Zero-Divisor Graphs of $\mathbb{Z}_n$ and Polynomial Quotient Rings over $\mathbb{Z}_n$

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Abstract

Critical to the understanding of a graph are its chromatic number and whether or not it is perfect. Here we prove when $\Gamma(\mathbb{Z}_n)$, the zero-divisor graph of $\mathbb{Z}_n$, is perfect and show an alternative method to [D] for determining the chromatic number in those cases. We go on to determine the chromatic number for $\Gamma(\mathbb{Z}_p[x]/\langle x^n\rangle)$ where p is prime and show that an isomorphism exists between this graph and $\Gamma(\mathbb{Z}_{p^n})$.

Introduction

We set out to study the zero-divisors of an algebraic ring. Graphically representing these elements leads to insight about their behavior and provides methods of categorizing the ring. In general, a *graph* G is a set of vertices $V(G)$ combined with a corresponding edge set $E(G)$ such that every element in $E(G)$ represents an unordered pairing of distinct elements in $V(G)$. The *order of* G, denoted $|G|$, is equal to the cardinality of the vertex set. Given a commutative ring R with unity, let $Z(R)$ be the non-zero zero-divisors of R. We define $\Gamma(R)$ as the zero-divisor graph of R whose vertex set is $Z(R)$. The edge set of $\Gamma(R)$ is the set of all pairs of distinct vertices (x, y) that are *adjacent*, that is $xy = 0$.

A graph G has a *chromatic number* $\chi(G)$ corresponding to the minimum number of colors required to color each vertex such that no two adjacent vertices are colored the same color. If every pair of vertices in G is adjacent then G is called *complete* and is denoted K_n where $n = |G|$. H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In particular, given $U \subseteq V(G)$ the subgraph *induced* by U is the graph with vertex set U and all edges in $E(G)$ whose end vertices are in U. Any subgraph of G is called a *clique* if it is complete, and the *clique number* $\omega(G)$ denotes the order of the largest clique in G. Clearly, the clique number provides a lower bound to the chromatic number of G ([CL], p. 279). Anna Duane determined the chromatic number of $\Gamma(\mathbb{Z}_n)$ for any n [D].

In this paper, we will first offer a second method of determining the chromatic number of $\Gamma(\mathbb{Z}_n)$ for certain n. Following this we will explore the chromatic number of the zero-divisor graph of certain polynomial quotient rings over $\mathbb{Z}_n$. Using these results we discover a connection between these two sets of zero-divisors.

1 Zero-Divisor Graphs of $\mathbb{Z}_n$

We begin by exploring the family of rings $\mathbb{Z}_n$.

In [D], the chromatic number of $\Gamma(\mathbb{Z}_n)$ was determined by defining a *proper coloring* of the vertices, where no two adjacent vertices share the same color. To establish the chromatic number, it was then shown that the graph could not be colored using fewer colors.

A second method of establishing the chromatic number takes advantage of the clique number of the graph. In order to use a clique to determine the exact chromatic number of a graph, the graph must first be proven to be perfect. A graph is defined to be *perfect* if for every subgraph $H \subseteq G$, $\omega(H) = \chi(H)$. The following theorem provides one tool for proving that a graph is perfect. Note that P_n is the graph of n vertices such that the vertices u_i and edges e_j form the alternating sequence $u_1, e_1, u_2, e_2, ..., u_{n-1}, e_{n-1}, u_n$, where $e_i = u_{i-1}u_i$ for $i = 1, 2, ..., n$ and $u_i \neq u_j$ for all $i \neq j$ as shown in Figure 1.

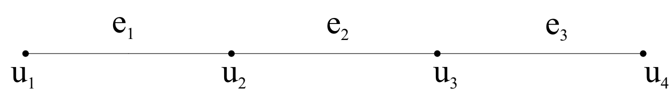

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Figure 1: P_4

Theorem 1.1 *[CL] If a graph G does not contain P_4 as an induced subgraph, then G is perfect.*

In most cases, $\mathbb{Z}_n$ does not yield a perfect zero-divisor graph. However, we now identify two cases which are perfect.

