 + \frac{c_1x + c_2}{x - x_0}D + \frac{(c_1 - c_1r_1)x - r_1^2 - (c_2 - 1)r_1 - c_1x_0}{(x - x_0)^2} \\ D^2 + \frac{c_1x + c_2}{x - x_0}D + \frac{-c_1x_0 + c_1x - c_1r_1x - r_1^2 - c_2r_1 + r_1}{(x - x_0)^2}\end{aligned}$$

We now add  $c_2 - c_2 + r_1 - r_1 + 1 - 1 = 0$  to our second coefficient in order to establish a form that is separable into a function multiplied by a constant and its derivative:

$$D^2 + \frac{c_1x + c_2}{x - x_0}D + \frac{-c_1x_0 - c_2 - r_1 + 1 + c_1x + c_2 + r_1 - 1 - c_1r_1x - r_1^2 - c_2r_1 + r_1}{(x - x_0)^2}$$

We can now manipulate this form to obtain a factorization just as we did in the first case. Thus,

$$\begin{aligned}D^2 + \left(\frac{c_1x + c_2 + r_1 - 1}{x - x_0} + \frac{1 - r_1}{x - x_0}\right)D - \frac{c_1x_0 + c_2 + r_1 - 1}{(x - x_0)^2} + \frac{c_1x + c_2 + r_1 - 1 - c_1r_1x - c_2r_1 - r_1^2 + r_1}{(x - x_0)^2} \\ D^2 - \frac{c_1x_0 + c_2 + r_1 - 1}{(x - x_0)^2} + \frac{c_1x + c_2 + r_1 - 1}{x - x_0}D + \frac{1 - r_1}{x - x_0}D + \frac{c_1x + c_2 + r_1 - 1 - r_1c_1x - c_2r_1 - r_1^2 + r_1}{(x - x_0)^2}.\end{aligned}$$

Now, we must add  $c_1x - c_1x = 0$  to the second term in the above operator in order to continue on with our derivation.

$$D^2 - \frac{-c_1x + c_1x_0 + c_1x + c_2 + r_1 - 1}{(x - x_0)^2} + \frac{c_1x + c_2 + r_1 - 1}{x - x_0}D + \frac{1 - r_1}{x - x_0}D + \frac{c_1x + c_2 + r_1 - 1 - r_1c_1x - c_2r_1 - r_1^2 + r_1}{(x - x_0)^2}$$

$$D^2 + \frac{c_1(x - x_0) - (c_1x + c_2 + r_1 - 1)}{(x - x_0)^2} + \frac{c_1x + c_2 + r_1 - 1}{x - x_0}D + \frac{1 - r_1}{x - x_0}D + \frac{c_1x + c_2 + r_1 - 1 - r_1c_1x - c_2r_1 - r_1^2 + r_1}{(x - x_0)^2}.$$

Once again, we notice  $\frac{d}{dx} \left( \frac{c_1x + c_2 + r_1 - 1}{x - x_0} (y) \right) = \frac{c_1(x - x_0) - (c_1x + c_2 + r_1 - 1)}{(x - x_0)^2} y + \frac{c_1x + c_2 + r_1 - 1}{x - x_0} \left( \frac{dy}{dx} \right)$ , thus we can factor our above operator into the following:

$$D \left( D + \frac{c_1x + c_2 + r_1 - 1}{x - x_0} \right) + \frac{1 - r_1}{x - x_0} \left( D + \frac{c_1x + c_2 + r_1 - 1}{x - x_0} \right) \left( D + \frac{1 - r_1}{x - x_0} \right) \left( D + \frac{c_1x + c_2 + r_1 - 1}{x - x_0} \right).$$

As in the proof of case i., we use our first factor to find a solution to our second order operator. Thus, we will consider the equation  $\left( D + \frac{c_1x + c_2 + r_1 - 1}{x - x_0} \right) (y) = 0$ :

$$\begin{aligned} \frac{dy}{dx} + \frac{c_1x + c_2 + r - 1}{x - x_0} y &= 0 \\ \frac{dy}{dx} &= -\frac{c_1x + c_2 + r - 1}{x - x_0} y \\ \frac{dy}{y} &= -\frac{c_1x + c_2 + r - 1}{x - x_0} dx \\ \ln(y) &= -(c_1x_0 + c_2 + r - 1) \ln(x - x_0) - c_1x + c_5 \\ \ln(y) &= \ln((x - x_0)^{-(c_1x_0 + c_2 + r - 1)}) - c_1x + c_5 \\ y &= c_6(x - x_0)^{-(c_1x_0 + c_2 + r - 1)} e^{-c_1x} \end{aligned}$$

Therefore,  $y = (x - x_0)^{-(c_1x_0 + c_2 + r - 1)} e^{-c_1x}$  is a solution to our second order operator and we are done.  $\square$

