THE BURNSIDE GROUP $B(3, 2)$ AS A TWO-RELATOR QUOTIENT OF $C_3 * C_3$

MATTHEW FARRELLY

Abstract. We prove that the free Burnside Group $B(3, 2)$ has order 27 and is isomorphic to $\langle a, b \mid a^3, b^3(ab)^3, (b^{-1}a)^3\rangle$. The technique of our proof is also used to show that $\langle a, b|a^3, b^3, a^2(ba)^nb^2\rangle$ is the semidirect product $C_{n^2+n+1} \rtimes C_3$.

1. Introduction

In 1902, William Burnside raised the question of whether a finitely generated group must be finite if each of its elements has order dividing a given natural number n. We call n the exponent of the group [Burn]. Along with this question Burnside provided cases in which it had an affirmative answer, namely for any group of exponent two or three and for all groups of exponent four that can be generated by two elements. The largest possible group of exponent n generated by m elements is denoted by $B(m, n) = \mathbb{F}_m/\mathbb{F}_m^n$. The groups $B(m, n)$ are called the free Burnside groups. Answering Burnside's question is equivalent to determining whether $B(m, n)$ is finite for any given m and n. This question is called the General Burnside Problem.

No further progress was made on this problem until 1940, when I.N. Sanov showed that all finitely generated groups of exponent four must be finite [Sanov]. Seventeen years later, Marshall Hall [Hall] demonstrated that this was also true for finitely generated groups of exponent six. Whether every two-generated group of exponent five is finite is still an open question. The best result in the exponent five direction is found in [HS5]. Hall and Sims used commutator relations to find a normal subgroup K_1 of index 5^{10} and order 5^{24}. They concluded with the result that $B(5, 2)$ is finite if K_1 is abelian.

In 1964 Golod discovered the first example of a finitely generated infinite group. This group has the property that every element in the group has finite order. Golod's finding suggested the existence of infinite groups of large exponent.

In 1968, Novikov and Adian published a ground breaking series of papers [AdNov] in which they proved that there are infinite periodic groups with odd exponent $n \geq 4381$. Their proof followed from a complicated inductive method. The method was to present the free Burnside groups $B(m, n) = \mathbb{F}_m/\mathbb{F}_m^n$ by relations of the form $A^n = 1$ with specially chosen elements A in $\mathbb{F}_m$. In 1975, Adian [Ad] improved the method and showed that there are infinite periodic groups of odd exponent $n \geq 665$. Adian and Novikov were able to prove much more than this. For example, they determined that the word and conjugacy problems are solvable in $B(m, n)$. They

Date: 19 June 2007.
Key words and phrases. Burnside Group, Groups, Presentations.

showed any finite or abelian subgroup of $B(m,n)$ is cyclic. They also proved that the centralizer of any non-identity element in $B(m,n)$ is a cyclic group of order n.

In this paper, we begin to investigate the structure of free Burnside groups from the ground up. One general method for finding the structure of a group is to decompose the group into products of well known groups such as cyclic groups. In the case of $B(3,2)$, we prove that such a decomposition is possible using semidirect products of cyclic groups. In his original paper, William Burnside proved that the largest $|B(3,2)|$ could be is 27. However, to actually prove that there is a group $B(3,2)$, one must prove that there exists a nonabelian group with order 27 and exponent three. It is easy to find an abelian group of order 27 and exponent three, namely $C_3 \times C_3 \times C_3$. When searching for a nonabelian group with the desired properties, one is motivated to try semidirect products of cyclic groups. In this paper we prove that $B(3,2)$ is the group $(C_3 \times C_3) \rtimes_{\alpha} C_3$ and has order 27.

Now consider infinite triangle groups of the form $\langle a, b | a^p, b^q, (ab)^r \rangle$ with $p, q, r \in \mathbb{Z}^+ \backslash \{1\}$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$ (cf. [HMT]). We will show that only one extra relator is needed to make the infinite triangle group $\langle a, b | a^3, b^3, (ab)^3 \rangle$ finite. More specifically, we add the single relation $(b^{-1}a)^3$ to obtain a group of order 27 and exponent three. Therefore this group is $B(3,2)$. This might motivate one to ask if $B(4,2)$ or $B(5,2)$ is a one-relator quotient of a triangle group[1].

I'd like to thank professors Jon Bannon and Kris Farwell for their guidance and encouragement.

1.1. Group Presentations.

For a more complete treatment of presentations and other group theory topics see [OLS].

We call the set $\mathcal{A} = \{a, b\}$ an alphabet. We can create finite concatenations of letters in $\mathcal{A}$ such as $ababbaaab$. This concatenation can be written equivalently as $abab^2a^3b$. Such concatenations of letters are called words in $\mathcal{A}$. We denote the trivial word (or empty word) with the symbol 1. We can also form words in $\mathcal{A} \cup \mathcal{A}^{-1} \cup \{1\}$. For example $aba^{-1}bb^{-1}bab$ is a word in $\mathcal{A} \cup \mathcal{A}^{-1} \cup \{1\}$. We say a word w in $\mathcal{A} \cup \mathcal{A}^{-1} \cup \{1\}$ is a reduced word if it meets one of the following conditions.

1) The word w in$\mathcal{A} \cup \mathcal{A}^{-1} \cup \{1\}$ is 1.
2) If the word w in $\mathcal{A} \cup \mathcal{A}^{-1} \cup \{1\}$is not 1, then no letter in w occurs adjacent to its inverse, and 1 does not occur in the word.

We call a letter adjacent to its inverse a cancelling pair. For example, $aba^{-1}bb^{-1}ba1b$ is not a reduced word for at least two different reasons. It contains both the cancelling pair bb^{-1} and the symbol 1. By successively removing cancelling pairs, and the symbol 1 from a word w we eventually obtain a unique reduced word w_0. The word w_0 is called the reduced word associated to w and the process is called reduction. Let $\mathbb{F}_2$ denote the set of all reduced words in $\mathcal{A} \cup \mathcal{A}^{-1} \cup \{1\}$. We make $\mathbb{F}_2$ into a group by equipping it with the multiplication given by concatenation followed by reduction. We demonstrate the multiplication and reduction in the example below.

Example 1. $(aba^{-1}ba) \cdot (a^{-1}b^{-1}aba) = ab^2a$ *in* $\mathbb{F}_2$

[1]Computer experiment suggests that $B(4,2)$ is not a one-relator quotient of the obvious triangle group. In fact, it appears that in order to get a candidate for $B(4,2)$ using short relators, we need at least 9 relators.