Theorem 1.2 *The graph $\Gamma(\mathbb{Z}_{p^n})$, where p is prime, is perfect.*

PROOF: Suppose $p = 2$ and $n = 2$ or $n = 3$. Then $\Gamma(\mathbb{Z}_{p^n})$ has $\{2\}$ and $\{2, 4, 6\}$ as vertex sets respectively. Also, if $p = 3$ and $n = 2$ then $\Gamma(\mathbb{Z}_{p^n})$ has vertices labeled 3 and 6. Thus in these cases, the order of $\Gamma(\mathbb{Z}_{p^n})$ is less than 4, and therefore it cannot contain P_4.

For all other cases of p and n, we must show that given four distinct elements $v_1, v_2, v_3, v_4 \in Z(\mathbb{Z}_{p^n})$ such that $v_1v_2 = 0$, $v_2v_3 = 0$, $v_3v_4 = 0$ there exists at least one more pair $v_qv_r = 0$ where $q, r \in \{1, 2, 3, 4\}$ and $q \neq r$. Each vertex v_k can be written as $m_kp^{i_k}$ where $\gcd(p, m_k) = 1$. When two vertices v_q and v_r are multiplied together we have $v_qv_r = m_qp^{i_q}m_rp^{i_r} = m_qm_rp^{i_q+i_r}$. This yields the condition $i_q + i_r \geq n$ for an edge to exist between v_q and v_r. Based on our definition of the vertices we know:

$$\begin{cases} i_1 + i_2 & \geq n \\ i_2 + i_3 & \geq n \\ i_3 + i_4 & \geq n \end{cases}.$$

We note that either $i_2 \geq i_3$ or $i_3 > i_2$. In the first case we can say that $i_2 + i_4 \geq i_3 + i_4 \geq n$ implies an edge between v_2 and v_4. In the second case we can say that $i_1 + i_3 \geq i_1 + i_2 \geq n$ implies an edge between v_1 and v_3. Thus $\Gamma(\mathbb{Z}_{p^n})$ cannot contain P_4 as an induced subgraph and is therefore perfect. ∎

Theorem 1.3 *The graph $\Gamma(\mathbb{Z}_{p_1p_2})$, where p_1 and p_2 are distinct primes, is perfect.*

PROOF: Suppose that p_1 or p_2 equals 2. Without loss of generality, assume $p_1 = 2$. Then $\Gamma(\mathbb{Z}_{p_1p_2})$ consists of one central vertex corresponding to p_2 which is connected to the remaining vertices of the form $2k$ such that $k < p_2$. The vertices that are divisible by 2 do not connect with each other since their product will not be divisible by p_2. Therefore, the graph cannot contain P_4 since the longest path in $\Gamma(\mathbb{Z}_{p_1p_2})$ contains two vertices divisible by 2 connected to the central vertex.

If $p_1 \neq 2$ and $p_2 \neq 2$ then let v_1, v_2, v_3 and v_4 be four distinct vertices in $\Gamma(\mathbb{Z}_{p_1p_2})$ such that $v_1v_2 = 0$, $v_2v_3 = 0$, and $v_3v_4 = 0$. Since $v_1v_2 = 0$, without loss of generality $p_1|v_1$ and $p_2|v_2$. It follows that p_1 must divide v_3 since $v_2v_3 = 0$. From this we then know p_2 must divide v_4 since $v_3v_4 = 0$. Therefore $v_1v_4 = 0$ and P_4 is not an induced subgraph of $\Gamma(\mathbb{Z}_{p_1p_2})$. ∎

Not only can we prove that $\Gamma(\mathbb{Z}_n)$ is perfect for these cases, but we also know these are the only cases when it is perfect.

Theorem 1.4 *The zero-divisor graph of $\mathbb{Z}_n$ is perfect if and only if $n = p^k$ for some prime p or $n = p_1p_2$ for some distinct primes p_1, p_2.*

PROOF: If $n = p^k$ or $n = p_1p_2$ we know $\Gamma(\mathbb{Z}_n)$ is perfect by Theorems 1.2 and 1.3. For all other cases of n, we know that $n = p_1p_2m$ for some distinct primes p_1 and p_2 and $m > 1$. Also, there exist four vertices in $\Gamma(\mathbb{Z}_n)$ defined as $v_1 = p_1$, $v_2 = p_2m$, $v_3 = p_1m$, and $v_4 = p_2$. Since $v_1v_2 = 0$, $v_2v_3 = 0$, and $v_3v_4 = 0$ but $v_1v_3 \neq 0$, $v_2v_4 \neq 0$, and $v_1v_4 \neq 0$ we know $\Gamma(\mathbb{Z}_n)$ contains P_4. Therefore, $\Gamma(\mathbb{Z}_n)$ is perfect if and only if $n = p^k$ or $n = p_1p_2$. ∎

Since we have proven that $\Gamma(\mathbb{Z}_{p^n})$ and $\Gamma(\mathbb{Z}_{p_1p_2})$ are perfect, we can use this property to determine the chromatic number. The proofs simply require finding the clique number of the graph.

