Intrinsically S^1 3-linked graphs and other aspects of S^1 embeddings

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Abstract

A graph can be embedded in various spaces. This paper examines S^1 embeddings of graphs. Just as links can be defined in spatial embeddings of graphs, links can be defined in S^1 embeddings. Because linking properties are preserved under vertex expansion, there exists a finite complete set of minor minimal graphs such that every S^1 embedding contains a non-split 3-link. This paper presents a list of minor minimal intrinsically S^1 3-linked graphs, along with methods used to find and verify the list, in hopes of obtaining the complete minor minimal set. Other aspects of S^1 embeddings are also examined.

1 Introduction

Any two disjoint circles embedded in space form a link. This link is said to be *splittable*, or just *split*, if the circles can be pulled apart without passing through each other. More precisely, a link L is split if there is an embedding of a 2-sphere F in $\mathbb{R}^3 - L$ such that each component of $\mathbb{R}^3 - F$ contains at least one component of L. A spatial embedding of a graph is *linked* if there is a pair of cycles that form a non-split link. A graph is *intrinsically linked* if every spatial embedding of the graph is linked. Conway and Gordon in [4] and Sachs in [11] show that the graph K_6 is intrinsically linked.

A graph H is a *minor* of a graph G if it can be obtained from G by a finite sequence of edge deletions, edge contractions, and vertex deletions. A graph G is said to be *minor minimal* with respect to a property if G has that property but no minor of G has that property. One can then consider the *complete minor minimal set* of graphs with respect to a property, that is, the set of all graphs with a given property such that no minor of these graphs has that property. In [9], Robertson and Seymour show that if a graph G has a certain property when any one of its minors has that property, then the complete set of minor minimal graphs with respect to that property is finite. For example, it is shown by Robertson, Seymour, and Thomas in [10] that the Petersen Family of graphs, those obtained by a series of $\triangle$-Y and Y-$\triangle$ exchanges on K_6, form the complete minor minimal set of intrinsically linked graphs.

In this paper we consider not spatial embeddings of graphs but rather S^1 embeddings of graphs. An S^1 embedding of a graph G is an injective map of the vertices of G into S^1. Each distinct S^1 embedding of a graph can be seen as a permutation of the vertices; the order of the vertices will determine the embedding. To help in describing the arrangement of vertices in an embedding, a vertex a is said to *neighbor* vertex b in an S^1 embedding of a graph G if there is a component of $S^1 - \{a, b\}$ that does not contain any vertices of G.

A *0-sphere* in an S^1 embedding of a graph G is composed of any two vertices that are the endpoints of a simple path in G. In this paper a 0-sphere is denoted by writing the vertices of the path in an ordered n-tuple. Just as a pair of disjoint cycles forms a link in a spatial embedding, a pair of disjoint 0-spheres forms a link in an S^1 embedding. A link (a, b) and (c, d) is said to be *split* if a and b lie on the same component of $S^1 - \{c, d\}$. Thus the link is *non-split* if a and b lie on different components of $S^1 - \{c, d\}$. For S^1 embeddings, the *mod 2 linking number* of two 0-spheres (a, b) and (c, d), denoted $lk_2((a, b), (c, d))$, is 0 if and only if (a, b) and (c, d) are split linked and is 1 if and only if (a, b) and (c, d) are non-split linked.

Just as some graphs are intrinsically linked in space, some graphs are intrinsically S^1 linked. A graph is intrinsically S^1 linked if every S^1 embedding contains a non-split link. It is shown by Cicotta et al. in [3] that the complete minor minimal set of intrinsically S^1 linked graphs is K_4 and $K_{3,2}$.

With the complete set of minor minimal intrinsically S^1 linked graphs known, we can look at other linking properties. One such property is the S^1 3-linking property. A graph is said to be intrinsically *n-linked* if every embedding has a non-split link of n components. The spatial analog of this property has been examined with limited success. In [6], Flapan, Naimi, and Pommersheim prove that K_{10} is the smallest complete graph to be intrinsically 3-linked. In [2], Bowlin and Foisy show that K_{10} is not minor minimal by exhibiting two subgraphs that are intrinsically 3-linked. In [8], O'Donnol shows that the complete bipartite graph $K_{2n,2n}$ is the smallest bipartite graph that is intrinsically n-linked. Each of these results comes in small steps, slowly building on the results before them.

The problem of identifying intrinsically n-linked graphs in space is difficult. To this end, we look at n-links in S^1 embeddings of graphs. An S^1 *n-link* in an S^1 embedding of a graph G is a set of n disjoint 0-spheres in the embedding of G. An n-link in an S^1 embedding of a graph G is said to be *split* if there are two points, x and y, on the circle such that both components of $S^1 - \{x, y\}$ contain at least one vertex involved in the n-link and every 0-sphere in the link lies entirely on one component of $S^1 - \{x, y\}$. A graph is said to be intrinsically S^1 n-linked if every S^1 embedding of the graph contains a non-split n-link. The primary concern of this paper is to investigate 3-links in S^1 embeddings of graphs. The goal of our research was to find the complete set of minor minimal intrinsically S^1 3-linked graphs. However, it is worth noting that many of our lemmas can be extended to hold for n-links. Although we present graphs that are minor minimal with respect to the intrinsic S^1 3-linking property, many of our methods can be used to find graphs that are n-linked for n greater than 3.

Another set of graphs, examined in [1] by Archdeacon et al., is the set of non-outer-cylindrical graphs. Non-outer-cylindrical graphs are defined as graphs that cannot be embedded in the plane so that there are at most two distinct faces whose boundaries contain all of the vertices. This set of graphs is of interest to this paper because it is very similar to our set of intrinsically S^1 3-linked graphs. Further, some of our proofs were inspired by this paper's methods.

In this paper, we will begin by discussing two graph operations, vertex expansion and $\triangle - Y$ exchange. We will show that both operations preserve certain S^1 linking properties. Further, we will prove several theorems and lemmata that were useful in our research. Then, we will present the set of graphs that we determined to be minor minimal intrinsically S^1 3-linked as well as some of the proofs that led to our result. Finally we will discuss two other varieties of S^1 linking: intrinsic n-links in $K_{n,n}$ and intrinsic pairwise non-split 3-links.