The set $\mathbb{F}_2$ of all reduced words in $\mathcal{A} \cup \mathcal{A}^{-1} \cup \{1\}$ with the multiplication given above is called the free non abelian group on two generators (or the free non abelian group on $\mathcal{A}$). We will not prove it here, but $\mathbb{F}_2$ is the free group on the set $\{a, b\}$. We mean this in the sense that given any group G and any (set) map $f{:}\{a, b\} \to G$, there exists a unique extension of f to a homomorphism $\widehat{f} : \mathbb{F}_2 \to G$. If we began with $\mathcal{A} = \{a, b, c\}$ we would obtain the free group $\mathbb{F}_3$ on three generators. Likewise if we start with $\mathcal{A} = \{a_1, a_2, ..., a_n\}$ we would obtain the free group $\mathbb{F}_n$ on n generators. In fact, if we begin with any set $\mathcal{A}$, we can obtain what is called the free group $\mathbb{F}_{\mathcal{A}}$ on the set $\mathcal{A}$.

Definition 1. *The group G is finitely generated if $\exists$ a finite subset $S \subseteq G$ such that every given element $g \in G$ can be written as a finite product of $s_1, s_2, ..., s_k$ with $s_i \in S \cup S^{-1} \cup \{1\}$.*

Definition 2. *Let G be a group and $S \subseteq G$. The **normal closure** S^G of S in G is the smallest normal subgroup of G that contains S.*

Definition 3. *Let $\mathcal{R} \subseteq \mathbb{F}_{\mathcal{A}}$ be a given subset. We define transformations that fix each coset of $\mathcal{R}^{\mathbb{F}_{\mathcal{A}}}$.*
1) $abb^{-1}a \to a^2$ (Cancellation)
2) $a^2 \to abb^{-1}a$ (Extension)
3) $w_1 r^{\pm 1} w_2 \to w_1 w_2$ with $r \in \mathcal{R}$ and $w_1, w_2 \in \mathbb{F}_{\mathcal{A}}$
4) $w_1 w_2 \to w_1 r^{\pm 1} w_2$ with $r \in \mathcal{R}$ and $w_1, w_2 \in \mathbb{F}_{\mathcal{A}}$
*Two words in $\mathbb{F}_{\mathcal{A}}$ are **$\mathcal{R}$-equivalent** if we can transform one to the other using finitely many applications of steps $1-4$.*

Definition 4. *An equation of the form $w = v$, where w and v are reduced words, is called a relation.*

Definition 5. *We say that $w = v$ is $\mathcal{R}$-deducible if w and v are $\mathcal{R}$-equivalent.*

Definition 6. *Let G be a group and $\{g_\alpha\}$ with $\alpha \in \mathcal{A}$ be a generating set. Suppose we replace each α occurring in a word by g_α and 1 by the identity in G. If the value of the word w equals 1 in G whenever the values of all $R \in \mathcal{R}$ are 1 in G, then we say that $w = 1$ with $w \in \mathbb{F}_{\mathcal{A}}$ **is a consequence** of $\{R = 1 \mid R \in \mathcal{R}\}$.*

Example 2. *Suppose G is a group generated by $\{g_1, g_2\}$ and the identity is denoted by 1. Furthermore suppose $g_1^3 = 1$ in G. It follows that $g_1^2 = g_1^{-1}$ in G. Thus we have that $g_1^2 = g_1^{-1}$ **is a consequence** of $\{g_1^3 = 1,\ g_2^3 = 1\}$.*

Definition 7. *A set of relations $\{R = 1 \mid R \in \mathcal{R}\}$ is said to **define a group** G **generated by** $\{g_a\}$ **with** $a \in \mathcal{A}$, if any relation in G among the generators g_a is a consequence of $\{R = 1 \mid R \in \mathcal{R}\}$. In this case any $R = 1$ with $R \in \mathcal{R}$ is then called a **defining relation** for G. We call each $R \in \mathcal{R}$ a **relator** of G.*

Suppose $\Phi : \mathbb{F}_{\mathcal{A}} \to G$. We claim this map is a surjective homomorphism: Suppose that $\{g_\alpha\}$ is a set of generators from G. Let $\alpha_1^{\varepsilon_1} \alpha_2^{\varepsilon_2} ... \alpha_k^{\varepsilon_k}$ with all $\varepsilon_k \in \mathbb{Z}$ be an arbitrary reduced word in $\mathbb{F}_{\mathcal{A}}$. Assume $\alpha_i^0 = 1$ and $\Phi(1) = 1$ then $\Phi(\alpha_1^{\varepsilon_1} \alpha_2^{\varepsilon_2} ... \alpha_k^{\varepsilon_k}) = g_{\alpha_1}^{\varepsilon_1} g_{\alpha_2}^{\varepsilon_2} ... g_{\alpha_k}^{\varepsilon_k}$.

Definition 8. *If $\{R = 1 \mid R \in \mathcal{R}\}$ defines the group G generated by $\{g_\alpha\}$, then $\{R = 1 \mid R \in \mathcal{R}\}$ together with the surjective homomorphism Φ is called a **presentation** for G. In this case we write $G = \langle \mathcal{A} \mid R = 1, R \in \mathcal{R} \rangle$.*

1.2. **Semidirect Product.** There are a few familiar ways to put two groups together to make a larger group. One such way is the direct product. Let H and K be groups. The external direct product of H and K is obtained by equipping the Cartesian product $H \times K$ with multiplication. The multiplication is given by $(h_1, k_1)(h_2, k_2) = (h_1 h_2, k_1 k_2)$ for every $h_1, h_2 \in H$ and $k_1, k_2 \in K$. The internal direct product provides a way to decompose a larger group into two smaller groups. Suppose we have a group G where H and K are normal subgroups of G. If $G = HK$ and $H \cap K = \{1\}$, then every $g \in G$ can be written uniquely as $g = hk$ for some $h \in H$ and $k \in K$. In this case the map that sends $g = hk$ to (h, k) gives and isomorphism from G to the external direct product of H and K. For this reason we say that G is the internal direct product of H and K. We may ask what happens when only one of the subgroups G is normal. This gives rise to the concept of semidirect product.