Theorem 1.5 *The graph $\Gamma(\mathbb{Z}_{p^n})$, where p is prime, has chromatic number $p^{\frac{n}{2}} - 1$ for n even and $p^{\frac{n-1}{2}}$ for n odd.*

PROOF: Let N_i be the subset of $Z(\mathbb{Z}_{p^n})$ defined by $N_i = \{kp^i | gcd(k,p) = 1, 0 < k < p^{n-i}\}$ for all $0 < i < n$. Given two elements $a, b \in N_i$, their product is $ab = k_ap^ik_bp^i = k_ak_bp^{2i}$. It follows that elements in N_i are connected in $\Gamma(\mathbb{Z}_{p^n})$ if and only if $2i \geq n$. Similarly given an element $c \in N_i$ and $d \in N_j$, their product is $cd = k_cp^ik_dp^j = k_ck_dp^{i+j}$. This shows elements of N_i connect with elements of N_j if and only if $i + j \geq n$. If $q = \lceil \frac{n}{2} \rceil$ then the set of vertices $S = \bigcup_{i=q}^{n-1} N_i$ induces a clique in $\Gamma(\mathbb{Z}_{p^n})$. Since S consists of all elements in $\mathbb{Z}_{p^n}$ divisible by p^q, elements in S may be written as tp^q where $0 < t < p^{n-q}$. Since there are $p^{n-q} - 1$ integer values for a between 0 and p^{n-q}, it follows that $|S| = p^{n-q} - 1$. For all N_i such that $i < q$ it follows that no two elements in N_i are connected since $2i < n$. Similarly, if $i < j < q$, for all $c \in N_i$ and $d \in N_j$, c and d are not connected in $\Gamma(\mathbb{Z}_{p^n})$. Thus, any complete subgraph in $\Gamma(\mathbb{Z}_{p^n})$ can contain at most one element not divisible by p^q. We now consider two cases for the value of n:
Case 1: If n is even, then $q = \frac{n}{2}$. In this case any element $a \in N_{q-1}$ does not connect to the elements in N_q since $q + q - 1 = n - 1$. Therefore the subgraph induced by $S \cup \{a\}$ is not a clique. This implies S is the largest clique in $\Gamma(\mathbb{Z}_{p^n})$. Thus $|S|$ is the clique number of $\Gamma(\mathbb{Z}_{p^n})$, that is $|S| = p^{\frac{n}{2}} - 1 = \omega(\Gamma(\mathbb{Z}_{p^n}))$.
Case 2: If n is odd, then $q = \frac{n+1}{2}$. In this case each vertex in N_{q-1} connects to all vertices in S. It follows that the subgraph induced by $S \cup \{a\}$ is a clique. Since we have proved that any clique in $\mathbb{Z}_{p^n}$ must not contain two elements not divisible by p^q we know $S \cup \{a\}$ is the largest clique. Therefore $|S \cup \{a\}| = p^{n-q} - 1 + 1 = p^{n-\frac{n+1}{2}} = p^{\frac{n-1}{2}} = \omega(\Gamma(\mathbb{Z}_{p^n}))$.

Since we know definitively the clique number of $\Gamma(\mathbb{Z}_{p^n})$, and Theorem 1.2 states that $\Gamma(\mathbb{Z}_{p^n})$ is perfect, we know $\chi(\Gamma(\mathbb{Z}_{p^n})) = \omega(\Gamma(\mathbb{Z}_{p^n}))$. ∎

The following proof uses this same method of determining the chromatic number by way of the graph's clique number. It should be noted that a much more straightforward proof uses the fact that all elements in $\Gamma(\mathbb{Z}_{p_1p_2})$ fall into two separate sets which are unconnected with themselves leaving a 2-color graph. However we include this proof for completeness in showing the significance of the graph's perfect property.