Note that in this paper we will often use "link" and similar terms such as "3-linked" to mean "non-split link" and so on. Further, in our pictures, we draw chords to represent edges in the S^1 embeddings. These chords are not actually part of the embedding, but they act as visual reminders.

2 Vertex Expansion Preserves S^1 Linking Properties

In [3], Cicotta et al. prove that vertex expansion preserves the intrinsic S^1 linking property. In order to show that the complete minor minimal set of intrinsically S^1 3-linked graphs is finite, it is necessary to show that vertex expansion also preserves intrinsic S^1 3-linking.

Theorem 2.1. *The operation vertex expansion preserves the intrinsic S^1 3-linking property.*

Proof. Let G be a graph with the intrinsic S^1 3-linking property, and let G' be the graph obtained by expanding the vertex v of G into vertices v_1 and v_2. Consider an arbitrary embedding of G'. We will show that there exists a 3-link in this embedding. Obtain an embedding of G from this embedding of G' by removing vertex v_2 and relabeling v_1 as v. Then, create an edge to v from each vertex y such that y was adjacent to v_2 in G'. This results in a vertex contraction of v_1 and v_2 to v. Therefore, we now have an embedding of G, and it must thus contain a 3-link.

Case 1: Suppose that v is not involved in the link. Then the same non-split 3-link exists in our original embedding of G'.

Case 2: Suppose that v is involved in the link. Let $(w_1, \ldots, w_i, v, w_{i+1}, \ldots, w_n)$ be the 0-sphere containing v that is a component of a link. Expand vertex v into vertices v_1 and v_2. There are four possibilities: either v_1 is adjacent to both w_i and w_{i+1}, v_1 is adjacent to w_i and v_2 is adjacent to w_{i+1}, v_1 is adjacent to w_{i+1} and v_2 is adjacent to w_i, or v_2 is adjacent to both w_i and w_{i+1}. In any of these cases, there is a path from w_i to w_{i+1} using at most vertices v_1 and v_2. Thus there is a 0-sphere from w_1 to w_n that is still disjoint from the other 0-spheres involved in the original non-split link, so the original non-split link exists.

In each case, there exists a 3-link in the S^1 embedding of G'. As this S^1 embedding of G' was arbitrary, there is a 3-link in every S^1 embedding. Thus, the operation vertex expansion preserves the intrinsic S^1 3-linking property. □

3 $\triangle - Y$ Exchange Preserves S^1 Linking Properties

The operation $\triangle - Y$ exchange is very important in our research. Because, as we will prove, $\triangle - Y$ exchange preserves intrinsic S^1 3-linking, we can construct families of graphs that are related by $\triangle - Y$ exchange. If the first graph (the one on which $\triangle - Y$ exchange is first performed) is intrinsically S^1 3-linked, then the entire family is as well.

Theorem 3.1. *The operation $\triangle - Y$ exchange preserves intrinsic S^1 3-linking.*

Proof. Let G be a graph that is intrinsically S^1 3-linked and contains at least one 3-cycle. Let G' be a graph obtained from G by a $\triangle - Y$ exchange. We will show that G' is intrinsically S^1 3-linked.

Consider an arbitrary S^1 embedding of G'. Consider the associated S^1 embedding of G that results from deleting vertex v and creating edges (a, b),(b, c), and (a, c). As G is intrinsically S^1 3-linked, there exists a 3-link in this S^1 embedding of G.

Case 1: The 0-spheres (a, b),(b, c), and (c, a) are not involved in the 3-link. Thus, the same 3-link exists in the S^1 embedding G' as exists in this S^1 embedding of G.

Case 2: One of the 0-spheres (a, b),(b, c), or (c, a) is involved in the 3-link. Note that at most one of these 0-spheres can be involved in the 3-link. Without loss of generality, the 0-sphere (a, b) is involved in the link. The other two 0-spheres of the 3-link will not be affected by the $\triangle - Y$ exchange. The 0-sphere from a to b will still exist in G' through the path (a, v, b). This path is disjoint from the other two 0-spheres in the link. The 3-link is unchanged except for the replacement of (a, b) with (a, v, b). Thus, there exists a 3-link in the S^1 embedding G'.

In every case, there exists a 3-link in the S^1 embedding of G'. As the embedding of G' was arbitrary, there exists a 3-link in every S^1 embedding of G'. Thus, the operation $\triangle - Y$ exchange preserves intrinsic S^1 3-linking. $\square$

An interesting question we considered was whether $\triangle - Y$ exchange preserves minor minimality. It turns out that it does not preserve this property.

However, the reverse operation, $Y - \triangle$ exchange, does preserve minor minimality under certain conditions.

Theorem 3.2. *Let Q be a property of a graph that is preserved under $\triangle - Y$ exchange. Let G be a graph that contains at least one degree three vertex and has property Q. Let G' be a graph obtained from G by a $Y - \triangle$ exchange. If G is minor minimal with respect to Q and if G' has property Q, then G' is also minor minimal with respect to Q.*

Proof. Suppose G as defined is minor minimal with respect to property Q and that G' has property Q. Suppose, for the sake of contradiction, that G' is not minor minimal with respect to property Q. Then there exists a graph H', which is a minor of G' and has property Q.

Case 1: Suppose that the triangle created in G' by a $Y - \triangle$ exchange of G is present in H'. Consider the graph H obtained from H' by $\triangle - Y$ exchange. As property Q is preserved under $\triangle - Y$ exchange and as H' has property Q, H also has property Q. However, H is a minor of G. As G is minor minimal with respect to Q, H cannot have property Q. This is a contradiction.

Case 2: Suppose that H' is obtained from G' in such a way that the triangle created in G' by a $Y - \triangle$ exchange of G is not wholly present in H'. Then H' is a minor of G. However, as G is minor minimal with respect to Q, H' cannot have property Q. This is a contradiction.

In each case we have reached a contradiction. Thus, G' is minor minimal with respect to Q. □

This result is interesting in that Q is not necessarily constrained to S^1 properties nor even to linking properties. This theorem is valid for any property preserved by $\triangle - Y$ exchange. Further, before this result, proofs of minor minimality could only be performed by tedious case checking. Now, if the last graph of a graph family (the graph on which $\triangle - Y$ exchange can no longer be performed) is proved to be minor minimal with respect to Q, all preceding graphs in that family are also minor minimal with respect to Q.