Definition 9. *Let G be a group with $H \trianglelefteq G$ and $K \leq G$. We say G is an (internal) semidirect product of H and K if we know the following:*
1) $H \cap K = \{1\}$
2) $HK = G$

We now consider some properties of semidirect products. Let G be a group with subgroups H and K. The properties $G = HK$ and $H \cap K = \{1\}$ tell us that any given $g \in G$ can be written as $g = hk$ for unique $h \in H$ and $k \in K$. Suppose we have $g_1 g_2 = h_1 k_1 h_2 k_2$ where $g_1, g_2 \in G$, $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Since $H \trianglelefteq G$ we know that $g^{-1} h_2 g \in H$ for every $g \in G$. Therefore $k_1 h_2 k_1^{-1} = h_2'$ for some $h_2' \in H$. This implies $k_1^{-1} h_2' k_1 = h_2$. A simple substitution shows that $g_1 g_2 = h_1 k_1 h_2 k_2 = h_1 k_1 k_1^{-1} h_2' k_1 k_2 = h_1 h_2' k_1 k_2$. Thus we have a way of commuting elements from H with elements from K. Writing $k_1 h_2 k_1^{-1} = \alpha(k_1)(h_2)$, we notice that α defines a map from K into $Aut(H)$. Since $\alpha(k_1 k_2)(h) = k_1 k_2 h k_2^{-1} k_1^{-1} = \alpha(k_1)(\alpha(k_2)(h) = (\alpha(k_1) \circ \alpha(k_2))(h)$ for every $h \in H$ we have that $\alpha(k_1 k_2) = \alpha(k_1) \circ \alpha(k_2)$ and α is a homomorphism. This suggests a way to construct an external semidirect product of H and K using a homomorphism α of K into $Aut(H)$. To this end, consider the Cartesian product $H \times K$ and a given homomorphism $\alpha : K \to Aut(H)$. We use the homomorphism α to define the product on $H \times K$ as $(h_1, k_1)(h_2, k_2) = (h_1 \alpha(k_1)(h_2), k_1 k_2)$ for every $h_1, h_2 \in H$ and $k_1, k_2 \in K$. The homomorphism property of α is needed to ensure that the product is associative. We then say that H with this multiplication is a semidirect product of K and denote it as $H \underset{\alpha}{\rtimes} K$.[2] The isomorphism class of the semidirect product depends on the choice of α. For example D_6 and $C_3 \times C_2$ are semidirect products of C_3 and C_2, but the latter is abelian and the former is not. In fact, if $\alpha : K \to \{id\} \leq Aut(H)$ where $id : h \to h$, then $H \underset{\alpha}{\rtimes} K = H \times K$.

1.3. **Presentations for Direct and Semidirect Products of Finite Cyclic Groups.** If $H = \langle a \mid \mathcal{R}_1 \rangle$, $K = \langle b \mid \mathcal{R}_2 \rangle$ and $\alpha : K \to Aut(H)$ then every relation in $H \underset{\alpha}{\rtimes} K$ is a consequence of $\mathcal{R}_1$, $\mathcal{R}_2$, and $\alpha(b)(a) b = ba$. Therefore we have that $\langle a, b \mid \mathcal{R}_1, \mathcal{R}_2, \alpha(b)(a) b = ba \rangle$ is a natural presentation for $H \underset{\alpha}{\rtimes} K$. Combining this and

[2]Note that it is possible for $\alpha_1 \neq \alpha_2$ but $H \underset{\alpha_1}{\rtimes} K \cong H \underset{\alpha_2}{\rtimes} K$.

$\alpha(y)(x) = x^n$ we see that $\langle x, y \mid x^{n^2+n+1}, y^3, yx = x^n y\rangle$ is a natural presentation for $C_{n^2+n+1} \rtimes_\alpha C_3$.

This process can be iterated. For example suppose $H \times K = \langle a, b \mid \mathcal{R}_1, \mathcal{R}_2, ab = ba\rangle$. If $G_3 = \langle c \mid \mathcal{R}_3\rangle$ and $\alpha : G_3 \to Aut(H \times K)$ then (identifying (a, b) with ab) every relation in $(H \times K) \rtimes_\alpha G$ is a consequence of $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, $ab = ba$, $ca = \alpha(c)(a * 1)c$, and $cb = \alpha(c)(1 * b)c$. Hence $(H \times K) \rtimes_\alpha G_3 = \langle a,\ b,\ c \mid \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, ab = ba, ca = \alpha(c)(a * 1)c, cb = \alpha(c)(1 * b)c\rangle$.

In this paper, we will need the following case. Let $C_3 = \langle a \mid a^3 = 1\rangle$ and $C_3 = \langle b \mid b^3 = 1\rangle$ be presentations of the cyclic group C_3. By the above, $C_3 \times C_3 = \langle a, b \mid a^3 = 1, b^3 = 1, aba^{-1}b^{-1}\rangle$. From the definition of cyclic group, every relation of C_3 (respectively C_3) is a consequence of $a^3 = 1$ (respectively $b^3 = 1$). We consider the homomorphism $\langle c \mid c^3 = 1\rangle = C_3 \to Aut(C_3 \times C_3)$ defined by $\alpha(c^r)(a^n, b^m) = a^n b^{r+m}$ with $r, n, m \in \{0, 1, 2\}$. It follows by the above discussion (again identifying (a, b) with ab) that $\langle a,\ b,\ c \mid \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, ab = ba, ca = abc, cb = bc\rangle$ is a presentation for $(C_3 \times C_3) \rtimes_\alpha C_3$.

2. Main Results

2.1. **The Burnside Group $B(3, 2)$ as a Semidirect Product.** The exponent of a group G is the smallest positive integer n such that $g^n = 1$ for all $g \in G$. Burnside raised the question as to whether a group of finite exponent is finite. This became known as the General Burnside Problem. In 1902, Burnside proved that for groups G with two generators and exponent three, the largest $|G|$ could be is 27. The group is known as the Burnside group of exponent three on two generators and is denoted $B(3, 2)$. We demonstrate a more detailed and concrete proof of the finiteness of all two generated groups of exponent three. In the proof we determine which pairs of reduced words of length two, three and so on are equal in $B(3, 2)$, adding various mild assumptions (such as non-abelian, etc.) along the way. From this point on when we say word we mean reduced word.

Theorem 1. *$B(3, 2)$ has at most order 27.*

Proof. If $ab = ba$ then $G \cong C_3 \oplus C_3$ and has order nine.

