Appendix to *Intrinsically S^1 3-linked graphs and other aspects of S^1 embeddings*

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This document contains the four proofs omitted in *Intrinsically S^1 3-linked graphs and other aspects of S^1 embeddings*. There are four proofs presented here. They correspond to Theorem 5.4, Theorem 5.7, Theorem 5.8, and Theorem 7.5. The proofs presented here follow a different notational style than that used in *Intrinsically S^1 3-linked graphs and other aspects of S^1 embeddings*, but that should not reduce their comprehensibility. Please note that these proofs have not been refereed.

1 Theorem 5.4

Theorem 5.4 *Let G' be a non-intrinsically S^1 3-linked graph and G'' be a non-intrinsically S^1-linked graph. Let G be the graph formed by pasting G' and G'' at a single edge, (a, b), where $(a, b) \in G'$ and $(a, b) \in G''$ and there is a non 3-linked embedding of G' and a non-linked embedding of G'' in which a and b are neighbors. Then G is not intrinsically S^1 3-linked.*

Proof. It will suffice to produce an S^1 embedding of G without a 3-link. Begin with a 3-linkless embedding of G'. Now embed the vertices of $G'' - \{a, b\}$ in a different component of $S^1 - \{a, b\}$ than the other vertices of G' such that the embedding of G'' is linkless. This is an S^1 embedding of G. It only remains to be shown that the embedding is 3-linkless; that is, every set of three disjoint 0-spheres forms a split 3-link.

Consider an arbitrary set of three disjoint 0-spheres in this embedding.

Case 1: If all three lie entirely in G', then they do not form a 3-link because the embedding of G' is 3-linkless.

Case 2: If two 0-spheres lie in G' and one lies in G'', or one 0-sphere lies in G' and two lie in G'', then they do not form a 3-link because each link containing a 0-sphere in each of G' and G'' is split.

Case 3: If all three lie in G'', then there is no 3-link because this embedding of G'' is linkless.

Case 4: Consider the case that exactly one of the 0-spheres contains vertices in both $G' - \{a, b\}$ and $G'' - \{a, b\}$.

Case 4a: Without loss of generality, let the 0-sphere include a in its path, but not b. Call this 0-sphere $(a_1, \ldots, a_i, a, a_{i+1}, \ldots, a_n)$ where $\{a_1, \ldots, a\} \in G'$ and $\{a, \ldots, a_n\} \in G''$. There are two subcases. First consider the case in which at least one of the other two 0-spheres lies entirely in G''. That 0-sphere forms split links with both $(a_1, \ldots, a)$ (because each link containing a 0-sphere in each of G' and G'' is split) and $(a, \ldots, a_n)$ (because G'' is linkless). Thus it forms a split link with $(a_1 \ldots, a_n)$. Similarly, that 0-sphere also forms a split link with the remaining 0-sphere. Therefore, the link formed by these three 0-spheres is a split 3-link.

Now consider the case in which the remaining two 0-spheres lie in G'. These 0-spheres do not form a 3-link with $(a_1, \ldots, a)$ because this embedding of G' is 3-linkless. Further, $(a, \ldots, a_n)$ does not form a link with either 0-sphere. Therefore, there is no 3-link.

Case 4b: Consider the case that the one 0-sphere contains both vertices a and b in its path. Call this 0-sphere $(v_1, \ldots, a, \ldots, b, \ldots, v_n)$. If v_1 and v_n are both in either G' or G'', then the 0-sphere $(v_1, \ldots, a, \ldots, b, \ldots, v_n)$ lies entirely in either G' or G'', which is covered in Cases 1 to 3. So suppose, without loss of generality, that $(v_1, \ldots, a) \in G'$ and $(b, \ldots, v_n) \in G''$. The 0-sphere $(b, \ldots, v_n)$ does not link with any other 0-sphere because it is contained in G'', which is linkless. If $(a, \ldots, b) \in G'$, then $(v_1, \ldots, a, \ldots, b) \in G'$, and it does not link with two other 0-spheres since G' is 3-linkless. Thus $(v_1, \ldots, a, \ldots, b, \ldots, v_n)$ does not form a 3-link with the other two 0-spheres. If $(a, \ldots, b) \in G''$, then $(a, \ldots, b, \ldots, v_n) \in G''$, and it does not link with any other 0-sphere, since G'' is linkless. Also, $(v_1, \ldots, a)$ can link with at most one of the 0-spheres because G'

is 3-linkless. Thus $(v_1, \ldots, a, \ldots, b, \ldots, v_n)$ does not form a 3-link with the other two 0-spheres.

Case 5: Consider the case that exactly two of the 0-spheres contain vertices in both $G' - \{a, b\}$ and $G'' - \{a, b\}$. This implies that one 0-sphere will contain a in its path and the other will contain b in its path. Call the 0-spheres $(a_1, \ldots, a, \ldots, a_n)$ and $(b_1, \ldots, b, \ldots, b_n)$ where $(a_1, \ldots, a) \in G'$, $(a, \ldots, a_n) \in G''$, $(b_1, \ldots, b) \in G'$, and $(b, \ldots, b_n) \in G''$. The 0-spheres $(a, \ldots, a_n)$ and $(b, \ldots, b_n)$ do not link with the remaining 0-sphere or with each other. The 0-spheres $(a_1, \ldots, a)$ and $(b_1, \ldots, b)$ do not contribute to a 3-link in G' because G' is 3-linkless. Since $(a_1, \ldots, a)$ is not involved in a 3-link in G' and $(a, \ldots, a_n)$ is not involved in any non-split link, then $(a_1, \ldots, a, \ldots, a_n)$ is not involved in a 3-link. Similarly, $(b_1, \ldots, b, \ldots, b_n)$ is not involved in a 3-link.

□

2 Theorem 5.7

Theorem 5.7 *There are no minor minimal intrinsically S^1 3-linked graphs with three adjacent degree two vertices.*

Proof. Assume that there is a graph G that is minor minimal intrinsically S^1 3-linked with 3 adjacent vertices of degree 2. Call these vertices v_1, v_2, and v_3.Without loss of generality, let v_1 be adjacent to a vertex x and vertex v_2, v_2 be adjacent to Vertices v_1 and v_3, and v_3 be adjacent to a vertex y and vertex v_2, as seen in Figure 1.