4 Preliminary Theorems

During the course of our research, we proved and then used various theorems and lemmata. The first theorem is an S^1 analog of a linking lemma in S^3, which may be found in [6]. Given two links and a set of conditions, it is possible to determine the existence of a 3-link without specifically knowing how the three link occurs. This theorem was useful in proving that graphs had the intrinsic S^1 3-linking property. Later in the paper, we will use this theorem to prove a result about graphs with a cut vertex.

Theorem 4.1. *Consider an S^1 embedding of a graph G with 0-spheres (x, y), (x, z), (s, t), and (p, q), such that $(s, t), (x, y)$, and (p, q) are mutually disjoint, $(s, t), (x, z)$, and (p, q) are mutually disjoint, and (x, y) and (x, z) intersect at vertex x. If $lk_2((s, t), (x, y)) = 1$ and $lk_2((p, q), (x, z)) = 1$, then one of the following is an S^1 3-link:*

1. $(s, t), (x, y), (p, q)$
2. $(s, t), (x, z), (p, q)$
3. $(s, t), (y, x, z), (p, q)$

Lemma 4.2. *Given 0-spheres $(a, b), (c, d), (c, e)$, and (d, e) in an S^1 embedding of graph G, $lk_2((a, b), (c, e)) = lk_2((a, b), (c, d)) + lk_2((a, b), (d, e))$.*

Proof. Consider an S^1 embedding of G with 0-spheres $(a, b), (c, d), (c, e)$, and (d, e). Observe that any 0-sphere disjoint from $(c, d), (d, e)$, and (c, e), e.g. (a, b), that forms a non-split link with at least one of $(c, d), (d, e)$, and (c, e) will form a non-split link with two and only two of $(c, d), (d, e)$, and (c, e).

Case 1: Suppose (a, b) does not link with $(c, d), (d, e)$, or (c, e). In this case, $lk_2((a, b), (c, e)) = lk_2((a, b), (c, d)) + lk_2((a, b), (d, e))$ follows, as $lk_2((a, b), (c, d)) = 0$, $lk_2((a, b), (d, e) = 0$, $lk_2((a, b), (c, e)) = 0$ and $0 \cong (0 + 0)_{mod2}$.

Case 2: Suppose (a, b) links with (c, d) and (d, e). In this case, $lk_2((a, b), (c, e)) = lk_2((a, b), (c, d)) + lk_2((a, b), (d, e))$ follows, as $lk_2((a, b), (c, d)) = 1$, $lk_2((a, b), (d, e) = 1$, $lk_2((a, b), (c, e)) = 0$ and $0 \cong (1 + 1)_{mod2}$.

Case 3: Suppose (a, b) links with (c, d) and (c, e). In this case, $lk_2((a, b), (c, e)) = lk_2((a, b), (c, d)) + lk_2((a, b), (d, e))$ follows, as $lk_2((a, b), (c, d)) = 1$, $lk_2((a, b), (d, e) = 0$, $lk_2((a, b), (c, e)) = 1$ and $1 \cong (1 + 0)_{mod2}$.

Case 4: Suppose (a, b) links with (d, e) and (c, e). This case follows as Case 3.

In each case, $lk_2((a, b), (c, e)) = lk_2((a, b), (c, d)) + lk_2((a, b), (d, e))$. □

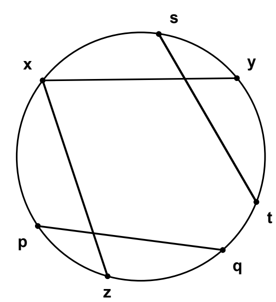

Figure 1: An S^1 embedding illustrating Theorem 4.1

Proof. (Of Theorem 4.1) Consider an S^1 embedding of a graph G with 0-spheres (x, y), (x, z), (s, t), and (p, q), such that $(s, t), (x, y)$, and (p, q) are mutually disjoint, $(s, t), (x, z)$, and (p, q) are mutually disjoint, and (x, y) and (x, z) intersect at vertex x. Assume $lk_2((s, t), (x, y)) = 1$, and $lk_2((p, q), (x, z)) = 1$.

Suppose that both $(s, t), (x, y), (p, q)$ and $(s, t), (x, z), (p, q)$ do not form 3-links as in Figure 1. Then, $lk_2((x, y), (p, q)) = 0$ and $lk_2((s, t), (x, z)) = 0$. By Lemma 4.2, $lk_2((s, t), (y, z)) = lk_2((s, t), (x, y)) + lk_2((s, t), (x, z)) \cong (1 + 0)_{mod2} \cong 1_{mod2} \cong 1$. Also, by Lemma 4.2, $lk_2((p, q), (y, x, z)) = lk_2((p, q), (x, y)) + lk_2((p, q), (x, z)) \cong (0 + 1)_{mod2} \cong 1_{mod2} \cong 1$. Thus $(s, t), (y, x, z)$, and (p, q) form a 3-link. Hence the result. $\square$

When looking for graphs that are intrinsically S^1 3-linked, it is helpful to be able to quickly identify graphs that are not S^1 3-linked. In essence, determining that a graph is not intrinsically S^1 3-linked relies on finding an embedding in which a 3-link is not present. The following theorem uses properties of certain kinds of graphs to provide 3-linkless embeddings.

Theorem 4.3. *Let G' be a graph that is not intrinsically S^1 3-linked and G'' be a graph that is not intrinsically S^1 linked. Let G be the graph formed by pasting G' and G'' at a vertex v. Then G is not intrinsically S^1 3-linked.*

Proof. It will suffice to produce an S^1 embedding of G without a 3-link. Begin with a 3-linkless embedding of G'. Let one of the vertices neighboring v be called x. Now embed G'' in a different component of $S^1 - \{v, x\}$ than the vertices of $G' - \{v, x\}$ such that the embedding of G'' is linkless. This is an S^1 embedding of G. It only remains to be shown that the embedding is 3-linkless; that is, every set of three disjoint 0-spheres forms a split 3-link.