Theorem 1.6 *The graph $\Gamma(\mathbb{Z}_{p_1p_2})$, where p_1 and p_2 are distinct primes, has chromatic number two.*

Proof: Clearly $\Gamma(\mathbb{Z}_{p_1p_2})$ has a clique of order two since any element divisible by p_1 is connected to an element divisible by p_2. Suppose there is a clique of order three. Then there must exist $q, r, s \in Z(\mathbb{Z}_{p_1p_2})$ such that $qr = 0$, $rs = 0$ and $qs = 0$. Since there are only two primes each element is divisible by only one. If all three elements are divisible by p_1 then the conditions will not hold. The conditions also fail if all are divisible by p_2. Thus we may assume without loss of generality that $p_1|q$ and $p_2|r$. This implies that $p_1|s$ by assumption that $rs = 0$. However, if both s and q are divisible by p_1 their product cannot be zero and we have a contradiction. Therefore $\Gamma(\mathbb{Z}_{p_1p_2})$ cannot have a clique of order three, and cannot have a clique of higher order since it would necessarily contain a clique of order three. This implies $\omega(\Gamma(\mathbb{Z}_{p_1p_2})) = 2$ and since $\Gamma(\mathbb{Z}_{p_1p_2})$ is perfect it has chromatic number two. ∎

2 Zero-Divisor Graphs of Polynomial Quotient Rings

To extend our study of zero-divisor graphs we examine the quotient ring $Q = \mathbb{Z}_p[x]/\langle x^n \rangle$ where p is prime and $n \geq 2$. The elements of this ring are congruence classes of polynomials modulo the principal ideal $\langle x^n \rangle$. For sake of simplifying notation, $q(x)$ will denote the congruence class of polynomials congruent to $q(x) \bmod \langle x^n \rangle$. Thus, an element in Q is of the form $q(x) = q_0 + q_1x + q_2x^2 + ... + q_{n-1}x^{n-1}$ where $q_i \in \mathbb{Z}_p$ for each i. Two non-zero polynomials $q(x)$ and $r(x)$ are zero-divisors in this quotient ring if $q(x)r(x) \equiv 0 \bmod \langle x^n \rangle$. To begin with, we demonstrate that using $\mathbb{Z}_p$ as the base of the quotient ring greatly simplifies the zero-divisor set of this ring.

Theorem 2.1 *If an element $a(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$ is in $Z(Q)$, then $a_0 \equiv 0 \mod p$.*

Proof: Let $a(x) \in Z(Q)$. Then there exists $b(x) \in Q$ such that $a(x)b(x) \equiv 0 \mod \langle x^n \rangle$. Let $b(x) = b_ix^i + b_{i+1}x^{i+1} + \cdots + b_{n-1}x^{n-1}$ where b_i is the first non-zero coefficient in the polynomial $b(x)$. The coefficient of x^i in the product $a(x)b(x)$ is a_0b_i. Since $a(x)b(x) \equiv 0 \mod \langle x^n \rangle$ and $i < n$, we must have $a_0b_i \equiv 0 \mod p$. Since $b_i \neq 0$, we know $a_0 \equiv 0 \mod p$. ∎

Corollary 2.2 *There are $p^{n-1} - 1$ zero-divisors in Q.*

Proof: Let $a(x)$ be in $Z(Q)$. Then by Theorem 2.1, $a_0 \equiv 0 \mod p$, so $a(x) = a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$ where $a_i \in \mathbb{Z}_p$. Every element of Q of this form is a zero-divisor because $a(x)x^{n-1} \equiv 0 \mod \langle x^n \rangle$. Since there are $n-1$ coefficients each with p choices, there are a total of p^{n-1} zero-divisors. However, the case where all coefficients are zero must be removed, so there are $p^{n-1} - 1$ nonzero zero-divisors in the ring Q if p is prime. ∎

Theorem 2.3 *The clique number of $\Gamma(Q)$ is $p^{\frac{n}{2}} - 1$ for n even and $p^{\frac{n-1}{2}}$ for n odd.*

PROOF: Define $A_i = \{a_i x^i + a_{i+1}x^{i+1} + \cdots + a_{n-1}x^{n-1} | a_i \in \mathbb{Z}_p, a_i \neq 0\}$. Since there are $(p-1)$ choices for the coefficient of x^i and p choices for the other $n-(i+1)$ coefficients, $|A_i| = (p-1)p^{n-i-1}$.