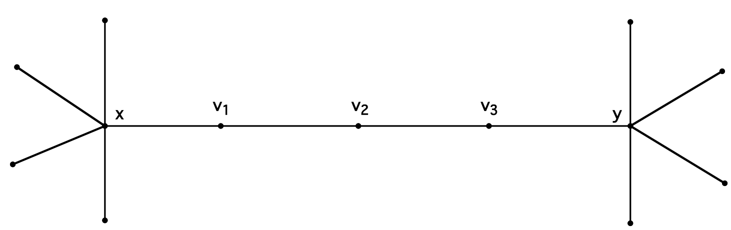

Figure 1: Three degree two vertices in a row.

Now consider the graph G' obtained by contracting edge (v_2, v_3) to vertex v_2, so that v_2 is now adjacent to v_1 and y. This graph is not intrinsically S^1 3-linked because it is a minor of G, which is minor minimal. Thus there is an S^1 embedding of G' without a 3-link. Consider this S^1 embedding for the following cases.

Case 1: Suppose neither v_1 nor v_2 are involved in any non-split link. Then expanding vertex v_2 to edge (v_2, v_3) so that vertices v_2 and v_3 are neighboring will not affect any links. This vertex expansion gives an S^1 embedding of G without a 3-link, which is a contradiction.

Case 2: Suppose, without loss of generality, that v_2 is involved in a non-split link, but v_1 is not. There are two cases to consider.

Case 2a: There exists exactly one edge linked with (v_2, y), as seen in Figure 2. Expanding vertex v_2 to edge (v_2, v_3) so that vertices v_2 and v_3 are neighboring will not create any new links because edge (v_3, y) will simply replace edge (v_2, y) in the link and nothing will cross (v_2, v_3) since they are neighbors. It will also not cause existing links to become disjoint since there was only one non-split link involving v_2. This vertex expansion gives an S^1 embedding of G without a 3-link, which is a contradiction since G is intrinsically S^1 3-linked.

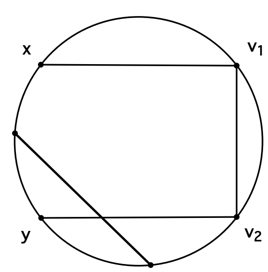

Figure 2: Case 2a

Case 2b: There exist two or more edges linked with (v_2, y). If the edges are disjoint, then there is a three link, which is a contradiction since this embedding of G' is S^1 3-linkless. If the edges are not disjoint, as seen in Figure 3, then expanding vertex v_2 to edge

(v_2, v_3) so that vertices v_2 and v_3 are neighboring will not create any new links because edge (v_3, y) will simply replace edge (v_2, y) in the links and nothing will cross (v_2, v_3) since they are neighbors. Also, the edges linked with (v_2, y) will remain non-disjoint. This vertex expansion gives an S^1 embedding of G without a 3-link, which is a contradiction since G is intrinsically S^1 3-linked.

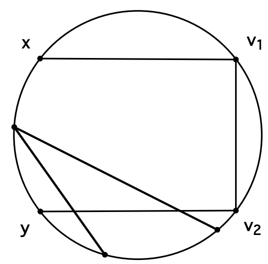

Figure 3: Case 2b

Case 3: Suppose both v_1 and v_2 are involved in a non-split link. There are the following cases to consider.

Case 3a: There exists exactly one edge linked with (v_1, v_2), and no edges linked with either (v_1, x) or (v_2, y), as seen in Figure 4. Expanding vertex v_2 to edge (v_2, v_3) so that vertices v_2 and v_3 are neighboring will not create any new links because nothing will cross (v_2, v_3) since they are neighbors. It will also not cause existing links to become disjoint since there was only one non-split link involving v_2. This vertex expansion gives an S^1 embedding of G without a 3-link, which is a contradiction since G is intrinsically S^1 3-linked.

Case 3b: There exists two or more edges linked with (v_1, v_2), and no edges linked with either (v_1, x) or (v_2, y). If the edges are disjoint, then there exists a 3-link, which is a contradiction since this embedding of G' is S^1 3-linkless. If the edges are not disjoint, as seen in Figure 5, then expanding vertex v_2 to edge (v_2, v_3) so that vertices v_2 and v_3 are neighboring will not create any new links because nothing will cross (v_2, v_3) since they are neighbors. The edges linked with (v_2, v_1) will remain non-disjoint. This vertex expansion gives an

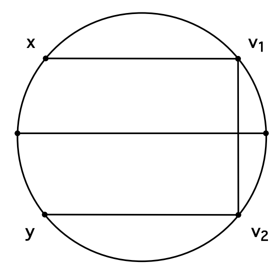

Figure 4: Case 3a

S^1 embedding of G without a 3-link, which is a contradiction since G is intrinsically S^1 3-linked.

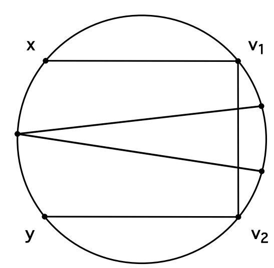

Figure 5: Case 3b

Case 3c: There exists an edge linked with (v_1, x) and an edge linked with (v_2, y), as seen in Figure 6. If the edges linked with (v_1, x) and (v_2, y) are the same or form a simple path, then there is a 3-link, which is a contradiction since this embedding of G' is S^1 3-linkless. If the edges linked with (v_1, x) and (v_2, y) are disjoint, then there is a 3-link involving the two edges and the 0-sphere (x, v_1, v_2, y). This is a contradiction since this embedding of G' is S^1 3-linkless.