## 6. OPERATORS WITH TWO REGULAR SINGULARITIES

In this section, we will consider operators of the form

$$D^2 + \frac{c_1}{(x - x_0)(x - x_1)} D + \frac{c_2x + c_3}{(x - x_0)^2(x - x_1)^2} \quad (21)$$

with  $c_1, c_2, c_3, x_0, x_1 \in \mathbb{C}$ . Unlike the operators in our first two cases, the operator above has two singularities. Hence, these operators cannot be solved by the original indicial equation from Frobenius. Therefore, we have modified equation (12) to the following form:

$$r(r - (x_0 + x_1)) + c_1r + q(x_0 + x_1) = 0. \quad (22)$$

The modification of the indicial equation (12) has a strong connection to our singular points  $x = x_0$  and  $x = x_1$ . We believe this relationship occurs because  $\frac{d}{dx}((x - x_0)(x - x_1)) = 2x - (x_0 + x_1)$ ;

however, their connection has not been proven. Theorem 10 contains two criteria for reducibility, both of which are necessary. Example 11 will show the necessity that  $r_1 = \frac{c_2}{2}$ . Theorem 9 required  $r_1$  to satisfy the original indicial equations and a modified indicial equation. In the following theorem, we will be able to relax this requirement to satisfying one modified indicial equation. Along with the requirement  $r_1 = \frac{c_2}{2}$ , this indicial equation is enough to determine the rest of the coefficients in our operator. Thus, we have the following theorem.

**Theorem 10:**

Let our operator be of the form  $D^2 + \frac{c_1}{(x-x_0)(x-x_1)}D + \frac{c_2x+c_3}{(x-x_0)^2(x-x_1)^2}$  and  $q(x_0 + x_1) = c_2(x_0 + x_1) + c_3$ , where  $c_1, c_2, c_3, x_0, x_1 \in \mathbb{C}$ . If our operator has a solution,  $r_1$ , to the modified indicial equation,

$$r(r - (x_0 + x_1)) + c_1r + q(x_0 + x_1) = 0,$$

such that  $r_1 = \frac{c_2}{2}$ , then our operator can be factored into  $(D + \frac{c_1+r_1}{(x-x_0)(x-x_1)})(D - \frac{r_1}{(x-x_0)(x-x_1)})$ . Furthermore,  $y = (x - x_1)^{\frac{r_1}{x_0-x_1}}(x - x_0)^{\frac{-r_1}{x_0-x_1}}$  is a solution to  $(D + \frac{c_1+r_1}{(x-x_0)(x-x_1)})(D - \frac{r_1}{(x-x_0)(x-x_1)})y = 0$ .

**Example 11:** Consider the operator  $D^2 + \frac{8}{x(x-1)}D + \frac{-4x+16}{x^2(x-1)^2}$

Since we have two singularities our indicial equation takes on the modified form in (22). Thus, we have

$$\begin{aligned} r(r-1) + 8r + 12 &= 0 \\ r^2 + 7r + 12 &= 0 \end{aligned}$$

Thus, our solutions are  $r = -3$  and  $r = -4$ . Hence our first factors would be  $(D + \frac{4}{x(x-1)})$  and  $(D + \frac{3}{x(x-1)})$ . From equation (6), our two factorizations would have to be

$$\begin{aligned} &\left(D + \frac{4}{x(x-1)}\right)\left(D + \frac{4}{x(x-1)}\right) \\ &\left(D + \frac{5}{x(x-1)}\right)\left(D + \frac{3}{x(x-1)}\right) \end{aligned}$$

which expand to

$$\begin{aligned} &D^2 + \frac{8}{x(x-1)}D + \frac{-8x+20}{x^2(x-1)^2} \\ &D^2 + \frac{8}{x(x-1)}D + \frac{-6x+18}{x^2(x-1)^2}. \end{aligned}$$

Neither of these forms are the first form we were considering. Thus our condition  $r_1 = \frac{c_2}{2}$  is necessary.

**Proof:** Let  $D^2 + \frac{c_1}{(x-x_0)(x-x_1)}D + \frac{c_2x+c_3}{(x-x_0)^2(x-x_1)^2}$ , with  $c_1, c_2, c_3, x_0, x_1 \in \mathbb{C}$ , have a solution,  $r_1$ , to

$$r(r - (x_0 + x_1)) + c_1r + q(x_0 + x_1) = 0,$$

where  $q(x_0 + x_1) = c_2(x_0 + x_1) + c_3$ . Assume  $2r_1 = c_2$ . Then,  $q(x_0 + x_1) = 2(x_0 + x_1)r_1 + c_3$ . From (22),  $q(x_0 + x_1) = -r_1^2 - (c_1 - (x_0 + x_1))r_1$  so that  $c_3 + 2(x_0 + x_1)r_1 = -r_1^2 - (c_1 - (x_0 + x_1))r_1 \Rightarrow c_3 = -r_1^2 - (c_1 + (x_0 + x_1))r_1$ . Therefore,