Suppose $G = B(3, 2) = \mathbb{F}_2/\mathbb{F}_2^3 = \langle a, b \mid g^3\rangle$ and assume that G is not abelian thus $ab \neq ba$. Further assume that a, b, a^{-1}, b^{-1} are distinct non trivial elements in G.

We first start out by developing some facts:

Fact 1 : $(ab)^3 = ababab = 1$ therefore $aba = b^{-1}a^{-1}b^{-1}$

Fact 2 : $(ba)^3 = bababa = 1$ therefore $bab = a^{-1}b^{-1}a^{-1}$

Fact 3 : $(ab^{-1}) = ab^{-1}ab^{-1}ab^{-1} = 1$ therefore $ab^{-1}a = ba^{-1}b$

Fact 4 : $(a^{-1}b) = a^{-1}ba^{-1}ba^{-1}b = 1$ therefore $a^{-1}ba^{-1} = b^{-1}ab^{-1}$

By Definition 6 we know $a^{-1}a^{-1} = a^2a^2 = a$ and $a^2 = a^{-1}$.are consequences of $a^3 = 1$. By similar reasoning the same is true for b.

We claim that no length two words are equal. We prove this by equating words of length two and use Facts $1 - 4$ to arrive at a contradiction. For example assume

$ab = b^{-1}a$ then:

$$\begin{aligned} ab &= b^{-1}a \\ bab &= a \\ a^{-1}b^{-1}a^{-1} &= a \text{ (by Fact 2)} \\ a^{-1}b^{-1} &= a^2 = a^{-1} \\ b^{-1} &= 1 \end{aligned}$$

This is a contradiction, therefore $ab \neq b^{-1}a$. Similarly the reader can deduce a contradiction for each length two equality. Therefore no length two words are equal. Counting up the elements thus far the $|G|$ is at least 13. The next step is to deduce which length three words are equal. From Facts $1-4$ we know that there are four length three words that are equal. We claim that other than Facts $1-4$, no length three words are equal. To prove this we repeat the process we used for the length two words. For example assume $ab^{-1}a^{-1} = bab^{-1}$ then:

$$\begin{aligned} ab^{-1}a^{-1} &= bab^{-1} \\ \underline{b^{-1}ab^{-1}}a^{-1} &= ab^{-1} \\ \underline{a^{-1}ba^{-1}}a^{-1} &= ab^{-1} \text{ (by Fact 4)} \\ aba &= b^{-1} \\ b^{-1}a^{-1}b^{-1} &= b^{-1} \text{ (by Fact 1)} \\ b^{-1}a^{-1} &= 1 \\ a^{-1} &= b \end{aligned}$$

Again we have a contradiction and see that $ab^{-1}a^{-1} \neq bab^{-1}$. The reader can repeat this for the other length three word equalities. This results in a confirmation of our claim. We now know that $|G|$ is at least 25.

Now words of length four must be considered. We start by looking at words beginning with a. A simple substitution will be used to obtain the length four words starting with b, a^{-1}, and b^{-1}. We reduce certain length four words starting with a into shorter words with our facts. We utilize a substitution of Fact 1 to reduce $abab^{-1}$ to $b^{-1}a^{-1}b$ and $ab^{-1}a^{-1}b^{-1}$ to $a^{-1}ba$. We now use a substitution of Fact 3 to reduce $ab^{-1}ab$ to $ba^{-1}b^{-1}$ and $aba^{-1}b$ to $a^{-1}b^{-1}a$. We also know that $abab = (ab)^2 = (ab)^{-1} = b^{-1}a^{-1}$. Facts 1-4 can't be used to reduce $ab^{-1}a^{-1}b$ and $aba^{-1}b^{-1}$ to shorter words. We can however reduce the other length four words by the symmetries between generators and their inverses. For example we exchange a and b in $abab^{-1}$ to yield $baba^{-1}$. We then substitute (Fact 1) $bab = a^{-1}b^{-1}a^{-1}$ into $baba^{-1}$ which results in $a^{-1}b^{-1}a^{-1}a^{-1}$. We see that this equals $a^{-1}b^{-1}a$. In this manner the length four words that can't be reduced to shorter words by using the facts are

$$\begin{aligned}&1)ab^{-1}a^{-1}b\\&2)aba^{-1}b^{-1}\\&3)a^{-1}bab^{-1}\\&4)a^{-1}b^{-1}ab\\&5)ba^{-1}b^{-1}a\\&6)bab^{-1}a^{-1}\\&7)b^{-1}aba^{-1}\\&8)b^{-1}a^{-1}ba\end{aligned}$$

Notice that $(b^{-1}aba^{-1})^{-1} = ab^{-1}a^{-1}b$. Thus 1 and 7 are inverses. Similarly 5 and 3 are inverses, 4 and 8 are inverses, and 2 and 6 are inverses. Note that 1 and 5 can be written differently. $bab^{-1}a = bba^{-1}b = b^{-1}a^{-1}b$. Therefore we have (1) $ab^{-1}a^{-1}b = abab^{-1}a$. By Fact 1 $\underline{aba}b^{-1}a = \underline{b^{-1}a^{-1}b^{-1}}b^{-1}a = b^{-1}a^{-1}ba$. Thus $1 = 8$. Above we had $aba^{-1}b = a^{-1}b^{-1}a$, therefore by symmetry $ba^{-1}b^{-1} = ab^{-1}ab$. This means that (5) $ba^{-1}b^{-1}a = ab^{-1}aba$. By Fact 1 we have $ab^{-1}\underline{aba} = ab^{-1}\underline{b^{-1}a^{-1}b^{-1}} = aba^{-1}b^{-1}$. Thus we see that $5 = 2$. By uniqueness of inverses we know that $7 = 4$ and $3 = 6$. Now consider $aba^{-1}b$. By Fact 3, $a\underline{ba^{-1}b} = a\underline{ab^{-1}a} = a^{-1}b^{-1}a$. Now we have $aba^{-1}b^{-1} = a^{-1}b^{-1}ab$. Therefore $4 = 2$. By uniqueness of inverses we have that $6 = 8$. In summary, we have $1 = 8 = 6 = 3$ and $7 = 4 = 2 = 5$. We see that there are only two words of length four. This makes $|G|$ at least 27.

We approach length five words similarly to how we approached length four words. Consider the length five words starting with a. We can do the same substitutions as we did in the length four case. From this we can obtain the other length five words. We claim that it is impossible to write a non-reducible word of length five. To help the reader we underline the substitutions and note where they came from.