Case 3d: There exists and edge linked with (v_2, y) and an edge linked with (v_1, v_2). If the edges linked with (v_1, x) and (v_2, y) are the same or are not disjoint, as seen in Figure 7, then expanding vertex v_2 to edge (v_2, v_3) so that vertices v_1 and v_2 are neighboring, and v_3 is placed on the S^1 where v_2 was will not create any new links because edge (v_3, v_2) will simply replace edge (v_2, v_1) in the links

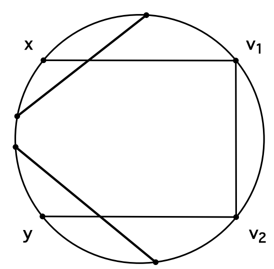

Figure 6: Case 3c

and nothing will cross (v_2, v_1) since they are neighbors. If the edges linked with (v_1, x) and (v_2, y) are disjoint, then there is a 3-link, which is a contradiction since this embedding of G' is S^1 3-linkless.

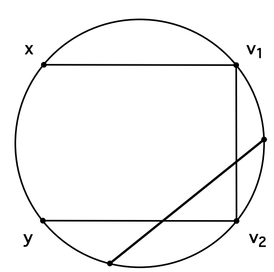

Figure 7: Case 3d

Case 3e: The 0-spheres (v_1, x) and (v_2, y) are linked with each other, as seen in Figure 8. Expanding vertex v_2 to edge (v_2, v_3) so that vertices v_2 and v_3 are neighboring will not create any new links because edge (v_3, y) will simply replace edge (v_2, y) in the link and nothing will cross (v_2, v_3) since they are neighbors. This vertex expansion gives an S^1 embedding of G without a 3-link, which is a contradiction since G is intrinsically S^1 3-linked.

All of the above cases result in a contradiction. Therefore, there are no minor minimal intrinsically S^1 3-linked graphs with three adjacent degree two vertices.

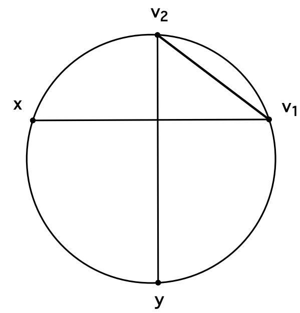

Figure 8: Case 3e

□

3 Theorem 5.8

Theorem 5.8 *No planar graph with six vertices is intrinsically S^1 3-linked.*

Proof. Consider the complete graph on six vertices, K_6. If one edge is removed, the graph contains K_5 and is thus non-planar. If two adjacent edges are removed, the graph contains a K_5. If two non-adjacent edges are removed, the graph contains a $K_{3,3}$ and is thus non-planar. There are five cases when three edges are removed.

Case 1 If three adjacent edges are removed, the graph contains a K_5.

Case 2 If three edges that form a triangle are removed, the graph contains a $K_{3,3}$.

Case 3 If three edges are removed such that two of the edges are adjacent and the last is not adjacent to the other two, the graph contains a $K_{3,3}$.

Case 4 If three edges are removed such that one edge is adjacent to two edges that are not adjacent to each other, the graph is not intrinsically S^1 3-linked, as can be seen in Figure 9.

Case 5 If three mutually non-adjacent edges are removed, the graph is not intrinsically S^1 3-linked, as can be seen in Figure 10.

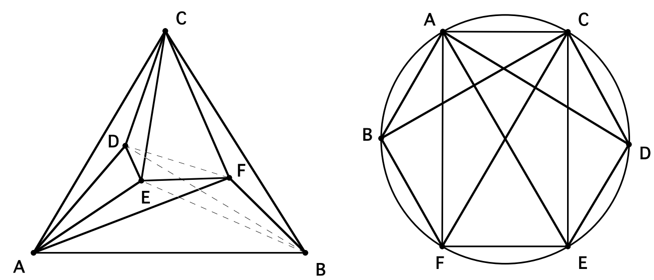

Figure 9:

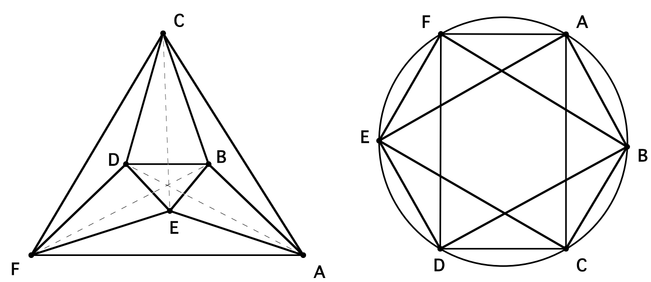

Figure 10:

There are four cases when four edges are removed such that the resultant graphs are not minors of graphs that are S^1 3-linkless.

Case 1 If four adjacent edges are removed, the graph contains a K_5.

Case 2 If a mutually non-adjacent pair of adjacent edges is removed, the graph contains a $K_{3,3}$.

Case 3 If a triangle and a disjoint edge are removed, the graph contains a $K_{3,3}$.

Case 4 If three adjacent edges and a disjoint edge is removed, then the graph contains K_5 as a minor.

There are two cases when five edges are removed such that the resultant graphs are not minors of graphs that are S^1 3-linkless.

Case 1 If five adjacent edges are removed, the graph contains a K_5.

Case 2 If a triangle and a pair of adjacent edges that are disjoint from the triangle are removed, the graph contains a $K_{3,3}$.

There is one case when six edges are removed such that the resultant graph is not a minor of a graph that is S^1 3-linkless. This graph is $K_{3,3}$.

There are no graphs when seven edges are removed such that the resultant graphs are not minors of graphs that are S^1 3-linkless.

Thus, every planar graph with six vertices is not intrinsically S^1 3-linked. □

4 Theorem 7.5

A previous REU showed that $K_{3,3}$ is minor minimally intrinsically S^1 3-linked. So far we have shown that the graphs T_7 and T_9 are intrinsically S^1 3-linked and that they are minor minimal. We claim that these three graphs are the complete minor minimal set of non-planar intrinsically S^1 3-linked graphs. To prove it we will put forth a series of lemmata, each building on the last.

The graph $R_8 + e$ is the graph shown in Figure 11 including the dashed edge. The graph without the dashed edge is the graph R_8.