Consider an arbitrary set of three disjoint 0-spheres in this embedding. If all three are contained entirely in G', then they do not form a 3-link because the embedding of G' is 3-linkless. If two 0-spheres are contained in G' and one is contained in G'', or one 0-sphere is contained in G' and two are contained in G'', then they do not form a 3-link because every link containing a 0-sphere in each of G' and G'' is split. Finally, if all three 0-spheres are contained in G'', then there is no 3-link because this embedding of G'' is linkless.

Consider the case that one of the 0-spheres contains vertices in both $G' - \{v\}$ and $G'' - \{v\}$. There is at most one such 0-sphere. Call this 0-sphere $(v_1, \ldots, v_i, v, v_{i+1}, \ldots, v_n)$ where $\{v_1, \ldots, v\} \in G'$ and $\{v, \ldots, v_n\} \in G''$. There are two subcases. First consider the case in which at least one of the other two 0-spheres lies entirely in G''. That 0-sphere forms split links with both $(v_1, \ldots, v)$ (because each link containing a 0-sphere in each of G' and G'' is split) and $(v, \ldots, v_n)$ (because G'' is linkless). Thus, it forms a split link with $(v_1 \ldots, v_n)$. Similarly, that 0-sphere also forms a split link with the remaining 0-sphere. Therefore, the link formed by these three 0-spheres is a split 3-link.

Now consider the case in which the remaining two 0-spheres lie in G'. These 0-spheres do not form a 3-link with $(v_1, \ldots, v)$ because this embedding of G' is 3-linkless. Further, $(v, \ldots, v_n)$ does not form a link with either 0-sphere. Therefore, there is no 3-link. □

Another similar theorem follows. The proof is omitted due to its similarity with the proof of Theorem 4.3.

Theorem 4.4. *Let G' be a graph that is not intrinsically S^1 3-linked and G'' be a graph that is not intrinsically S^1 linked. Let G be the graph formed by pasting G' and G'' along a single edge, (a, b), where the edge (a, b) is in G' and G''. If there exists a 3-linkless S^1 embedding of G' and a linkless S^1 embedding of G'' such that a and b are neighbors in both embeddings, then G is not intrinsically S^1 3-linked.*

Here we present several other theorems that we proved in the course of our research and were helpful in pursuing our results. We have omitted the proofs of Theorem 4.7 and Theorem 4.8 because they are overly technical and do not provide insight.

Theorem 4.5. *If a graph G is intrinsically S^1 3-linked and v and w are any two vertices in G, then $G - \{v, w\}$ contains a cycle or a vertex of degree at least three.*

Proof. We will prove the contrapositive. Consider a graph G as defined above. Suppose $G - \{v, w\}$ does not contain a cycle or a vertex of degree 3 or greater. Then $G - \{v, w\}$ can be embedded in $\mathbb{R}^1$. As $G - \{v, w\}$ can be embedded in $\mathbb{R}^1$, it can be embedded into S^1 in such a way that any vertex is adjacent only to vertices that it neighbors. (Note that neighboring does not imply adjacency.) Also, as there is no cycle in $G - \{v, w\}$, there exist neighboring vertices a and b such that a and b are not adjacent. Embed vertices v and w into the S^1 embedding of $G - \{v, w\}$ as neighbors such that v neighbors a and w, and w neighbors x and b. Any set of three disjoint 0-spheres contains a 0-sphere without v or w in its associated path, call it $(p, \ldots, q)$. Because each vertex in the path $\{p, \ldots, q\}$ is only adjacent to its neighbors, there is a component of $S^1 - \{p, q\}$ that contains only vertices in the associated path of $(p, \ldots, q)$, and therefore cannot be involved in a non-split link. Thus, in this embedding, any set of three disjoint 0-spheres forms a split link. Therefore, G is not intrinsically S^1 3-linked. Hence, the result by contrapositive. □

Theorem 4.6. *Let G be a minor minimal graph with the intrinsic S^1 3-linking property. G does not contain a 3-cycle with degree 2 vertex.*

Proof. Suppose that G contains a 3-cycle with degree 2 vertex. Let c be one such vertex, and let a and b be the other two vertices that form the 3-cycle. Consider an arbitrary embedding of G. In this embedding, either the 0-sphere (a, b) is a component of a 3-link, or it is not. If it is not, then its deletion will not affect the 3-link. If (a, b) is a component of a 3-link, note that the edges (a, c) and (b, c) are not disjoint from (a, b) and are thus not part of the 3-link. Thus, vertex c is disjoint from the 3-link. Delete edge (a, b) from this embedding. The 3-link is unchanged except for the replacement of 0-sphere (a, b) with (a, c, b). So any S^1 embedding of G with edge (a, b) deleted has a 3-link. This is a contradiction as G is minor minimal with respect to the intrinsic S^1 3-linking property. Thus no minor minimal intrinsically S^1 3-linked graph contains a 3-cycle with a degree 2 vertex. □

Theorem 4.7. *There are no minor minimal intrinsically S^1 3-linked graphs with a degree two vertex adjacent to two degree two vertices.*

Theorem 4.8. *No planar graph with six vertices is intrinsically S^1 3-linked.*

5 The Graphs

Using these results, we were able to prove that the graphs in Figures 2 and 3 are minor minimal intrinsically S^1 3-linked graphs. Further, the non-planar graphs shown in Figure 3 are the complete set of non-planar minor minimal S^1 3-linked graphs. The proof is overly long and tedious, and we will omit it. Note that in Figures 2 and 3 the arrows represent $\triangle - Y$ exchange.