I)

$$\begin{aligned}ab\underline{ab^{-1}a} &= ab\underline{ba^{-1}b} \text{ (by Fact 3)}\\&= ab^{-1}a^{-1}b\end{aligned}$$

II)

$$\begin{aligned}ab^{-1}\underline{aba} &= ab^{-1}\underline{b^{-1}a^{-1}b^{-1}} \text{ (by Fact 1)}\\&= aba^{-1}b^{-1}\end{aligned}$$

III)

$$\begin{aligned}ab^{-1}a^{-1}ba &= ab\underline{ba^{-1}b}a = ab\underline{ab^{-1}a}a \text{ (by Fact 3)}\\&= \underline{aba}b^{-1}a^{-1} = \underline{b^{-1}a^{-1}b^{-1}}b^{-1}a^{-1} \text{ (by Fact 1)}\\&= b^{-1}a^{-1}ba^{-1}\end{aligned}$$

IV)

$$a\underline{b^{-1}a^{-1}b^{-1}}a = a\underline{aba}a \text{ (by Fact 1)} = a^{-1}ba^{-1}$$

V)

$$\begin{aligned}a\underline{ba^{-1}b}a &= a\underline{ab^{-1}a}a \text{ (by Fact 3)}\\&= a^{-1}b^{-1}a^{-1}\end{aligned}$$

VI)

$$\begin{aligned} aba^{-1}b^{-1}a &= \underline{aba}ab^{-1}a \\ &= \underline{b^{-1}a^{-1}b^{-1}}ab^{-1}a \text{ (by Fact 1)} \\ b^{-1}a^{-1}\underline{b^{-1}ab^{-1}}a &= b^{-1}a^{-1}\underline{a^{-1}ba^{-1}}a \text{ (by Fact 4)} \\ &= b^{-1}ab \end{aligned}$$

VII)

$$\begin{aligned} ab\underline{a^{-1}b^{-1}a^{-1}} &= ab\underline{bab} \text{ (by Fact 2)} \\ &= ab^{-1}ab \end{aligned}$$

VIII)

$$\begin{aligned} \underline{aba}b^{-1}a^{-1} &= \underline{b^{-1}a^{-1}b^{-1}}b^{-1}a^{-1} \text{ (by Fact 1)} \\ &= b^{-1}a^{-1}ba^{-1} \end{aligned}$$

IX)

$$\begin{aligned} \underline{aba}ba &= \underline{b^{-1}a^{-1}b^{-1}}ba \text{ (by Fact 1)} \\ &= b^{-1} \end{aligned}$$

X)

$$\begin{aligned} a\underline{b^{-1}a^{-1}b^{-1}}a^{-1} &= a\underline{aba}a^{-1} \text{ (by Fact 1)} \\ &= a^{-1}b \end{aligned}$$

We see that we can't write a length five word starting with a. Therefore every reduced word in G of length five or higher can be represented by a word of length strictly less than 5. Thus $|G|$ is at most 27. □

We have shown that $B(3,2)$ has order at most 27. We aim now to show that $B(3,2)$ has at least order 27 by showing that the group $(C_3 \times C_3) \rtimes_\alpha C_3$ is non abelian, exponent three, and generated by two elements.[3]

Theorem 2. *$B(3,2)$ has order at least 27.*

Proof. Let $G = \langle a, b, c \mid a^3,\ b^3,\ c^3,\ aba^{-1}b^{-1},\ cac^{-1}(\alpha(c)(a))^{-1},\ cbc^{-1}(\alpha(c)(b))^{-1}\rangle$ with $\alpha(c^r)(a^nb^m) = (a^nb^{nr+m})$ for all $r, n, m \in \{0,1,2\}$ where this defines the homomorphism $\alpha : \langle c\rangle \to Aut(\langle a, b\rangle)$. Consider $(C_3 \times C_3) \rtimes_\alpha C_3 \cong G$. We know this group has order 27. We will show this group is generated by two elements and has exponent three. This will show that the above group is actually $B(3,2)$. Since $ca = abc$ in G we have $a^{-1}cac^{-1} = a^{-1}abcc^{-1} = b$. Since $ab = ba$ and $cb = bc$, every element of G can be written as $a^nb^mc^r$ where $n, m, r \in \{0,1,2\}$. The product for all $a^nb^mc^r$ and $a^sb^tc^q$ in G with $n, m, r, s, t, q \in \{0,1,2\}$ is given by

$$\begin{aligned} (a^nb^mc^r)(a^sb^tc^q) &= a^nb^m\alpha(c^r)(a^sb^t)c^{r+q} \\ &= a^nb^ma^sb^{rs+t}c^{r+q} \\ &= a^{n+s}b^{m+rs+t}c^{r+q} \end{aligned}$$

[3]Note that $B(3,2)$ is the largest non abelian group on two generators with exponent three.

By the above, we have $(a^n b^m c^r)^2 = a^{2n} b^{2m+nr} c^{2r}$ and

$$\begin{aligned}(a^n b^m c^r)^3 &= (a^{2n} b^{2m+nr} c^{2r})(a^n b^m c^r) \\ &= a^{3n} b^{3m+3nr} c^{3r} \\ &= (a^3)^n (b^3)^{m+nr} (c^3)^r = 1\end{aligned}$$

Therefore G has exponent three. Since $ca = abc$ we have that $b = cac^{-1}a^{-1}$. We then can substitute this in for b and obtain $(C_3 \times C_3) \rtimes_\alpha C_3 = \langle a, c \mid a^3,\ c^3,\ (a^{-1}cac^{-1})^3,\ cac^{-1}a^{-1}ca^{-1}c^{-1}a,\ ca^{-1}cac^{-1}a^{-1}c^{-1}a \rangle$. Since G has exponent three and is generated by a and c, we have that $G \cong B(3,2)$ and has order at least 27. □

Combining Theorem 1 and Theorem 2 we have that $B(3,2)$ has order 27.

2.2. **The Burnside Group $B(3,2)$ as a One-Relator Quotient of a Triangle Group.** Suppose that G_1 and G_2 are groups. If $\varphi : G_1 \to G_2$ is a homomorphism and $N_1 \trianglelefteq G_1$ then $\varphi(N_1) \trianglelefteq G_2$. Since $\varphi(gN_1) = \varphi(g)\varphi(N_1) = \varphi(g)N_2$, the map $\Phi(gN_1) = \varphi(g)N_2$ gives a homomorphism between G_1/N_1 and G_2/N_2. We can easily check that if φ is onto, then Φ is onto. We collect these into a theorem.