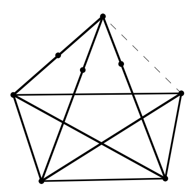

Figure 11: A picture of the graph $R_8 + e$

Lemma 4.1. *The graphs $R_8 + e$, T_8, and T_9 are minors of any intrinsically S^1 3-linked graph that can be obtained from K_5 by degree 2 vertex expansion.*

Proof. To do this, we will simply list every case of degree 2 vertex expansion on K_5. Label the vertices of K_5 as a, b, c, d, and e. The first is a single degree 2 vertex expansion, with the new vertex v adjacent to vertices a and b. In this case, embed K_5 with a and b neighbors, then apply Theorem 5.4.

The next case is two degree 2 vertex expansions. First consider the case that the second vertex added, w, is adjacent to v. In this case, we can apply Theorem 5.4 using the embedding of $K_5 + v$ obtained by applying Theorem 5.4 in the first vertex expansion. Note that for further degree 2 vertex expansions, the resulting graph will, by the same logic, not be 3-linked if the new vertex is adjacent to a previous added vertex. In light of this, we will only consider the cases in which a new degree 2 vertex expansion occurs that is not adjacent to any of the added degree 2 vertices.

Now let w be adjacent to c and d. Then embed K_5 so that a and b are neighbors, and c and d are neighbors, and apply Theorem 5.4 twice. Similarly, in the case that v neighbors a and b and w neighbors b and c, embed K_5 such that b neighbors both a and c and apply the Folding Lemma twice. Thus there are no graphs obtained from two degree 2 vertex expansions on K_5 that are intrinsically S^1 3-linked.

Now consider doing three degree 2 vertex expansions on a K_5. Note that we have already shown that $R_8 + e$ and T_8 are intrinsically S^1 3-linked and note that both of these graphs are obtained from a K_5 by doing three degree 2 vertex expansions. If v is adjacent to vertices a and b, if w is adjacent to b and c, and if x is adjacent to c and d, then embed K_5 so that b neighbors a and c and c neighbors d. Then apply the Folding Lemma three times. If v is adjacent to vertices a and b, if w is adjacent to b and c, and if x is adjacent to d and e, then embed K_5 so that b neighbors a and c and so that d neighbors e. Then apply the Folding Lemma three times. Thus, we have considered every way to add three degree 2 vertices to a K_5. The graphs $R_8 + e$ and T_8 are the only intrinsically S^1 3-linked graphs obtained by adding three degree 2 vertices.

Now consider doing four degree 2 vertex expansions on a K_5. Note that we have already shown that T_9 is intrinsically S^1 3-linked and note that this graph is obtained from a K_5 by doing four degree 2 vertex expansions. Further, the cases in which the graph obtained by performing four degree 2 vertex expansions on K_5 contains $R_8 + e$ or T_8 as a minor will be intrinsically S^1 3-linked. The only other case is that v is adjacent to vertices a and b, w is adjacent to b and c, x is adjacent to c and d, and y is adjacent to d and e. In this case, embed K_5 so that b neighbors a and c and d neighbors c and e. Then apply Theorem 5.4 four times. □

Lemma 4.2. *Suppose G is intrinsically S^1 3-linked. If G is a graph with K_5 as a minor and if G also has a subgraph G' attached at a cut vertex to K_5,* *then the subgraph G' is intrinsically S^1 linked.*

Proof. If G' is not intrinsically S^1 linked, then by the Folding Lemma, the entire graph G will not be intrinsically S^1 3-linked. $\square$

Lemma 4.3. *The graph intrinsically S^1 3-linked graph G obtained by attaching graphs to a vertex of the K_5 is not a non-planar, minor minimally intrinsically S^1 3-linked graph other than $K_{3,3}$, T_7, and T_9.*

Proof. By Lemma 4.2, all of the possible graphs attached by a cut vertex to K_5 are intrinsically S^1 linked. Therefore, if G' is a subgraph attached by vertex v in K_5, G' contains either K_4 or $K_{3,2}$ as a minor. The graph G obtained by pasting G' along vertex v to K_5 has either a K_4 pasted to a K_4 along vertex v or a $K_{3,2}$ pasted to a K_4 along vertex v since K_4 can be obtained as a minor of K_5 by contracting or deleting a vertex other than v.

The graph G obtained by attaching a graph G' to a degree 2 vertex of the K_5 can be contracted so that it has the graph G which consists of the graph G' attached to a K_5 at one (call it v) of the five degree 4 vertices of the K_5 as a minor and so it has either a K_4 pasted to a K_4 along vertex v or a $K_{3,2}$ pasted to a K_4 along vertex v. $\square$

Definition 4.4. *A graph G is intrinsically S^1 (a,b)-neighborly linked if G is linked in all S^1 embeddings that have vertex a as a neighbor to vertex b.*

Lemma 4.5. *If G is a connected graph that is not intrinsically S^1 linked, but that is intrinsically S^1 (a, b)-neighborly linked, then G contains a 4-cycle with a and b non-adjacent on the 4-cycle as a minor. We will call this 4-cycle with a and b as non-adjacent vertices C_4*.*

Proof. We will prove the lemma by the contrapositive. Assume that G does not have C_4* with vertices a and b as opposite vertices as a minor. Then, there are not two or more paths with completely disjoint vertices from a to b since any two completely distinct paths from a to b can be contracted to the graph C_4*. Also, there is at least 1 distinct path from a to b since G is a connected graph.

Assume that there is more than one distinct path from a to b with maximum length. Any two of these paths must have at least one vertex in common since there cannot be two paths with completely disjoint vertices. Call one such vertex, v. Assume that there are n (for some integer n, $n \geq 1$) vertices on each of the distinct paths with maximum length from a to b between a vertex v (not a or b) and some vertex w, which is possibly the same as one of the vertices, a or b (n is an integer ≥ 1 because two paths, (a, v, b), are not distinct). Because n is ≥ 1, C_4* is a minor. See Figure 12. This contradicts the hypothesis, so there is exactly one distinct path from a to b with maximum length.