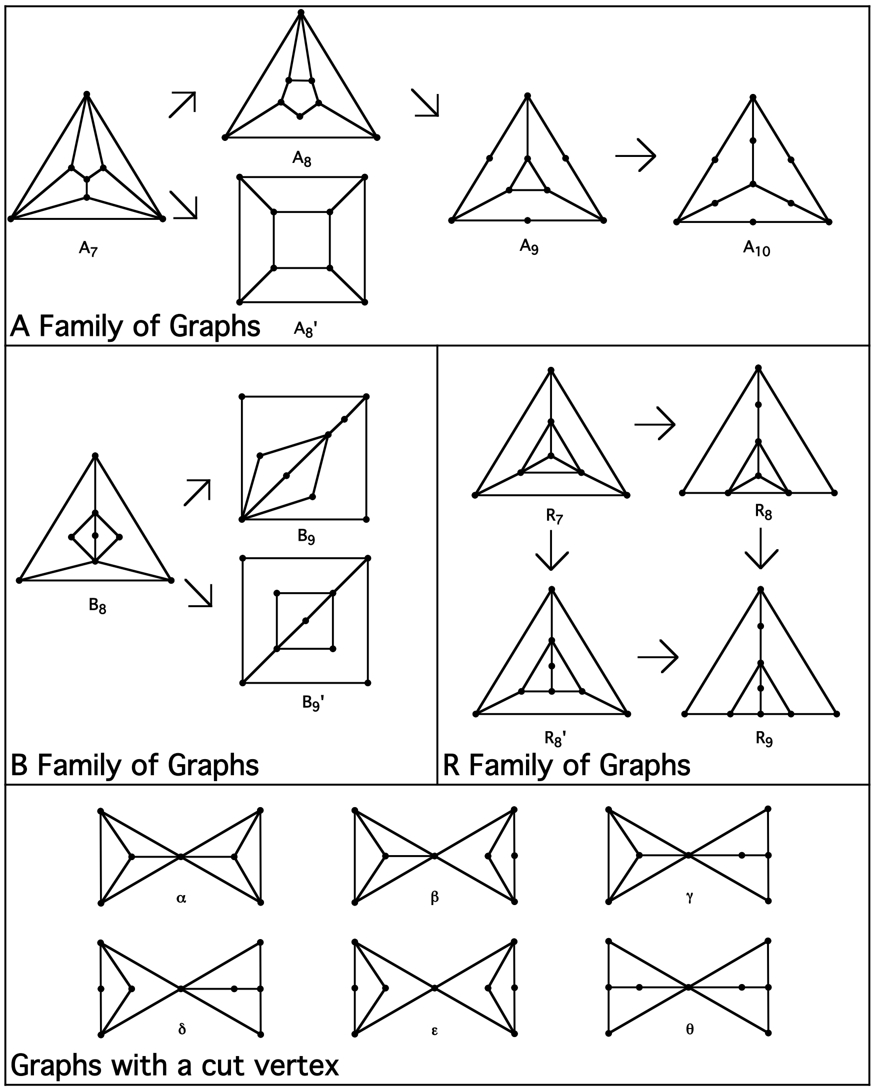

Figure 2: Minor minimal intrinsically S^1 3-linked graphs

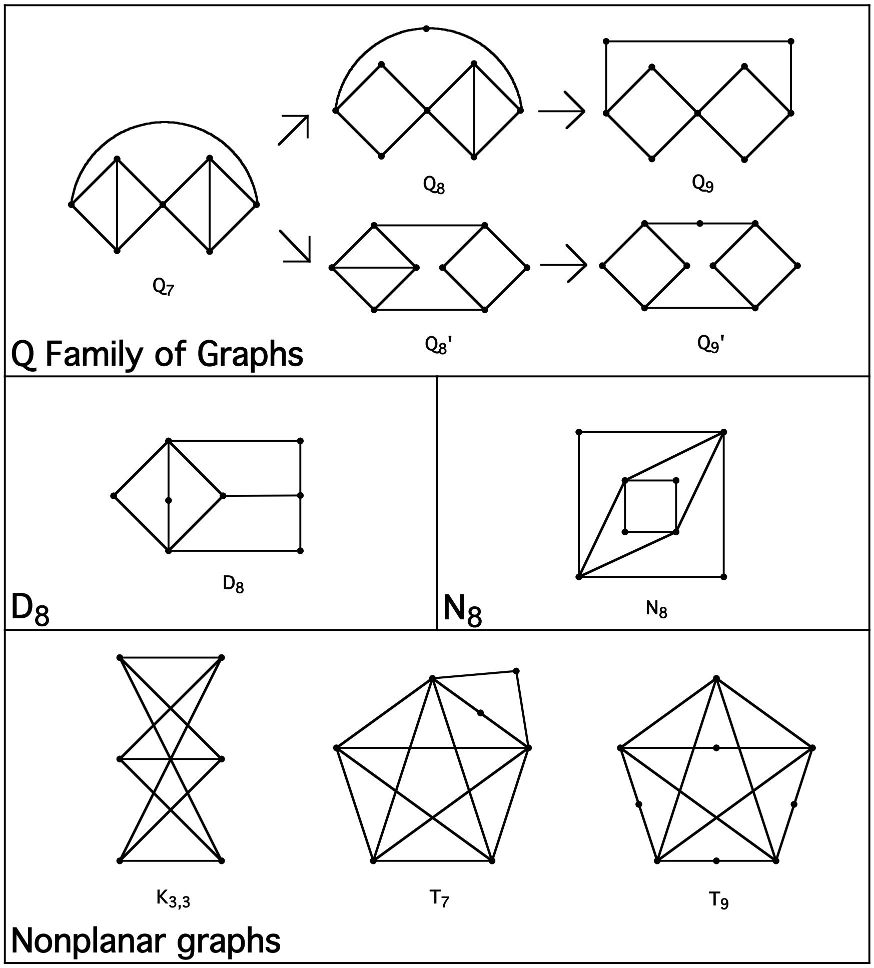

Figure 3: Minor minimal intrinsically S^1 3-linked graphs, continued

6 Selected Proofs

Although we have rigorously shown that each of the listed graphs is minor minimal intrinsically S^1 3-linked, we do not present every proof here. Many of the proofs are a result of tedious case checking, which is not insightful. We will, however, include the proof that the graph R_7 is intrinsically S^1 3-linked as a short example of such a proof. We also will show that the graphs $\alpha, \beta, \gamma, \delta, \varepsilon$, and θ are intrinsically S^1 3-linked, and that they are the only minor minimal intrinsically S^1 3-linked graphs with a cut vertex.

Refer to Figure 4 for the labeling of the vertices used in the proof that R_7 is intrinsically S^1 3-linked.