Theorem 3. *Let G_1 and G_2 be groups. If $\varphi : G_1 \to G_2$ is a homomorphism, $N_1 \trianglelefteq G_1$ and $N_2 = \varphi(N_1)$, then $\Phi(gN_1) = \varphi(g)N_2$ defines a homomorphism $\Phi : G_1/N_1 \to G_2/N_2$. If φ is onto, then Φ is onto.*

Corollary 1. *If N_1 and N_2 are normal subgroups of $\mathbb{F}_2$ and $\Phi : \mathbb{F}_2 \to \mathbb{F}_2$ be an isomorphism such that $\Phi(N_1) = N_2$, then $\varphi(gN_1) = \Phi(g)N_2$ defines an isomorphism from $\mathbb{F}_2/N_1$ onto $\mathbb{F}_2/N_2$.*

Corollary 2. *Let R_1 and R_2 be the relators of G_1 and G_2 respectively. Let $\Phi : \mathbb{F}_2 \to \mathbb{F}_2$, be an isomorphism. Suppose $G_1 = \langle a, b | R_1 \rangle$ and $G_2 = \langle a, b | R_2 \rangle$. For all $r_1 \in R_1$, we have $\Phi(r_1) \in R_2^{\mathbb{F}_2}$ (the normal subgroup of $\mathbb{F}_2$ generated by R_2). For all $r_2 \in R_2$, we have $\Phi^{-1}(r_2) \in R_1^{\mathbb{F}_2}$ (the normal subgroup of $\mathbb{F}_2$ generated by R_1). We then have that $G_1 \cong G_2$.*

Proof. Suppose $\Phi(R_1) \subseteq R_2$ and $\Phi^{-1}(R_2) \subseteq R_1$. Then $R_2 \subseteq \Phi(R_1)$, so $\Phi(R_1) = R_2$, and hence $\Phi(R_1^{\mathbb{F}_2}) = R_2^{\mathbb{F}_2}$. Since Φ is a homomorphism we can apply Corollary 1, with $G_2 = \mathbb{F}_2$, $N_1 = R_1^{\mathbb{F}_2}$ and $N_2 = R_2^{\mathbb{F}_2}$. □

Corollary 3. *Let n be a positive integer.*

$$G_1 = \langle a, b | a^3, b^3, a^2(ba)^n b^2 \rangle \cong \langle x, y | x^{n^2+n+1}, y^3, yx = x^n y \rangle = G_2.$$

Proof. We will show that the map $\Phi : \mathbb{F}_2 \to \mathbb{F}_2$ sending $a \to y$, and $b \to y^{-1}x$ is an isomorphism of $\mathbb{F}_2$ onto itself. This isomorphism sends $\{a^3, b^3, a^2(ba)^n b^2\}^{\mathbb{F}_2}$ onto $\{\ x^{n^2+n+1},\ y^3\ ,\ yxy^{-1}x^{-n}\}^{\mathbb{F}_2}$. The result will then follow from Corollary 2. We know $\{y, y^{-1}x\}$ freely generate $\mathbb{F}(\{x, y\})$. We also know that the map $\varphi(a) = x$ and $\varphi(b) = y^{-1}x$ is one to one and onto. Therefore there is a unique isomorphism of $\mathbb{F}(\{x, y\})$ onto $\mathbb{F}(\{a, b\})$ extending φ (by the standard map extension property of free groups). We have that a^3, b^3, and $a^2(ba)^n b^2$ generate $\{a^3, b^3, a^2(ba)^n b^2\}^{\mathbb{F}_2}$. If we show $\Phi(a^3)$, $\Phi(b^3)$ and $\Phi(a^2(ba)^n b^2)$ are all 1 in

$$\mathbb{F}_2/\{x^{n^2+n+1}, y^3, yxy^{-1}x^{-n}\}^{\mathbb{F}_2} = \langle x, y | x^{n^2+n+1}, y^3, yx = x^n y \rangle$$

then we'll have that

$$\Phi(\{a^3, b^3, a^2(ba)^n b^2\}^{\mathbb{F}_2}) \subseteq \{x^{n^2+n+1}, y^3, yxy^{-1}x^{-n}\}^{\mathbb{F}_2}.$$

I)

$$\Phi(a^3) = y^3 = 1;$$

II) From repeated use of $yx = x^n y$ we have that

$$\begin{aligned}\Phi(b^3) &= (y^{-1}x)^3 = y^2xy^2xy^2x \\ &= yx^ny^3xyx^ny = yx^{n+1}yx^ny \\ &= x^{n^2+n}y^2x^ny = x^{n^2+n}y^2yx \\ &= x^{n^2+n+1} = 1\end{aligned}$$

III) Note we use $yx = x^n y$ again.

$$\begin{aligned}\Phi(a^2(ba)^nb^2) &= y^2(y^{-1}xy)^n(y^{-1}x)^2 \\ &= y^2(y^{-1}x^ny)(y^{-1}xy^{-1}x) = yx^{n+1}y^{-1}x \\ &= x^{n^2+n}yy^{-1}x = x^{n^2+n+1} = 1.\end{aligned}$$

We now show that

$$\Phi^{-1}(\{x^{n^2+n+1}, y^3, yxy^{-1}x^{-n}\}^{\mathbb{F}_2}) \subseteq \{a^3, b^3, a^2(ba)^nb^2\}^{\mathbb{F}_2}$$

and therefore

$$\Phi(\{a^3, b^3, a^2(ba)^nb^2\}^{\mathbb{F}_2}) \supseteq \{x^{n^2+n+1}, y^3, yxy^{-1}x^{-n}\}^{\mathbb{F}_2}$$

and so

$$\Phi(\{a^3, b^3, a^2(ba)^nb^2\}^{\mathbb{F}_2}) = \{x^{n^2+n+1}, y^3, yxy^{-1}x^{-n}\}^{\mathbb{F}_2}$$

To this end, note that, in G_1 :