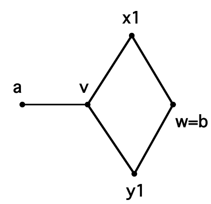

Figure 12: The graph C_4* is a minor if there is more than one distinct path from a to b with maximum length

Consider the unique path of maximum length from a to b that includes vertices a and b. Call this path p. Embed p in S^1 so that if there are n (n is possibly 0) vertices on the unique path of maximum length from a to b other than a and b, then a neighbors b. Also if $n = 1$, v_1 neighbors a and b. If n > 1, v_1 neighbors a and v_2, v_i neighbors v_{i-1} and v_{i+1} for all integers i, $1 <$ $i < n$ and v_n neighbors v_{n-1} and b.

Embed any vertex that is connected by a path to vertex a by an edge or a path (but not by any edge or path that contains another vertex of p) so that it is on the opposite component of $S^1 - \{a, v_1\}$ as vertex b. The reason for considering such vertices of G that do not connect by an edge or path to any other vertex of p is that if one of these points (call it w) connects by a path to a and to v_1, then $(a, w, v_1, v_2, \ldots, v_n, b)$ has a greater length than p, which is the unique path with maximum length from a to b by assumption. Also, if w connects by a path to a and to one of the other vertices along p (besides a and v_1) then the graph G has C_4* as a minor by contracting the edges of either (a, w, v_i, q_1) (where $2 \leq i \leq n$) or (a, w, b, q_2) where q_1 represents the path along p from v_i back to a and q_2 represents the path along p in the direction from b to a.

Continue to embed any vertex that is connected by an edge or a path to vertex v_i (but not by any edge or path that contains another vertex of p), for some i ($1 \leq i \leq n - 1$), so that it is on the opposite component of $S^1 - \{v_i, v_{i+1}\}$ as vertex b. Also, embed any vertex that is connected by an edge or a path to vertex v_n (but not by any edge or path that contains another vertex of p) so that it is on the opposite component of $S^1 - \{v_n, b\}$ as vertex a. Notice that none of these vertices are connected by a path that does not include v_i ($1 \leq i \leq n$) along p to any of the other vertices along p by the same argument as that given for vertex w. Call the vertex that now neighbors b at this point in embedding the graph G, $w*$. Also, embed any vertex that is connected by an edge or a path to vertex b (but not by any edge or path that contains another vertex of p) so that it is on the opposite component of $S^1 - \{w*, b\}$ as vertex a. Notice that none of these vertices are connected by a path that excludes the vertex b along p to any of the other vertices along p by the same argument as that given for vertex w. Therefore, no two of the vertices that are connected by an edge or a path to a vertex in p (but not by any edge or path that contains another vertex of p) form an edge that links with an edge along p.

Also, the vertices that are connected by an edge or a path to p_i (a vertex in p, possibly a or b), but not by any edge or path that contains another vertex of p along with the vertex p_i itself do not form edges that are linked with each other. This is because the graph G is not intrinsically S^1 linked, so no minor of G is intrinsically S^1 linked. Therefore, the subgraphs that are embedded all on the same component of $S^1 - \{p_i, p_{i+1}\}$ (p_i and p_{i+1} are two adjacent vertices along p) can be folded so that there is no link in the part of the graph that is connected at a vertex along p by Theorem 5.4. The edges formed by vertices connected by an edge or a path to b (but not by any edge or path that contains another vertex of p) and the same kind of edges formed by vertices connected to v_n do not form links because the vertices are on different components of $S^1 - \{w*, b\}$ and there cannot exist such a path from b to $w*$ because this would be part of a longer path from a to b than p. Also, the vertices on different components of $S^1 - \{p_i, p_{i+1}\}$ cannot have edges between them because then the vertices on different components of $S^1 - \{p_i, p_{i+1}\}$ are connected by an edge or a path to more than one vertex of p.

The edges formed strictly using the vertices along p also are not linked by construction.

Because G is a connected graph, all the vertices are connected by some path to the vertices along p. The edges formed using strictly vertices (such as w) that are connected by a path to a vertex a, b, or v_i in p by an edge or a path (but not by any path that contains another vertex of p) do not form edges that link. These vertices also do not form edges that link with edges in

p. Finally, none of the edges formed strictly using the vertices in p contain a link.

Note that G is not intrinsically S^1 a,b-neighborly linked, since we have constructed an embedding of G without a link and with a as a neighbor to b. □

Definition 4.6. *Attaching a graph G to a connected graph G' along two vertices (v, w) of K_5 is an operation performed by choosing two vertices in G', say a and b, and adding edges (a, v) and (b, w) and then contracting edges (a, v) and (b, w). Attaching a connected graph G' along an edge of the K_5 is a particular case where the two vertices (a and b) are adjacent degree 4 vertices in the subdivision of K_5.*

We will only consider attaching a graph, G', to a K_5 if G' is a connected graph because otherwise G is not a connected graph and therefore will not be minor-minimal.

Lemma 4.7. *Attaching a connected graph G' along an edge (a, b) of K_5 does not create a minor minimally non-planar intrinsically S^1 3-linked graph other than $K_{3,3}$, T_7, and T_9.*

Proof. We want to consider all of the ways that a connected graph G' can be attached to the K_5 along an edge (a, b). There are three possibilities that we will consider for G': (i) G' is not intrinsically S^1 linked and is not intrinsically (a, b)-neighborly linked, (ii) G' is not intrinsically S^1 linked and is intrinsically (a, b)-neighborly linked, and (iii) G' is intrinsically $S^!$ linked.

Case 1: If G' is not intrinsically S^1 linked and is not intrinsically (a, b)-neighborly linked, then there is a linkless embedding of G' with a and b as neighbors. There is also a 3-linkless embedding of a K_5 with any two adjacent degree 4 vertices, v and w as neighbors. If G' contains edge (a, b), then by Theorem 5.4 the graph G formed by attaching G' to a K_5 along edge (a, b) is not intrinsically S^1 3-linked.

If G' does not contain edge (a, b), then since G' is not intrinsically (a, b)-neighborly linked, there is an embedding of G' with a and b as neighbors and without a link. Adding edge (a, b) to this embedding of the graph does not create a link. Therefore, the graph G formed by attaching G' $+(a, b)$ to a K_5 along edge (v, w) is not intrinsically S^1 3-linked by the Folding Lemma.