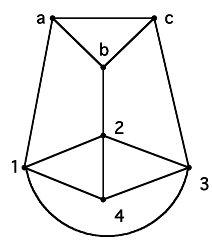

Figure 4: The graph R_7

Theorem 6.1. *The graph R_7 is intrinsically S^1 3-linked.*

Proof. R_7 contains K_4 (vertices 1-4 in Figure 4) as a subgraph. Embed the K_4 in S^1. There is, up to symmetry, only one embedding of the K_4 subgraph. This embedding shown in Figure 5. Then consider the possible placement of vertices a, b, and c. If any of the two of these vertices are in different components of $S^1 - \{1, 2, 3, 4\}$, they will form a 3-link with $(1, 3)$ and $(2, 4)$. Now consider the case that all three vertices are in one component of $S^1 - \{1, 2, 3, 4\}$. Let x denote the vertex of a, b, c that neighbors both others. This vertex is adjacent to one of vertices 1, 2, 3, or 4. Thus there is an edge (x, y) for some $y \in \{1, 2, 3, 4\}$ that forms a non-split link with the edge (w, z) where w and z are the neighbors of x. Note also that y is an endpoint of the non-split link $\{(1, 3), (2, 4)\}$. Then by Theorem 4.1 there is a 3-link. □

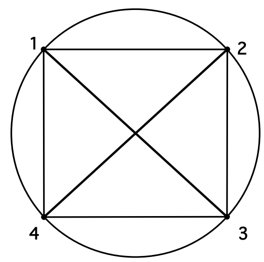

Figure 5: An embedding of vertices 1-4 of R_7

One simple way to form intrinsically S^1 3-linked graphs is by pasting together two intrinsically S^1 linked graphs at one vertex, forming a graph with a cut vertex. The 3-link in the final graph will be composed of the two 2-links in each of the component graphs. This will follow from Theorem 4.1. Because there are only two minor minimal intrinsically S^1 linked graphs, there are only a small number of ways to paste them together to get a minor minimal intrinsically S^1 3-linked graph.

Theorem 6.2. *Let G be a graph formed by pasting together graphs A and B, where A and B are each either a K_4 or $K_{3,2}$, at a vertex. The graph G is intrinsically S^1 3-linked.*

Proof. Given an embedding of G in which the cut vertex is the endpoint of a 0-sphere in a link in both subembeddings of A and B, then G is 3-linked by Theorem 4.1. Thus it will suffice to show that in every embedding of K_4 and $K_{3,2}$, each vertex can be written as the endpoint of a 0-sphere in a link. For K_4, this follows immediately. For $K_{3,2}$ it requires a little more work. Let the vertices of the $K_{3,2}$ be labeled as $\{1, 2, a, b, c\}$, where the lettered vertices are adjacent to each of the numbered vertices. In each embedding of this graph there is a link, say, without loss of generality, $(1, a)$ and $(2, b)$. It remains to be shown that c can be written as the endpoint of a 0-sphere in a link. If c is on the same component of $S^1 - \{2, b\}$ as 1, then $(c, 1, a)$ and $(2, b)$ form a link. If c is on the same component of $S^1 - \{2, b\}$ as a, then $(1, c)$ and $(2, b)$ form a link. □

Further, these graphs are minor minimal. This will follow from Theorem 4.3.

Theorem 6.3. *Each of the above graphs with a cut vertex is minor minimal with respect to intrinsic S^1 3-linking.*

Proof. Let the G be the composite graph as defined in the previous theorem. Consider any edge contraction, edge removal, or vertex removal performed on G. Without loss of generality, performing any of those three operations on G to obtain G' is equivalent to performing the operation on A to produce A' and then pasting A' and B to get G'. But because A is minor minimal with respect to intrinsic S^1 linking, A' must have an S^1 embedding without a non-split link. Then, by Theorem 4.3, there exists a 3-linkless S^1 embedding of G'. □

Now it just remains to show that these are the only minor minimal intrinsically S^1 3-linked graphs that have a cut vertex. This will follow from Theorems 4.1 and 4.3.

Theorem 6.4. *If a graph G is minor minimal intrinsically S^1 3-linked and has a cut vertex v, then G is composed of two minor minimal intrinsically S^1 linked graphs pasted at v.*

Proof. Denote the components of $G - \{v\}$ as $\{X_i\}_{i=1}^n$. Then let H_i be the subgraph of G such that $H_i = G - \sum_{k \neq i} X_k$. Note that each H_i is intrinsically S^1 linked; if H_i were not intrinsically linked, then by Theorem 4.3 $G - H_i$ is intrinsically S^1 3-linked, which is a contradiction. Given that each H_i is intrinsically S^1 linked, any H_i and H_j form an intrinsically S^1 3-linked graph by Theorem 4.1. Therefore, G must be precisely $H_i \amalg H_j$ because G is minor minimal. Therefore G is composed of exactly two intrinsically S^1 linked graphs, denoted H and H'.

It remains to show that both H and H' are minor minimal with respect to the intrinsic S^1 linking property. For the sake of contradiction, assume there is a minor, $\bar{H}$ of H that is intrinsically S^1 linked. Then consider the graph formed by pasting $\bar{H}$ and H' at a vertex. By Theorem 4.1 this graph is intrinsically S^1 3-linked. But this graph is a minor of G, so this is a contradiction. □

7 n-Component Links in S^1

Another property considered is intrinsic S^1 n-linking. A graph G is intrinsically S^1 n-linked if every S^1 embedding of G contains a non-split link of

n-components. We will show that the graph $K_{n,n}$ is intrinsically S^1 n-linked for $n \geq 3$.

Theorem 7.1. *The graph $K_{n,n}$ is intrinsically S^1 n-linked for all $n \geq 3$*

Lemma 7.2. *Given an S^1 embedding of $K_{n,n}$ for some $n > 3$ such that adjacent vertices a and b of $K_{n,n}$ are not neighbors, if $K_{n-1,n-1}$ is intrinsically $(n-1)$-linked, then the 0-sphere (a, b) is a component of an n-link in the S^1 embedding of $K_{n,n}$.*

Proof. Consider an S^1 embedding of $K_{n,n}$ for some $n > 3$ such that adjacent vertices a and b of $K_{n,n}$ are not neighbors. Suppose that $K_{n-1,n-1}$ is intrinsically $(n-1)$-linked. Then, the S^1 embedding of $K_{n-1,n-1}$ obtained by deleting a and b from $K_{n,n}$ has a non-split link of $(n-1)$-components, denoted L. If all of the components of L lie in the same component of $S^1 - \{a, b\}$, then a and b are neighbors. As a and b are not neighbors, this is a contradiction. Then there is a component of L that is non-split linked with (a, b). Thus the 0-sphere (a, b) is a component of an n-component link in the S^1 embedding of $K_{n,n}$.

