On Fourier Series Using Functions Other than Sine and Cosine

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Abstract

An important aspect of Fourier series is that $\sin x, \cos x$, and all of their dilations ($\sin jx$ and $\cos jx$ for all j) create an orthogonal basis of the Hilbert space of periodic square-integrable functions with period 2π. In this paper, we define the notion of dilation basis and prove that only a pair of orthogonal sinusoidal functions can generate an orthogonal dilation basis of this space.

1 Introduction

Fourier series are representations of periodic functions that use infinite sums of trigonometric functions. See, for example, [2] or [4]. For example, the sawtooth function seen below has Fourier series $2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$.

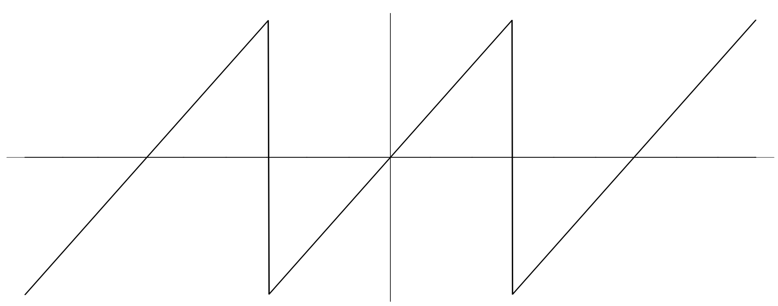

Figure 1: The Sawtooth Function

An important aspect of Fourier series is the orthogonality of its respective parts - that is, they use an orthogonal basis $\{1, \sin x, \cos x, \sin 2x, \cos 2x, ...\}$ of the set of periodic functions with period 2π. Define a *dilation basis* of a Hilbert space of square-integrable functions to be a set of functions $f_1(x), f_2(x), ..., f_n(x)$ such that

$$\{f_1(x), f_2(x), ..., f_n(x), f_1(2x), ..., f_n(2x), ..., f_1(jx), ..., f_n(jx), ...\}$$

is a Hilbert space basis. Note: we may allow a constant, as its dilations are the same constant function, and repeated elements do not add to a set. This leads to the question - are there any other orthogonal dilation bases for the real-valued square-integrable periodic functions?

The answer is that all orthogonal dilation bases of the periodic functions with common period 2π consist of two orthogonal sinusoidal functions and a constant - only sets of the form $\{c_1,\, a_1 \sin x + b_1 \cos x,\, c_2(a_1 \cos x - b_1 \sin x)\}$ can create an orthogonal dilation basis. We see a similar situation in the theory of wavelets (see [1] or [3]), where often a function $\psi(x) \in L^2(\mathbf{R})$ is chosen such that the family of functions $\psi_{m,n}(x) = 2^{m/2}\psi(2^m x - n)$, with m, n ranging through the integers, forms an orthogonal basis for $L^2(\mathbf{R})$. There, however, the dilations are by powers of 2, rather than by all positive integers.

2 Orthogonality

To begin to analyze this question, we must first look at orthogonality. Within the set of periodic functions, two functions $f(x)$ and $g(x)$ are considered orthogonal if

$$\langle f,\, g\rangle = \int_0^{2\pi} f(x)g(x)dx = 0.$$

The integral is well-known to give an inner product for functions on $[0, 2\pi]$. If we allowed complex-valued functions, then we would have to take the complex conjugate of $g(x)$ in the integral. For simplicity, we are restricting to real-valued functions. In the question given above, every distinct pair of functions in the set $f_1(x), f_1(2x), ..., f_1(jx), ...$ must be orthogonal. If $\langle f(jx),\, f(kx)\rangle = 0$ for all $j \neq k$, we say that f is *dilation-orthogonal.* An interesting question, then, is whether this is possible for functions that are not translations of the sin and cos functions.

The answer is yes - a simple example being $\sin x + \cos 2x$. In fact, there is an infinite variety of functions (for example, $\sin x + \cos ax$ for all positive integers a) that are dilation-orthogonal. However, this property is not sufficient to create the basis, as shown by the example in the next section.