Case 2: Consider attaching a connected graph G' that is not intrinsically S^1 linked and that is intrinsically (a,b)-neighborly linked to K_5 along edge (v,w). G' can be contracted to C_4* (with a and b as non-adjacent vertices) since it contains it as a minor (by Lemma 4.5). When attaching G' to K_5, the edges, (a,v) and (b,w) are not part of the C_4* minor of G' because they are attached to G' before contracting. Also, edge (a,b) is not part of the C_4* minor of G' because a and b are on non-adjacent edges of the 4-cycle. Therefore, when edges (a,v) and (b,w) are contracted, C_4* is still a minor of the graph G and (a,b) is not an edge of C_4* because now $a = v$ and $b = w$. Thus, the graph G with connected graph G' attached contains K_5 with a graph C_4* attached along edge (v,w) (so that edge (v,w)i is not an edge in the C_4*). This is the graph T_7 with the extra edge (v,w). Therefore, T_7 is a minor of any graph G with a K_5 and a graph such as G' attached along an edge to the graph K_5.

Case 3: If a connected graph, G', is intrinsically S^1 linked, then the graph G formed by attaching G' to the graph K_5 along an edge (a,b) contains a minor minimally intrinsically S^1 3-linked planar graph with a cut vertex as a minor. This is because if G' is intrinsically S^1 linked, then it contains $K_{3,2}$ or K_4 as a minor. The graph G obtained by attaching G' along edge (a,b) to K_5 has either a K_4 pasted to a K_4 along vertex a or b or a $K_{3,2}$ pasted to a K_4 along vertex a or b as a minor.

□

We will now consider all of the other ways to attach another edge or graph G' to a subdivided K_5 at 2 vertices (where at least one of the vertices is not a degree 4 vertex).

Lemma 4.8. *Consider the graph G formed by attaching a connected graph G' to a K_5 at either two degree 2 vertices on an incident edge of the K_5 (See Figure 13) or by attaching a connected graph G' at a vertex a in the K_5 and at a degree 2 vertex on the same edge ((a,b) of the original K_5(See Figure 14)). The graph G is not a non-planar, minor minimally intrinsically S^1 3-linked graph other than $K_{3,3}$, T_7, and T_9 .*

Proof. For the graph G formed by attaching the degree 2 vertex on the same edge ((a,b) of the original K_5) as a down to b, add edge (a,b). This

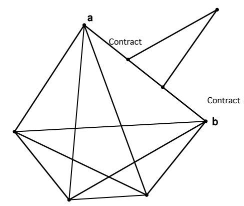

Figure 13: One way a graph G' might be connected to a K_5 at two degree 2 vertices

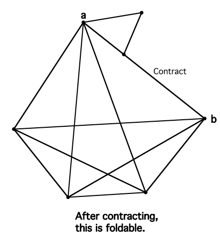

Figure 14: One way a graph G' might be connected to K_5 at two degree 2 vertices

graph is now K_5 with a graph G'' pasted along an edge. The graph G'' is not intrinsically linked and not (v, w)-neighbor-linked where v and w are the vertices in G'' that are attached along edge (a, b) (as shown by the embedding in Figure 15). This graph is not S^1 3-linked so the graph without edge (a, b) is also not S^1 3-linked. This graph is not a non-planar, minor minimally intrinsically S^1 3-linked graph other than $K_{3,3}$, T_7, and T_9.

For the graph G formed by attaching a connected graph G' at a vertex a in the K_5 and at a degree 2 vertex on the same edge ((a, b) of the original K_5, add edge (a, b). This graph is now K_5 with a graph G'' pasted along an edge. The graph G'' is not intrinsically linked and not (v, w)-neighbor-linked where v and w are the vertices in G'' that are attached along edge (a, b). This graph is not S^1 3-linked so the graph without edge (a, b) is also not S^1 3-linked. This graph is not a non-planar, minor minimally intrinsically S^1 3-linked graph other than $K_{3,3}$, T_7, and T_9. □

Lemma 4.9. *Consider the graph G formed by either attaching a connected graph G' or an edge to a K_5 at a vertex a in the K_5 and at a degree 2 vertex on*

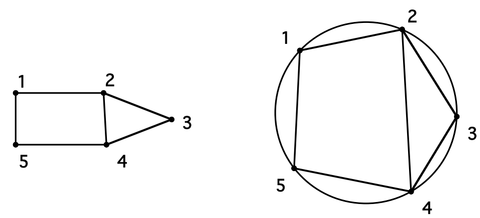

Figure 15: A picture of G” and a linkless S^1 embedding of this graph

a different edge (not incident to a in the original K_5), at two degree 2 vertices on adjacent edges of the K_5, or at two degree 2 vertices on non-adjacent edges of the K_5. The graph G contains $K_{3,3}$ as a minor.

Proof. Each of the three types of graphs described contain a $K_{3,3}$ as illustrated in Figures 16, 17, and 18.

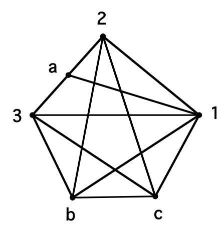

Figure 16: The graph G is obtained from a K_5 by adding a graph G' or an edge at a vertex and a degree 2 vertex on a different edge

□

Lemma 4.10. *Consider the graph G formed by either attaching a connected graph G' or an edge to a K_5 at non-adjacent degree 4 vertices. The graph G contains T_7 as a minor.*

Proof. Contract the graph G' to a single vertex that is still attached to a K_5 at non-adjacent degree 4 vertices. Contract the edges between the two degree 4 vertices so that there is exactly one degree 2 vertex between the two degree 4 vertices. This graph is T_7 and is a minor of G. □

We know that attaching one connected graph G' along a edge (a, b) of K_5 does not create a minor minimally non-planar intrinsically S^1 3-linked graph

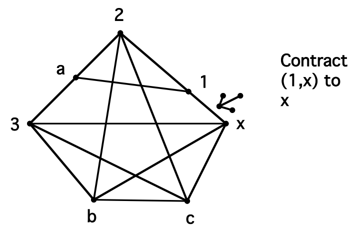

Figure 17: The graph G is obtained from K_5 by adding a graph G' or an edge at two degree 2 vertices on adjacent edges of the K_5

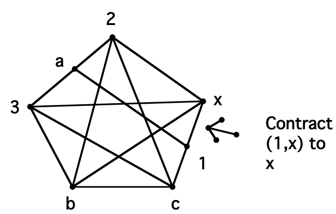

Figure 18: The graph G is obtained from K_5 by adding a graph G' or an edge at two degree 2 vertices on non-adjacent edges of the K_5

other than $K_{3,3}$, T_7, and T_9. We will now consider attaching more than one graph G' along an edge in the original K_5.