I)

$$\Phi^{-1}(y^3) = a^3 = 1$$

II)

$$\Phi^{-1}(x^{n^2+n+1}) = b^3 = 1$$

III)

$$\begin{aligned}\Phi^{-1}(yxy^2x^{n^2+1}) &= aaba^2(ab)^{n^2+1} \\ &= a^2ba^2a(ba)^{n^2}b = a^2b\underline{(ba)^{n^2}}b \\ &= a^2b\underline{(ab)}^nb \text{ (note } ab = (ba)^n \text{ from the hypothesis)} \\ &= a^2(ba)^nb^2 = 1.\end{aligned}$$

The result follows. Since we know that $C_{n^2+n+1} \rtimes C_3 = \langle x, y | x^{n^2+n+1}, y^3, yx = x^n y\rangle$ we have that $C_{n^2+n+1} \rtimes C_3 \cong \langle a, b | a^3, b^3, a^2(ba)^nb^2\rangle$. □

Definition 10. *Let $G = \langle x, y \mid R\rangle$. We say a group K is a one-relator quotient of G if there exists a reduced word w in $\mathbb{F}_2$ such that $K \cong \langle x, y \mid R \cup \{w\}\rangle$.*

Corollary 4. *The Burnside group $B(3, 2)$ is a one relator quotient of the triangle group $\langle a, b \mid a^3, b^3, (ab)^3\rangle$. More specifically, if $G = (C_3 \times C_3) \rtimes_{\alpha} C_3 = \langle a, c \mid a^3, c^3, (a^{-1}cac^{-1})^3, cac^{-1}a^{-1}ca^{-1}c^{-1}a, ca^{-1}cac^{-1}a^{-1}c^{-1}a\rangle$ and $H = \langle a, b \mid a^3, b^3(ab)^3, (b^{-1}a)^3\rangle$, then $G \cong H$.*

Proof. We show that the map $\Phi : \mathbb{F}_2 \to \mathbb{F}_2$ sending $a \to a$ and $b \to c$ is an isomorphism of $\mathbb{F}_2$ onto itself. This isomorphism sends $\{a^3, b^3, (ab)^3, (b^{-1}a)^3\}^{\mathbb{F}_2}$ onto $\{a^3, c^3, (a^{-1}cac^{-1})^3,\ cac^{-1}a^{-1}ca^{-1}c^{-1}a,\ ca^{-1}cac^{-1}a^{-1}c^{-1}a\}^{\mathbb{F}_2}$. The result will then follow from Corollary 2. We know $\{a, c\}$ freely generate $\mathbb{F}(\{a, c\})$ and the map $\varphi(a) = a$ and $\varphi(b) = c$ is one to one and onto. Therefore there is a unique isomorphism of $\mathbb{F}(\{a, c\})$ onto $\mathbb{F}(\{a, b\})$ extending φ (by the standard map extension property of free groups). We have that a^3, b^3, $(ab)^3$, and $(b^{-1}a)^3$ generate $\{a^3, b^3, (ab)^3, (b^{-1}a)^3\}^{\mathbb{F}_2}$. If we show $\Phi(a^3)$, $\Phi(b^3)$, $\Phi((ab)^3)$, and $\Phi((b^{-1}a)^3)$ are all 1 in

$$\begin{aligned}&\mathbb{F}_2/\{a^3, c^3, (a^{-1}cac^{-1})^3, cac^{-1}a^{-1}ca^{-1}c^{-1}a, ca^{-1}cac^{-1}a^{-1}c^{-1}a\}^{\mathbb{F}_2}\\ &= \langle a, c | a^3, c^3, (a^{-1}cac^{-1})^3, cac^{-1}a^{-1}ca^{-1}c^{-1}a, ca^{-1}cac^{-1}a^{-1}c^{-1}a\rangle\end{aligned}$$

then we'll have that

$$\begin{aligned}\Phi(\{a^3, b^3, (ab)^3, (b^{-1}a)^3\}^{\mathbb{F}_2}) \quad \subseteq \quad & \{a^3, c^3, (a^{-1}cac^{-1})^3,\\ & cac^{-1}a^{-1}ca^{-1}c^{-1}a, ca^{-1}cac^{-1}a^{-1}c^{-1}a\}^{\mathbb{F}_2}.\end{aligned}$$

I)

$$\Phi(a^3) \to a^3 = 1$$

II)

$$\Phi(b^3) \to c^3 = 1$$

III) From $cac^{-1}a^{-1}ca^{-1}c^{-1}a = 1$ and $ca^{-1}cac^{-1}a^{-1}c^{-1}a = 1$ we obtain $cac^{-1}a^{-1} = a^{-1}cac^{-1} = c^{-1}a^{-1}ca$. This implies $ca = c^{-1}a^{-1}ca^{-1}c$. We have underlined this substitution. We use $cac^{-1}a^{-1}ca^{-1}c^{-1}a$ to obtain $cac^{-1}a^{-1} = a^{-1}cac^{-1}$. This implies $ac^{-1}a^{-1}c = c^{-1}a^{-1}ca$. We use brackets to show the substitution.

$$\begin{aligned}\Phi(ab)^3 \quad &\to \quad (ac)^3 = a\underline{ca}cac\\ &= \quad \{a\underline{c^{-1}a^{-1}c\}a^{-1}c}cac\\ &= \quad \{c^{-1}a^{-1}ca\}a^{-1}c^{-1}ac = 1\end{aligned}$$

IV) From $ac^{-1}a^{-1}c = cac^{-1}a^{-1}$ we obtain $ac^{-1} = c^{-1}ac^{-1}a^{-1}ca$. We underline this substitution for the reader. We also underline the substitution of $cac^{-1}a^{-1} = a^{-1}cac^{-1}$ for the reader.

$$\begin{aligned}\Phi(b^{-1}a)^3 \quad &\to \quad (c^{-1}a)^3\\ &= \quad c^{-1}\underline{ac^{-1}}ac^{-1}a\\ &= \quad c^{-1}\underline{c^{-1}ac^{-1}a^{-1}ca}ac^{-1}a\\ &= \quad \underline{cac^{-1}a^{-1}}ca^{-1}c^{-1}a\\ &= \quad \underline{a^{-1}cac^{-1}}ca^{-1}c^{-1}a = 1\end{aligned}$$