□

Proof. (Of Theorem 7.1) We will prove by induction for $n \geq 3$.

For the base case, let $n = 3$. Up to symmetry, there are three distinct S^1 embeddings of $K_{3,3}$. As there exists a 3-link in each of these S^1 embeddings, as seen in Figure 6, $K_{3,3}$ is intrinsically S^1 3-linked.

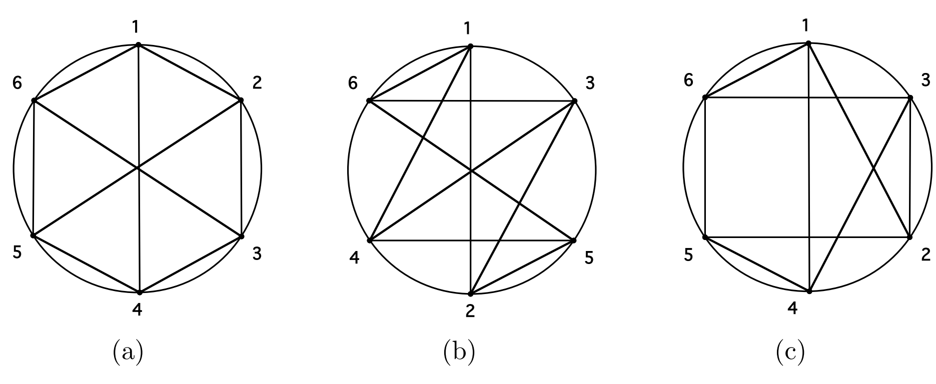

Figure 6: The three S^1 embeddings of $K_{3,3}$, up to symmetry

For our induction step, given $n > 3$, we will assume that $K_{n-1,n-1}$ is intrinsically S^1 $(n-1)$-linked.

Consider an arbitrary S^1 embedding of $K_{n,n}$. As each vertex in $K_{n,n}$ has degree n, where $n \geq 3$, each vertex is adjacent to at least one non-neighboring vertex. Then, consider adjacent non-neighboring vertices a and b. By Lemma 7.2, as $K_{n-1,n-1}$ is intrinsically $S^1(n-1)$-linked (by our assumption), the 0-sphere (a, b) is a component of an n-component link in the S^1 embedding of $K_{n,n}$. As this was an arbitrary S^1 embedding of $K_{n,n}$, every S^1 embedding contains a non-split link of n-components. Therefore, $K_{n,n}$ is intrinsically S^1 n-linked.

By the principle of mathematical induction, as our base case and induction step are true, $K_{n,n}$ is intrinsically S^1 n-linked for all $n \geq 3$. □

8 Intrinsically Pairwise Non-Split S^1 3-Linked Graphs

A *pairwise non-split 3-link* is a three component link in which each component of the link is non-split linked with each of the other two components. In [5], Flapan et al. describe a graph that is intrinsically pairwise non-split 3-linked in space. In this section we are concerned with graphs that are pairwise non-split 3-linked in S^1. We will prove that the graphs K_6 and $K_{3,3,1}$ are intrinsically pairwise non-split S^1 3-linked and that the property is preserved by $\triangle - Y$ exchange. From this, we will show that the graphs of the Petersen Family are all intrinsically pairwise non-split S^1 linked. Finally, using this result we show that intrinsically linked in space implies pairwise non-split 3-linked in S^1.

Theorem 8.1. *The graph K_6 is intrinsically pairwise non-split S^1 3-linked.*

Proof. There is only one distinct S^1 embedding of K_6, as seen in Figure 7.

There is a pairwise non-split 3-link in this embedding. Thus, in every S^1 embedding there is a pairwise non-split 3-link. So K_6 is intrinsically pairwise non-split S^1 3-linked. □

Not only is K_6 intrinsically pairwise non-split S^1 3-linked, but it is also minor minimal with respect to this property. The proof is simple and we will omit it.

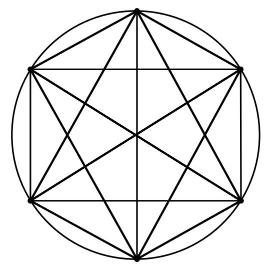

Figure 7: The S^1 embedding of K_6

Theorem 8.2. *The graph $K_{3,3,1}$ is intrinsically pairwise non-split S^1 3-linked.*

Proof. Consider the S^1 embeddings of the graph $K_{3,3}$. There are three cases up to symmetry. Refer back to Figure 6 for the different embeddings.

Case 1: Embed $K_{3,3}$ as in Figure 6(a).

As can be seen, this embedding has a pairwise non-split 3-link. This link will exist no matter where the last vertex of the $K_{3,3,1}$ is embedded.

Case 2: Embed $K_{3,3}$ as in Figure 6(b).

As can be seen, this embedding has a pairwise non-split 3-link. This link will exist no matter where the last vertex of the $K_{3,3,1}$ is embedded.

Case 3: Embed $K_{3,3}$ as in Figure 6(c)

There are now four possible places, up to symmetry, for the last vertex of the $K_{3,3,1}$ to be embedded. Refer to Figure 8 for the following cases.

Case 3a: Embed the last vertex (vertex 7) as in Figure 8(a).

The 0-spheres $(1,4)$, $(2,7)$, and $(5,6,3)$ form a pairwise non-spit 3-link.

Case 3b: Embed the last vertex (vertex 7) as in Figure 8(b).

The 0-spheres $(1,4)$, $(3,7)$, and $(2,5)$ form a pairwise non-spit 3-link.

Case 3c: Embed the last vertex (vertex 7) as in Figure 8(c).