Lemma 4.11. *Consider the graph G obtained by attaching more than one connected graph along an edge in the original K_5 where one of the graphs is intrinsically S^1 linked or where one of the graphs is not intrinsically S^1 linked, but that is intrinsically (x, y) neighbor-linked (where vertices x and y which are being attached to edge (x, y) in the K_5). The graph G is not minor minimal with respect to being intrinsically S^1 3-linked.*

Proof. This graph where one of the graphs being attached is intrinsically S^1 linked has a minor minimally intrinsically S^1 3-linked, planar graph with a cut vertex as a minor. For the graph where one of the graphs is not intrinsically S^1 linked but that is intrinsically (x, y) neighbor-linked, G contains T_7 as a minor. □

Lemma 4.12. *If the graph G, obtained by attaching more than one graph that is neither intrinsically S^1 linked nor (x, y)-neighborly-linked along edges* *in the original K_5 (so that each x and y are vertices in the K_5), is intrinsically S^1 3-linked, then G contains $R_8 + e$, T_8, or T_9 as a minor.*

Proof. First assume that all of the vertices of the edges (that a graph will be attached to in the K_5) can be simultaneously embedded as neighbors in an S^1 embedding of the K_5. Because all of the graphs being attached to the K_5 along an edge are not (x, y)-neighborly linked where (x, y) is the edge that is attached to an edge in the orignial K_5 and because each of the pairs of vertices needed for these edges are neighbors in a particular S^1 embedding of the K_5, the folding lemma can be applied for each of these graphs. Therefore, G is not intrinsically S^1 3-linked.

Now assume that all of the vertices of the edges (that a graph will be attached to in the K_5) cannot be simultaneously embedded as neighbors when embedding the K_5 by itself in S^1. Note that if a degree 2 vertex is placed on each such edge, then G has either $R_8 + e$, T_8, or T_9 as a minor. Consider the graph G formed by attaching each of the graphs along an edge in the K_5. Contract all of these graphs attached along an edge to the original K_5 to a vertex adjacent to the two vertices of the edge and remove the original edges in the K_5 that have a graph attached to them. Notice that this is the same as performing a degree 2 vertex expansion of the K_5 and that this graph is a minor of G. Therefore, all such graphs G have a graph with degree 2 vertex expansions of the K_5 as a minor.

By Lemma 4.1, the graphs $R_8 + e$, T_8, and T_9 are the minors of any intrinsically S^1 3-linked graph that can be obtained from K_5 by degree 2 vertex expansion. Therefore, the graph G obtained by attaching more than one graph that is neither intrinsically S^1 linked nor (x, y)-neighborly-linked along edges in the original K_5 (so that each x and y are vertices in the K_5) is intrinsically S^1 3-linked, it will have one of these graphs as a minor. □

Lemma 4.13. *The graph G obtained by attaching one or more connected graphs at three or more vertices of the original K_5 contains a $K_{3,3}$ as a minor.*

Proof. Contract one of the graphs that is attached to the K_5 by three or more vertices to a single vertex that is still adjacent to the vertices of the original K_5. Label this single vertex 1. Label three of the vertices that 1 is connected to in the K_5 as a, b, and c. Then label the two remaining vertices of the K_5 as 2 and 3. The vertices labeled 1, 2, 3, a, b, and c have the requisite edges for the graph, $K_{3,3}$. □

Lemma 4.14. *The graph G obtained by attaching at least one connected graph at three or more vertices (to a subdivided K_5) is not a minor minimally, non-planar intrinsically S^1 3-linked graph other than $K_{3,3}$, T_7, and T_9. In particular, each of these graphs has either $K_{3,3}$ or T_7 as a minor.*

Proof. By Lemma 4.13, if G is a graph obtained by attaching one or more connected graphs at three or more vertices of the original K_5 contains a $K_{3,3}$ as a minor. If G is a graph obtained by attaching one or more graphs at any three or more degree 2 vertices on two distinct edges of the original K_5, then the graph has $K_{3,3}$ as a minor by Lemma 4.9.

If G is a graph obtained by attaching one or more graphs at two of the original vertices of the K_5 and at a degree 2 vertex not on the edge between them, then G contains $K_{3,3}$ as a minor by Lemma 4.9. If G is a graph obtained by attaching one or more graphs at two of the original vertices of the K_5 and at a degree 2 vertex on the edge between them, then G contains T_7 as a minor.

If G is a graph obtained by attaching one or more connected graphs at one of the degree 4 vertices, v, in the subdivided K_5 and at two degree 2 vertices on edges of the original K_5 not incident to v or on distinct edges, then G contains $K_{3,3}$ as a minor by Lemma 4.9. If G is a graph obtained by connecting one or more graphs at one of the degree 4 vertices, v, in the subdivided K_5 and at two degree 2 vertices on the same edge incident to v, then G contains T_7 as a minor.