We now show that

$$\begin{aligned}&\Phi^{-1}(\{a^3, c^3, (a^{-1}cac^{-1})^3, cac^{-1}a^{-1}ca^{-1}c^{-1}a, ca^{-1}cac^{-1}a^{-1}c^{-1}a\}^{\mathbb{F}_2})\\ &\subseteq \{a^3, b^3, (ab)^3, (b^{-1}a)^3\}^{\mathbb{F}_2}\end{aligned}$$

and therefore

$$\begin{aligned}&\Phi(\{a^3, b^3, (ab)^3, (b^{-1}a)^3\}^{\mathbb{F}_2})\\ &\supseteq \{a^3, c^3, (a^{-1}cac^{-1})^3, cac^{-1}a^{-1}ca^{-1}c^{-1}a, ca^{-1}cac^{-1}a^{-1}c^{-1}a\}^{\mathbb{F}_2}\end{aligned}$$

and so

$$\begin{aligned}&\Phi(\{a^3, b^3, (ab)^3, (b^{-1}a)^3\}^{\mathbb{F}_2})\\&= \{a^3, c^3, (a^{-1}cac^{-1})^3, cac^{-1}a^{-1}ca^{-1}c^{-1}a, ca^{-1}cac^{-1}a^{-1}c^{-1}a\}^{\mathbb{F}_2}\end{aligned}$$

To this end, note that

I)

$$\Phi^{-1}(a^3) \to a^3 = 1$$

II) From $(ba)^3$ we obtain $ba = (a^{-1}b^{-1})^2$. We use the brackets to show this substitution. We use $(ba)^3$ to obtain $b^{-1}a^{-1} = (ab)^2$. We have underlined the substitution of $b^{-1}a^{-1}$ for the reader.

$$\begin{aligned}\Phi^{-1}(a^{-1}cac^{-1})^3 &\to (a^{-1}bab^{-1})^3\\&= a^{-1}\{ba\}b^{-1}a^{-1}bab^{-1}a^{-1}bab^{-1}\\&= a^{-1}\{a^{-1}b^{-1}a^{-1}b^{-1}\}b^{-1}a^{-1}bab^{-1}a^{-1}bab^{-1}\\&= a\underline{b^{-1}a^{-1}}b^{-1}\underline{b^{-1}a^{-1}}bab^{-1}a^{-1}bab^{-1}\\&= a\underline{abab}b^{-1}\underline{abab}bab^{-1}a^{-1}bab^{-1}\\&= a^{-1}ba^{-1}bab^{-1}a\underline{b^{-1}a^{-1}}bab^{-1}\\&= a^{-1}ba^{-1}bab^{-1}a\underline{abab}bab^{-1}\\&= a^{-1}ba^{-1}ba\underline{b^{-1}a^{-1}}bab^{-1}ab^{-1}\\&= a^{-1}ba^{-1}ba\underline{abab}bab^{-1}ab^{-1}\\&= a^{-1}ba^{-1}ba^{-1}bab^{-1}ab^{-1}ab^{-1}\\&= (a^{-1}b)^3(ab^{-1})^3 = 1\end{aligned}$$

III)

$$\Phi^{-1}(c^3) \to b^3 = 1$$

IV)We use $(ab)^3$ and $(ba)^3$ to obtain $b^{-1}a^{-1} = (ab)^2$ and $a^{-1}b^{-1} = (ba)^2$ respectively.

$$\begin{aligned}\Phi^{-1}(cac^{-1}a^{-1}ca^{-1}c^{-1}a) &\to ba\underline{b^{-1}a^{-1}}b\underline{a^{-1}b^{-1}}a\\&= ba(ab)^2b(ba)^2a\\&= baababbbabaa\\&= ba^{-1}ba^{-1}ba^{-1}\\&= (ba^{-1})^3 = 1\end{aligned}$$

V) We underline the substitution of $a^{-1}b^{-1} = (ba)^2$.

$$\begin{aligned}\Phi^{-1}(ca^{-1}cac^{-1}a^{-1}c^{-1}a) &\to ba^{-1}bab^{-1}a^{-1}b^{-1}a\\&= ba^{-1}bab^{-1}\underline{a^{-1}b^{-1}}a\\&= ba^{-1}bab^{-1}\underline{baba}a\\&= ba^{-1}ba^{-1}ba^{-1} = 1\end{aligned}$$

We have shown that there is an isomorphism between $(C_3 \times C_3) \rtimes_\alpha C_3$ and $H = \langle a, b \mid a^3, b^3(ab)^3, (b^{-1}a)^3\rangle$. We know that $(C_3 \times C_3) \rtimes_\alpha C_3$ is a two generated group of order 27 and exponent three. This means it is the Burnside group $B(3, 2)$. Furthermore we have shown that $H = \langle a, b \mid a^3, b^3(ab)^3, (b^{-1}a)^3\rangle$ is isomorphic to $(C_3 \times C_3) \rtimes_\alpha C_3$. □

3. Conclusion

In summary we have proven that $B(3,2)$ has order 27 and is isomorphic to $\langle a, b \mid a^3, b^3(ab)^3, (b^{-1}a)^3\rangle$ and $(C_3 \times C_3) \underset{\alpha}{\rtimes} C_3$. We have also proven that

$$\langle a, b | a^3, b^3, a^2(ba)^n b^2\rangle \cong \langle x, y | x^{n^2+n+1}, y^3, yx = x^n y\rangle$$

which is isomorphic to $C_{n^2+n+1} \rtimes C_3$. Furthermore we have shown that $B(3,2)$ is a one relator quotient of the triangle group $\langle a, b | a^3,\ b^3,\ (ab)^3\rangle$. We conclude this paper with the following definition.

Definition 11. *Let p be a prime and G be a group of exponent p with two generators a and b. A p-presentation is a presentation of the form $\langle a, b \mid \mathcal{R}\rangle$ where $\mathcal{R}$is irredundant and $w = r^p$ for all $w \in \mathcal{R}$, with r a cyclically reduced word that is not a p^{th} power. The length of a p-presentation is defined to be the cardinality of $\mathcal{R}$. The length of a minimal p-presentation is denoted by $\mathcal{L}(G)$ and is an isomorphism invariant for groups of exponent p.*

In terms of the above definition, Corollary 4 establishes that $\mathcal{L}(B(3,2)) = 4$, since any group with presentation $\langle a, b \mid a^3, b^3, w^3\rangle$ is a generalized triangle group and all finite generalized triangle groups have been completely classified. (cf [FHR] and [LRS]). We have yet to compute $\mathcal{L}(B(4,2))$ and $\mathcal{L}(B(5,2))$.

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515 Loudon Road, Loudonville, NY 12211
E-mail address: mc13farr@siena.edu