The 0-spheres $(1,4)$, $(6,7)$, and $(2,5)$ form a pairwise non-spit 3-link.

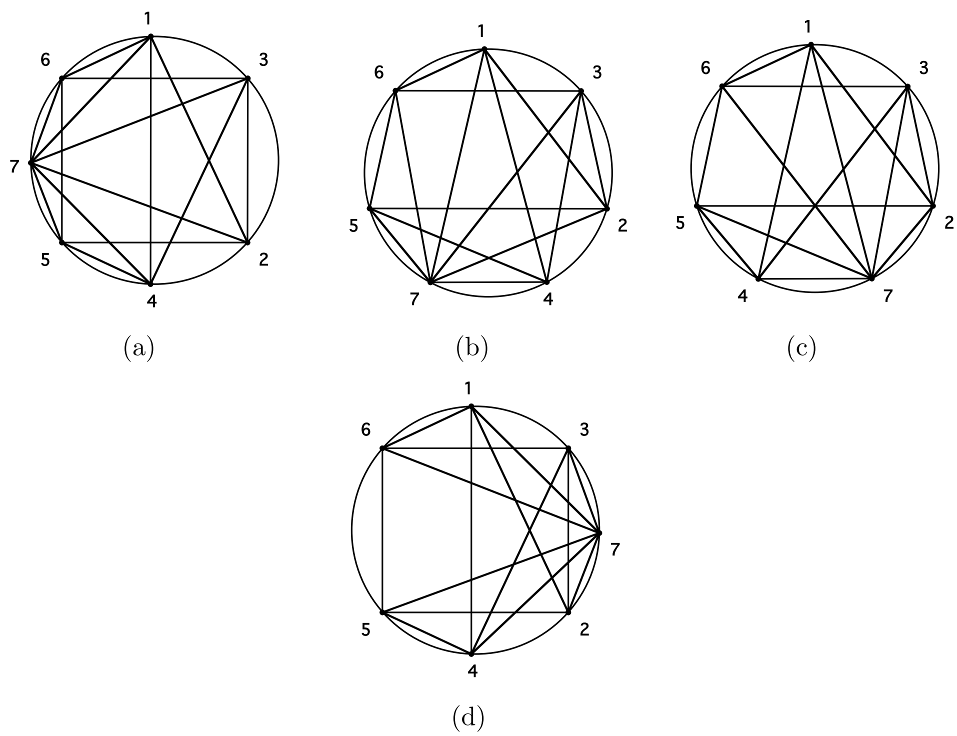

Figure 8: Four S^1 embeddings of $K_{3,3,1}$

Case 3d: Embed the last vertex (vertex 7) as in Figure 8(c).
The 0-spheres $(1, 2)$, $(3, 4)$, and $(6, 7)$ form a pairwise non-spit 3-link.

In every case there is a pairwise non-split S^1 3-link. Thus, $K_{3,3,1}$ is intrinsically pairwise non-split S^1 3-linked. □

Now that we have that K_6 and $K_{3,3,1}$ are intrinsically pairwise non-split 3-linked, we are concerned with creating a larger set of graphs with the same property. To this end, we will show that the operation $\triangle - Y$ exchange preserves the intrinsic pairwise non-split S^1 3-linking property.

Theorem 8.3. *The operation $\triangle - Y$ exchange preserves the intrinsic pairwise non-split S^1 3-linking property.*

Proof. Let G be a graph that is intrinsically pairwise non-split S^1 3-linked and contains a 3-cycle. Let G' be a graph obtained from G by a $\triangle - Y$ exchange. We will show that G' is intrinsically pairwise non-split S^1 3-linked.

Consider an arbitrary S^1 embedding of G'. Consider the associated S^1 embedding of G that results from deleting vertex v and creating edges (a, b),(b, c), and (a, c). As G is intrinsically pairwise non-split S^1 3-linked, there exists a pairwise non-split 3-link in this S^1 embedding of G.

Case 1: The 0-spheres (a, b),(b, c), and (c, a) are not involved in the three component link. Thus, the same pairwise non-split 3-link exists in the S^1 embedding G' as exists in this S^1 embedding of G.

Case 2: One of the 0-spheres (a, b),(b, c), or (c, a) is involved in the pairwise non-split 3-link. Note that at most one of these 0-spheres can be involved in this link. Without loss of generality, the 0-sphere (a, b) is involved in the link. The other two 0-spheres of the pairwise non-split 3-link will not be affected by the $\triangle - Y$ exchange. The 0-sphere (a, v, b) will exist in G' because there is a simple path $\{a, v, b\}$. This path is disjoint from the other two 0-spheres in the link. The pairwise non-split 3-link remains unchanged except for the replacement of (a, b) with (a, v, b). Thus, there exists a pairwise non-split 3-link in the S^1 embedding G'.

In every case, there exists pairwise non-split 3-link in the S^1 embedding of G'. As this embedding was arbitrary, every S^1 embedding of G' will have a pairwise non-split S^1 3-link. Thus, the operation $\triangle - Y$ exchange preserves the intrinsic pairwise non-split S^1 3-linking property. □

Theorem 8.4. *The graphs of the Petersen Family are all intrinsically pairwise non-split S^1 3-linked.*

Proof. As K_6 and $K_{3,3,1}$ are intrinsically pairwise non-split S^1 3-linked, as the Petersen Family is formed from K_6 and $K_{3,3,1}$ by a series of $\triangle - Y$ exchanges, and as $\triangle - Y$ exchange preserves the intrinsic pairwise non-split S^1 3-linking property, it follows that the graphs of the Petersen Family are all intrinsically pairwise non-split 3-linked. □

Corollary 8.5. *If a graph G is intrinsically linked in space, then it is intrinsically pairwise non-split 3-linked in S^1.*

Proof. Let G be a graph that is intrinsically linked in space. Then, as the Petersen Family of graphs forms the complete minor minimal set of graphs that are intrinsically S^3 linked, G contains a Petersen graph as a minor. Note that any supergraph of a graph with a certain linking property will have that same property. Then, as G contains a Petersen graph, which is intrinsically pairwise non-split S^1 3-linked by the previous theorem, as a minor, G is intrinsically pairwise non-split S^1 3-linked. □

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