If G is a graph obtained by attaching one or more connected graphs at any three or more degree 2 vertices on one edge of the original K_5, then the graph has T_7 as a minor. □

Lemma 4.15. *The graph G obtained by a combination of degree 2 vertex expansions and attaching graphs along a vertex, an edge, or along 3 or more vertices of the K_5 is not a minor minimally, non-planar intrinsically S^1 3-linked graph other than $K_{3,3}$, T_7, and T_9.*

Proof. Assume that G is a non-planar, minor minimally intrinsically S^1 3-linked graph other than $K_{3,3}$, T_7, and T_9. If one of the graphs that is attached is intrinsically S^1 linked or if one of the graphs is not intrinsically S^1 linked, but is intrinsically (x, y) neighborly linked for the vertices x and y being attached to an edge of the K_5, then by Lemma 4.11 G is not minor minimally intrinsically S^1 3-linked. Also by Lemmas 4.13 and 4.14, if a graph G' is attached at three or more vertices of the K_5, G contains $K_{3,3}$ or T_7 as a minor. Therefore, G must be obtained by degree 2 vertex expansions and adding graphs that are not intrinsically S^1 linked and not intrinsically (x, y)-neighborly linked for the vertices x and y being attached to an edge of the K_5. By the proof of Lemma 4.12, either G is not intrinsically S^1 3-linked or G has one of $R_8 + e$, T_8, and T_9 as a minor. The graph T_9 is the only one of these graphs that is a non-planar minor minimally intrinsically S^1 3-linked graph. □

Theorem 7.5 *The complete minor minimal set of non-planar intrinsically S^1 3-linked graphs is the set of the three graphs, $K_{3,3}$, T_7, and T_9.*

Proof. By Kuratowski's Theorem ([1]) we know that any non-planar graph contains $K_{3,3}$ or K_5 as a minor. Because $K_{3,3}$ is minor minimally intrinsically S^1 3-linked, any other non-planar, minor minimally intrinsically S^1 3-linked graphs contain K_5 as a minor, but do not contain $K_{3,3}$ as a minor. The graph K_5 itself is not intrinsically S^1 3-linked.

Any proper supergraph, G, of K_5 (that does not contain $K_{3,3}$ as a minor) is related to K_5 in one of the following ways: (i) G is obtained from a K_5 by degree 2 vertex expansions only, (ii) G has a graph attached at a vertex along the K_5, (iii) G has a graph attached along 2 vertices of the original K_5, (iv) G has a graph attached along 3 or more vertices of the original K_5, or (v) G has a combination of degree 2 vertices and graphs attached along a vertex, along 2 vertices, and along 3 or more vertices of the K_5.

First consider the supergraphs of K_5 that only have degree 2 vertex expansions. By Lemma 4.1, the graphs $R_8 + e$, T_8, and T_9 are minors of any graph that can be obtained from K_5 by degree 2 vertex expansion. Note that $R_8 + e$ and T_8 are not in the set of non-planar, minor minimally intrinsically S^1 3-linked graphs. The graph $R_8 + e$ has R_8 as a minor, which is planar and which is minor minimally intrinsically S^1 3-linked. The graph T_8 has D_8 as a minor, which is planar and which is minor minimally intrinsically S^1 3-linked. The graph T_9 is non-planar and minor minimally intrinsically S^1 3-linked.

Any supergraph of K_5 that has a cut vertex is not a non-planar, minor minimally intrinsically S^1 3-linked graph other than $K_{3,3}$, T_7, and T_9 by Lemma 4.3. All such graphs contain a planar, minor minimal intrinsically S^1 3-linked graph as a minor.

The only ways to attach a graph G' along 2 vertices in a subdivision of K_5 are to attach the G' along an edge of the original K_5, to attach G' to one vertex a of the original K_5 and to a degree 2 vertex on an edge that is not incident to a, to attach G' to one vertex of the original K_5, a and at a degree 2 vertex on an incident edge, to attach G' at two degree 2 vertices on the same edge of the original K_5, to attach G' at two degree 2 vertices on adjacent edges of the original K_5, to attach G' at two degree 2 vertices on non-adjacent edges of the original K_5, or to attach G' at two non-adjacent degree 4 vertices in a subdivided K_5.

Any supergraph of K_5 obtained by attaching a graph G' along an edge (a, b) of K_5 does not create a minor minimally non-planar intrinsically S^1 3-linked graph other than $K_{3,3}$, T_7, and T_9 by Lemma 4.7.

By Lemma 4.8, the graph G formed by attaching a graph G' to a K_5 at either two degree 2 vertices on the same edge of the K_5 (See Figure 13) or by attaching a graph G' at a vertex a in the K_5 and at a degree 2 vertex on the same edge ((a, b) of the original K_5(See Figure 14)) is not a non-planar, minor minimally intrinsically S^1 3-linked graph other than $K_{3,3}$, T_7, and T_9 .

By Lemma 4.9, the graph G formed by either attaching a graph G' or an edge to a K_5 at a vertex a in the K_5 and at a degree 2 vertex on a different edge (not containing a in the original K_5), at two degree 2 vertices on adjacent edges of the K_5, or at two degree 2 vertices on non-adjacent edges of the K_5, contains $K_{3,3}$ as a minor.

By Lemma 4.10, the graph G formed by either attaching a connected graph G' or an edge to a subdivided K_5 at non-adjacent degree 4 vertices contains T_7 as a minor.

By Lemma 4.13, the graph G obtained by attaching one or more graphs at three or more vertices of the original K_5 contains a $K_{3,3}$ as a minor.

By Lemma 4.14, the graph G obtained by attaching at least one graph at three or more vertices (possibly not in the original K_5) is not a non-planar, minor minimally intrinsically S^1 3-linked graph other than $K_{3,3}$, T_7, and T_9.

By Lemma 4.15, the graph G obtained by a combination of degree 2 vertex expansions and attaching graphs along a vertex, along 2 vertices, and along 3 or more vertices in the K_5 is not a minor minimally, non-planar intrinsically S^1 3-linked graph other than $K_{3,3}$, T_7, and T_9. □

References

[1] K. Kuratowski. *Sur le probleme des courbes gauches en topologie,* Fund. Math. **15** (1930), 271-283.

[2] A. Brouwer, R. Davis, A. Larkin, D. Studenmund, and C. Tucker. *Intrinsically S^1 3-linked graphs and other aspects of S^1 embeddings*, preprint.