Case 1: If n is even, then $n = 2m$ for some $m \in \mathbb{Z}$. Then $x^m x^m = x^n \equiv 0$, and $x^{m+j}x^{m+k} \equiv 0 \mod \langle x^n \rangle$ for any $j, k \in \mathbb{Z}^+$. Thus, when $i \geq m$, any two elements $a(x), b(x) \in A_i$ are connected. Moreover, all elements in A_i are connected to all elements in A_j when $i, j \geq m$. This gives a complete subgraph with

$$\sum_{i=m}^{n-1} |A_i| = \sum_{i=m}^{n-1} (p-1)p^{n-i-1} = (p-1)\sum_{k=1}^{n-m} p^{n-m-k} = (p-1)\sum_{k=0}^{n-m-1} p^k = (p-1)\frac{p^{n-m}-1}{p-1} = p^m - 1$$

elements. The remaining elements in A_i where $i < m$ will not contribute to the clique since $i + m < n$. Furthermore there is no disjoint clique since none of the elements in A_i are connected to elements in A_j for $i \leq j < m$. Therefore, $\bigcup_{i=m}^{n-1} N_i$ is the largest clique and $\omega(\Gamma(Q)) = p^m - 1 = p^{\frac{n}{2}} - 1$ when n is even.

Case 2: If n is odd, then $n = 2m+1$ for some $m \in \mathbb{Z}$. Then $x^m x^m \not\equiv 0$, but $x^{m+1}x^m \equiv 0$, and $x^{m+1}x^{m+1} \equiv 0$. Thus, when $i > m$, any two elements $a(x), b(x) \in A_i$ are connected. Moreover, all elements in A_i are connected to all elements in A_j when $i, j > m$. This gives a complete subgraph with

$$\sum_{i=m+1}^{n-1} |A_i| = \sum_{i=m+1}^{n-1} (p-1)p^{n-i-1} = (p-1)\sum_{k=0}^{n-m-2} p^k = (p-1)\frac{p^{n-m-1}-1}{p-1} = p^m - 1$$

elements. The elements in the set A_m do not connect to each other but they do connect to all the elements in A_i where $i > m$, so only one element from A_m can be included in the clique. Therefore, there is a complete subgraph connecting the p^m elements. The remaining elements in A_i where $i < m$ and all elements except one from A_m will not contribute to the clique since $i + m < n$. Therefore, $\bigcup_{i=m}^{n-1} N_i \cup a$, for a single $a \in A_m$, is the largest clique and $\omega(\Gamma(Q)) = p^m = p^{\frac{n-1}{2}}$ when n is odd. ∎

Corollary 2.4 *The chromatic number of* $\Gamma(Q)$ *is equal to the clique number of* $\Gamma(Q)$.

PROOF: Let A_i be as in the proof of Theorem 2.3.

Case 1: Let n be even. A clique is formed from the elements in A_i where $i \geq \frac{n}{2}$. The remaining elements in A_i where $i < \frac{n}{2}$ can be assigned the same color as the elements in $A_{\frac{n}{2}}$ since $i + \frac{n}{2} < n$. Therefore, $\omega(\Gamma(Q)) = \chi(\Gamma(Q))$ when n is even.

Case 2: Let n be odd. The clique is formed from the elements in A_i where $i > \frac{n-1}{2}$ in addition to one element from $A_{\frac{n-1}{2}}$. The remaining elements in A_i where $i \leq \frac{n-1}{2}$ can be assigned the same color since $i + \frac{n-1}{2} < n$. Therefore, $\omega(\Gamma(Q)) = \chi(\Gamma(Q))$ when n is odd. ∎

Returning to our work in Section 1, we notice that $\Gamma(Q)$ is similar to $\Gamma(\mathbb{Z}_{p^n})$, having the same clique number and chromatic number. In fact, we can prove the graphs are isomorphic. Note that two graphs are *isomorphic* if there exists a one-to-one function from one vertex set onto the other such that adjacency is preserved by the function.

Lemma 2.5 *[L] Let b be greater than 1. Then every $a > 0$ can be uniquely represented in the form*

$$a = c_n b^n + c_{n-1}b^{n-1} + \cdots + c_1 b + c_0$$

with $c_n \neq 0$, $n \geq 0$, and $0 \geq c_i < b$ for $i = 0, 1, 2, \ldots, n$.