$$\begin{aligned} D^2 + \frac{c_1}{(x-x_0)(x-x_1)}D + \frac{c_2x+c_3}{(x-x_0)^2(x-x_1)^2} &= D^2 + \frac{c_1}{(x-x_0)(x-x_1)}D + \frac{2r_1x - r_1^2 - (c_1 + (x_0 + x_1))r_1}{(x-x_0)^2(x-x_1)^2} \\ &= D^2 + \frac{c_1}{(x-x_0)(x-x_1)}D + \frac{2r_1x - (x_0 + x_1)r_1 - c_1r_1 - r_1^2}{(x-x_0)^2(x-x_1)^2} \\ &= D^2 + \frac{c_1}{(x-x_0)(x-x_1)}D + \frac{r_1(2x - (x_0 + x_1))}{(x-x_0)^2(x-x_1)^2} - \frac{r_1(c_1 + r_1)}{(x-x_0)^2(x-x_1)^2} \\ &= D^2 + \frac{r_1(2x - (x_0 + x_1))}{(x-x_0)^2(x-x_1)^2} - \frac{r_1}{(x-x_0)(x-x_1)}D + \frac{c_1 + r_1}{(x-x_0)(x-x_1)}D - \frac{r_1(c_1 + r_1)}{(x-x_0)^2(x-x_1)^2}. \end{aligned}$$

Noticing  $\frac{d}{dx} \left( \frac{-r_1}{(x-x_0)(x-x_1)}(y) \right) = \frac{r_1(2x-(x_0+x_1))}{(x-x_0)^2(x-x_1)^2}y - \frac{r_1}{(x-x_0)(x-x_1)} \left( \frac{dy}{dx} \right)$ , we can then factor the above equation into the form below:

$$\begin{aligned} &= D \left( D - \frac{r_1}{(x-x_0)(x-x_1)} \right) + \frac{c_1 + r_1}{(x-x_0)(x-x_1)} \left( D - \frac{r_1}{(x-x_0)(x-x_1)} \right) \\ &= \left( D + \frac{c_1 + r_1}{(x-x_0)(x-x_1)} \right) \left( D - \frac{r_1}{(x-x_0)(x-x_1)} \right). \end{aligned}$$

From here we can once again manipulate our first factor and apply it to a function  $y(x)$  to find a solution to this equation.

$$\begin{aligned} \left( D - \frac{r_1}{(x-x_0)(x-x_1)} \right)(y(x)) &= 0 \\ \frac{dy}{dx} &= \frac{r_1}{(x-x_0)(x-x_1)}y(x) \\ \frac{dy}{y(x)} &= \frac{r_1}{(x-x_0)(x-x_1)}dx \\ \ln(y(x)) &= \frac{r_1}{x_0 - x_1} \ln\left(\frac{x-x_0}{x-x_1}\right) + c_6 \\ y(x) &= c_7(x-x_0)^{\frac{r_1}{x_0-x_1}}(x-x_1)^{\frac{-r_1}{x_0-x_1}} \end{aligned}$$

Thus, the proof is complete.  $\square$

Example 12: Consider the operator  $D^2 + \frac{5}{(x-1)(x-3)}D - \frac{6x-18}{(x-1)^2(x-3)^2}$ . Then, our indicial equation would be

$$r(r-4) + 5r - 6 = 0,$$

which has roots  $r_1 = -3$  and  $r_2 = 2$ . Testing for our requirement on our root, we see that  $r_1 = 3 = \frac{6}{2}$ , so our first factor is of the form  $D + \frac{3}{(x-1)(x-3)}$ . From (6), we see that our factorization is  $\left( D + \frac{2}{(x-1)(x-3)} \right) \left( D + \frac{3}{(x-1)(x-3)} \right)$ . Furthermore, from Theorem 10,  $y = (x-1)^{\frac{3}{2}}(x-3)^{\frac{-3}{2}}$  is a solution to

$$(D^2 + \frac{5}{(x-1)(x-3)}D - \frac{6x-18}{(x-1)^2(x-3)^2})y = 0.$$

## 7. CONCLUSION:

All of the results in this paper are for very specific differential operators; however, these results have developed methods for solving homogeneous equations whose operators are factorable. The following are avenues for development of this field:

1. The results within this paper only consider operators of order two. Considering operators of order three changes the relationships between coefficients in our factored form and our expanded form. To our knowledge, no general form for a factored  $n^{\text{th}}$ -order operator exists.
2. Also, further investigation concerning second order operators with polynomial coefficients that factor into two first order operators with rational function coefficients would extend the results of Section 4. This case is very specific, but could provide understanding of cancelation within the factorization of differential operators.
3. Within Section 5, tests were given for second order operators with one singularity and a numerator containing a linear function of  $x$ . There are many forms left to consider. Increasing the degree of the numerator or even the number of singularities in these operators would allow further development of this field.
4. In Section 6, we explored differential operators with two singularities. The form of our operator was very specific; however, the indicial equation within this section could provide the form of an indicial equation for all second order differential operators with two singularities. Further development of this equation could provide interesting results.

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