Proposition 1 *Let $f(x)$ be a dilation-orthogonal non-constant square-integrable periodic function with period 2π. Then $\int_0^{2\pi} f(x)\, dx = 0$.*

Proof Let $f(x)$ have the Fourier series expansion

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,.$$

The integral in the proposition is 0 if and only if $a_0 = 0$. As $f(x)$ is dilation-orthogonal,

$$0 = \langle f(kx),\, f(x) \rangle$$

for all integers $k > 1$. Therefore,

$$0 = a_0^2 + \sum_{n=1}^{\infty} \left(a_n a_{kn} + b_n b_{kn} \right).$$

Since f is square-integrable,

$$\sum_{n=1}^{\infty} \left(a_n^2 + b_n^2 \right)$$

converges.

The Cauchy-Schwarz inequality states that

$$\left| \sum_{n=1}^{\infty} a_n a_{kn} \right|^2 \leq \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} a_{kn}^2.$$

The right side is equal to a finite number multiplied by a number that approaches 0 as $k \to \infty$, so the left side converges to 0 as $k \to \infty$. This works similarly for the b_n's. However,

$$a_0^2 = -\sum_{n=1}^{\infty} \left(a_n a_{kn} + b_n b_{kn} \right).$$

Since a_0 is independent of k and the right side can be arbitrarily small, $a_0 = 0$.

This shows that within any orthogonal dilation basis for the set of all square-integrable periodic functions, a nonzero constant must be included, since the constant functions are periodic but cannot be generated any other way by an orthogonal dilation basis. The other functions are all orthogonal to a constant, and so must have integral 0. We can now restrict ourselves to looking at a basis of the set of 0-integral functions, as any dilation basis of the entire set of periodic functions corresponds to a basis of 0-integral functions by removal of the constant function.

3 An Example

Not every function with dilation-orthogonality can be a part of an orthogonal dilation basis. For example, $\sin x + \cos 2x$ cannot be part of a basis generated by a dilation basis. If it could be, then it could not be $f_i(jx)$, with $j > 1$, since the period of $f_i(x)$ would be $2\pi/j$ (instead of 2π). Therefore, $\sin x + \cos 2x = f_{i_0}(x)$ for some i_0. We can expand $\cos(x)$ in terms of the basis. This yields the following:

$$\cos x = \sum_{i=1}^{n} \sum_{j=1}^{\infty} a_{ij} f_i(jx)$$

$$\cos 2x = \sum_{i=1}^{n} \sum_{j=1}^{\infty} a_{ij} f_i(2jx)$$

$$\int_0^{2\pi} (\sin x + \cos 2x) \cos 2x dx = \int_0^{2\pi} (\sin x + \cos 2x) \sum_{i=1}^{n} \sum_{j=1}^{\infty} a_{ij} f_i(2jx) dx$$

Since $f_{i_0} = \sin x + \cos 2x$,

$$\pi = \sum_{i=1}^{n} \sum_{j=1}^{\infty} a_{ij} \int_0^{2\pi} f_i(2jx) f_{i_0}(x) dx = 0$$

because of the mutual orthogonality of every member of the basis. This is a contradiction, so $\sin x + \cos 2x$ cannot be in such a basis. This proof can be applied to all functions that are finite sums of sines and cosines, except those that are of the form $a \cos x + b \sin x$, or, equivalently, $A \sin(x + B)$. The next section shows that even infinite sums do not work.

4 Main Result

Theorem 1 *Let* $\{f_1(x), f_2(x), ..., f_n(x)\}$ *be an orthogonal dilation basis of the set of square-integrable periodic functions on* $[0, 2\pi]$ *with integral 0. Then* $n = 2$*; moreover,* $f_1(x) = a_1 \sin x + b_1 \cos x$ *and* $f_2(x) = c(a_1 \cos x - b_1 \sin x)$ *for some constants* a_1*,* b_1*, and* c*.*

Proof Since we are assuming that $\{f_i(jx)\}_{i,j}$ is a basis, there is an expansion

$$\sin(x) = \sum_{i=1}^{n} \sum_{j=1}^{\infty} a_{ij} f_i(jx).$$

The coefficients a_{ij} are given by

$$a_{ij} = \frac{\langle \sin x,\ f_i(jx) \rangle}{\langle f_i(jx),\ f_i(jx) \rangle} = \frac{\int_0^{2\pi} \sin x\, f_i(jx) dx}{\int_0^{2\pi} f_i^2(jx) dx}, \tag{1}$$

since an element in a Hilbert space with an orthogonal basis is equal to the sum of its projections on the basis vectors. Expand $f_i(x)$ in a Fourier series:

$$f_i(x) = \sum_{k=1}^{\infty} \left(A_{ik} \cos kx + B_{ik} \sin kx \right).$$

Let us look at the integral in the numerator of (1). When $j > 1$,

$$\int_0^{2\pi} \sin x\, f_i(jx) dx = \int_0^{2\pi} \sin x \sum_{k=1}^{\infty} \left(A_{ik} \cos jkx + B_{ik} \sin jkx \right) dx$$

$$= \sum_{k=1}^{\infty} \int_0^{2\pi} \sin x \left(A_{ik} \cos jkx + B_{ik} \sin jkx \right) dx = \sum_{k=1}^{\infty} 0 = 0,$$

since $\sin x$ is orthogonal to $\sin jx$ and $\cos jx$ for $j \geq 2$. It follows that $a_{ij} = 0$ when $j \geq 2$. Therefore,

$$\sin x = \sum_{i=1}^{n} d_i f_i(x)$$

for some constants d_i. Similarly,

$$\cos x = \sum_{i=1}^{n} c_i f_i(x)$$

for some constants c_i.

$$A_{ik} = \frac{1}{\pi} \int_0^{2\pi} f_i(x) \cos kx \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} f_i(x) (\sum_{j=1}^{n} c_j f_j(kx)) dx$$

$$= \frac{1}{\pi} \sum_{j=1}^{n} c_j \int_0^{2\pi} f_i(x) f_j(kx) dx.$$

If $k > 1$, then

$$A_{ik} = \sum_{j=1}^{n} 0 = 0.$$

A similar argument can be made for B_{ik} - that is, $B_{ik} = 0$ for all $k > 1$. Therefore,

$$f_i(x) = a_i \cos x + b_i \sin x = A_i \sin(x + B_i)$$

for some a_i, b_i, A_i, and B_i. However, that means that each of these functions is a linear combination of $\sin x$ and $\cos x$. Since this is a two-dimensional space, there can be only two functions, f_1 and f_2. The fact that f_1 are f_2 are orthogonal easily implies the theorem. This completes the proof.

Putting the constant function back in, as described at the end of section 2, we obtain the following.

Corollary 1 *Let $f_1(x), f_2(x), ..., f_n(x)$ be an orthogonal dilation basis of the set of square-integrable periodic functions on $[0, 2\pi]$. Then $n = 3$, and the functions are given, up to order, by $f_1(x) = c_1$, $f_2(x) = a_1 \sin x + b_1 \cos x$, and $f_2(x) = c_2(a_1 \cos x - b_1 \sin x)$ for some constants c_1, a_1, b_1, and c_2.*

5 Generalizations

The above theorem can be restated in the following way: Let S be the semigroup of dilation-functions from S^1 to itself - that is, the set of functions $x, 2x, 3x, ...$ on S^1. Let T be a set of functions from S^1 to $\mathbb{R}$. Consider the set of functions

$B = \{f \circ g | f \in T, g \in S\}$. If B is an orthogonal basis of L^2, then every element of T is either constant or a linear combination of sin and cos.

The result from this paper can easily be generalized to many different situations. In general: Let V be an inner product function space over $\mathbb{R}$. Let S be a semigroup of bijective, continuous functions from S^1 to itself, let S_0 be the set of those functions in S such that their inverses are also in S, and let T be a set of functions from S^1 to $\mathbb{R}$. Consider the set of functions $B = \{f \circ g | f \in T, g \in S\}$. If B is a basis of V, let us call T a semigroup-dilation basis of V over S. They are called orthogonal if and only if B is an orthogonal basis.

Let us say we have two orthogonal semigroup-dilation bases T and T' of V over S . Then any element h of T' is a linear combination of function in $\{f \circ g | f \in T, g \in S_0\}$. The proof of this is exactly the same as the proof of the main result in the earlier section. The only adjustments that need to be made are that every mention of $sin(ix)$ must be replaced with $f(g(x))$ for some f in T and g in S, while all mentions of $f_i(jx)$ should be replaced with $h(g(x))$ for some h in T' and g in S. After that, all mentions of the integral should be replaced with the inner product. Then, the case when $j \geq 2$ corresponds to those functions g that do not have inverses in S. Therefore, the proof that there are no terms for the functions that do not have inverses in S still works, and so the proof is effectively done.

References

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