Theorem 2.6 *If p is prime then $\Gamma(\mathbb{Z}_{p^n}) \cong \Gamma(Q)$.*

PROOF: Let $C = \mathbb{Z}_{p^n}$ and Q be defined as above. By Lemma 2.5 for each $q \in Z(C)$ we may uniquely write $q = q_k p^k + q_{k+1} p^{k+1} + \cdots + q_{n-1} p^{n-1}$ where $k < n$ and $0 \le q_i < p$ for each $i = k, \ldots, n-1$. Then we define $\phi_p : Z(C) \to Z(Q)$ by $\phi_p(q) = q(x) = q_k x^k + q_{k+1} x^{k+1} + \cdots + q_{n-1} x^{n-1}$.

Suppose q and r are equivalent in $Z(C)$, then $q = r + sp^n$ for some $s \in \mathbb{Z}$. Now $q(x) = r(x) + sx^n$ and we see that $\phi_p(q)$ is equivalent to $\phi_p(r)$ in $Z(Q)$. Hence ϕ_p is a well-defined function.

Let $q(x) \in Z(Q)$. By Theorem 2.1 we know $q(x) = q_1 x + \cdots + q_{n-1} x^{n-1}$ where $0 \le q_i < p$ for all $i = 1, \ldots, n$ and $q_i \ne 0$ for at least one i. Clearly, $q = q_1 p + \cdots + q_{n-1} p^{n-1} \in Z(C)$ since $p | q$. Thus, $\phi_p(q) = q(x)$ and ϕ_p is onto.

To establish the order of $Z(C)$ we note that there are $p^{n-1} - 1$ non-zero integers less than p^n which are divisible by p. Since these integers comprise $Z(C)$ we have $|Z(C)| = p^{n-1} - 1$. In the case of $Z(Q)$ Theorem 2.1 guarantees us that any $q(x) \in Z(Q)$ can be written as $q(x) = q_1 x + \cdots + q_{n-1} x^{n-1}$ where $0 \le q_i < p$ for all $i = 1, \ldots, n$ and $q_i \ne 0$ for at least one i. Since there are $p^{n-1} - 1$ choices for the coefficients modulo p, the order of $Z(Q)$ must also be $p^{n-1} - 1$. Thus by the pigeon hole principle, since ϕ_p is onto we know that it must also be one-to-one.

Finally, we will show that ϕ_p preserves adjacency. Let $q \in Z(C)$ and write $q = ap^k$ for some $0 < k < n$ and $0 < a < p^{n-k}$ with $\gcd(a, p) = 1$. If $r \in Z(C)$ is adjacent to q, then $qr \equiv 0 \mod p^n$. Hence, we can write $r = bp^{n-k}$ for some $0 < b < p^k$ with $\gcd(b, p) = 1$. Now Lemma 2.5 assures us that $a = a_0 + a_1 p + \cdots + a_{n-k-1} p^{n-k-1}$ and $b = b_0 + b_1 p + \cdots + b_{k-1} p^{k-1}$ where $0 \le a_i, b_j < p$ whenever $i = 1, \ldots, n-k-1$ and $j = 1, \ldots, k-1$. By definition, $\phi_p(q) = q(x) = a_0 x^k + a_1 x^{k+1} + \cdots + a_{n-k-1} x^{n-1}$ and $\phi_p(r) = r(x) = b_0 x^{n-k} + b_1 x^{n-k+1} + \cdots + b_{k-1} x^{n-1}$. Multiplying these elements we see $q(x) r(x) = f(x) x^n$ for some polynomial $f(x) \in \mathbb{Z}_p[x]$. But clearly then $q(x) r(x) \equiv 0 \mod \langle x^n \rangle$. Thus, ϕ_p preserves adjacency. ∎

Having provided this final isomorphism we can conclude from earlier work that $\Gamma(Q)$ is a perfect graph.

We propose further study to explore isomorphisms of $\Gamma(\mathbb{Z}_{(p_1 p_2)^n})$ with $\Gamma(\mathbb{Z}_{p_1 p_2}[x]/\langle x^n \rangle)$. This task may be substantially more complex given the fact that $\Gamma(\mathbb{Z}_{p_1 p_2}[x]/\langle x^n \rangle)$ is not perfect for $n > 1$ as seen by inducing P_4 with the vertex set $\{p_1 + p_2 x^{n-1}, p_2 x^{n-1}, p_1 x^{n-1}, p_2 + p_1 x^{n-1}\}$.

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