# On Fourier Series Using Functions Other than Sine and Cosine

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**Abstract**

An important aspect of Fourier series is that $\sin x, \cos x$, and all of their dilations ($\sin jx$ and $\cos jx$ for all $j$) create an orthogonal basis of the Hilbert space of periodic square-integrable functions with period $2\pi$. In this paper, we define the notion of dilation basis and prove that only a pair of orthogonal sinusoidal functions can generate an orthogonal dilation basis of this space.

## 1 Introduction

Fourier series are representations of periodic functions that use infinite sums of trigonometric functions. See, for example, [2] or [4]. For example, the sawtooth function seen below has Fourier series

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx.$$

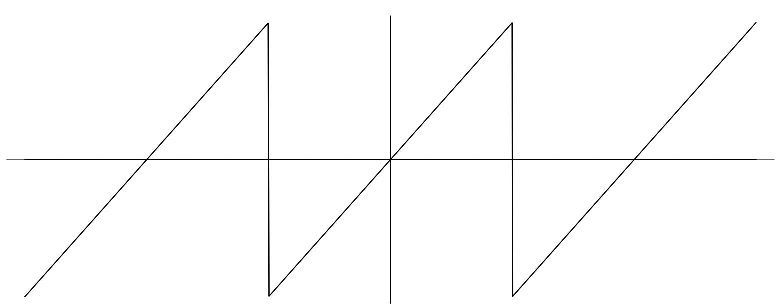

Figure 1: The Sawtooth Function

An important aspect of Fourier series is the orthogonality of its respective parts - that is, they use an orthogonal basis $\{1, \sin x, \cos x, \sin 2x, \cos 2x, ...\}$ of the set of periodic functions with period $2\pi$. Define a *dilation basis* of a Hilbert space of square-integrable functions to be a set of functions $f_1(x), f_2(x), ..., f_n(x)$ such that

$$\{f_1(x), f_2(x), ..., f_n(x), f_1(2x), ..., f_n(2x), ..., f_1(jx), ..., f_n(jx), ...\}$$

is a Hilbert space basis. Note: we may allow a constant, as its dilations are the same constant function, and repeated elements do not add to a set. This leads to the question - are there any other orthogonal dilation bases for the real-valued square-integrable periodic functions? The answer is that all orthogonal dilation bases of the periodic functions with common period $2\pi$ consist of two orthogonal sinusoidal functions and a constant - only sets of the form $\{c_1,\, a_1 \sin x + b_1 \cos x,\, c_2(a_1 \cos x - b_1 \sin x)\}$ can create an orthogonal dilation basis.

We see a similar situation in the theory of wavelets (see [1] or [3]), where often a function $\psi(x) \in L^2(\mathbf{R})$ is chosen such that the family of functions $\psi_{m,n}(x) = 2^{m/2}\psi(2^m x - n)$, with $m, n$ ranging through the integers, forms an orthogonal basis for $L^2(\mathbf{R})$. There, however, the dilations are by powers of 2, rather than by all positive integers.

## 2 Orthogonality

To begin to analyze this question, we must first look at orthogonality. Within the set of periodic functions, two functions $f(x)$ and $g(x)$ are considered orthogonal if

$$\langle f,\, g\rangle = \int_0^{2\pi} f(x)g(x)dx = 0. \tag{1}$$

The integral is well-known to give an inner product for functions on $[0, 2\pi]$. If we allowed complex-valued functions, then we would have to take the complex conjugate of $g(x)$ in the integral. For simplicity, we are restricting to real-valued functions.

In the question given above, every distinct pair of functions in the set $f_1(x), f_1(2x), ..., f_1(jx), ...$ must be orthogonal. If $\langle f(jx),\, f(kx)\rangle = 0$ for all $j \neq k$, we say that $f$ is *dilation-orthogonal*. An interesting question, then, is whether this is possible for functions that are not translations of the sin and cos functions.

The answer is yes - a simple example being $\sin x + \cos 2x$. In fact, there is an infinite variety of functions (for example, $\sin x + \cos ax$ for all positive integers $a$) that are dilation-orthogonal. However, this property is not sufficient to create the basis, as shown by the example in the next section.

**Proposition 1** *Let $f(x)$ be a dilation-orthogonal square-integrable periodic function with period $2\pi$. Then $\int_0^{2\pi} f(x)\, dx = 0$.*

**Proof** Let $f(x)$ have the Fourier series expansion

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

The integral in the proposition is 0 if and only if $a_0 = 0$. As $f(x)$ is dilation-orthogonal,

$$0 = \langle f(kx),\, f(x)\rangle$$

for all integers $k > 1$. Therefore,

$$0 = a_0^2 + \sum_{n=1}^{\infty} a_n a_{kn} + b_n b_{kn}.$$

Since $f$ is square-integrable,

$$\sum_{n=1}^{\infty} a_n^2 + b_n^2$$

converges. The Cauchy-Schwarz inequality states that

$$\left|\sum_{n=1}^{\infty} a_n a_{kn}\right|^2 \leq \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} a_{kn}^2.$$

The right side is equal to a finite number multiplied by a number that approaches 0 as $k \to \infty$, so the left side converges to 0 as $k \to \infty$. This works similarly for the $b_n$'s. However,

$$a_0^2 = -\sum_{n=1}^{\infty} a_n a_{kn} + b_n b_{kn}.$$

Since $a_0$ is independent of $k$ and the right side can be arbitrarily small, $a_0 = 0$.

This shows that within any orthogonal dilation basis for the set of all square-integrable periodic functions, a nonzero constant must be included, since the constant functions are periodic but cannot be generated any other way by an orthogonal dilation basis. The other functions are all orthogonal to a constant, and so must have integral 0. We can now restrict ourselves to looking at a basis of the set of 0-integral functions, as any basis of the entire set of periodic functions corresponds to a basis of 0-integral functions by removal of the constant function.

## 3 An Example

Not every function with dilation-orthogonality can be a part of an orthogonal dilation basis. For example, $\sin x + \cos 2x$ cannot be part of a basis generated by a dilation basis. If it could be, then it could not be $f_i(jx)$, with $j > 1$, since the period of $f_i(x)$ would be $2\pi/j$ (instead of $2\pi$). Therefore, $\sin x + \cos 2x = f_{i_0}(x)$ for some $i_0$. We can expand $\cos(x)$ in terms of the basis. This yields the following:

$$\cos x = \sum_{i=1}^{n} \sum_{j=1}^{\infty} a_{ij} f_i(jx) \qquad (2)$$

$$\cos 2x = \sum_{i=1}^{n} \sum_{j=1}^{\infty} a_{ij} f_i(2jx) \qquad (3)$$

$$\int_0^{2\pi} (\sin x + \cos 2x) \cos 2x dx = \int_0^{2\pi} (\sin x + \cos 2x) \sum_{i=1}^{n} \sum_{j=1}^{\infty} a_{ij} f_i(2jx) dx \quad (4)$$

Since $f_{i_0} = \sin x + \cos 2x$,

$$\pi = \sum_{i=1}^{n} \sum_{j=1}^{\infty} a_{ij} \int_0^{2\pi} f_i(2jx) f_{i_0}(x) dx = 0 \quad (5)$$

because of the mutual orthogonality of every member of the basis. This is a contradiction, so $\sin x + \cos 2x$ cannot be in such a basis. This proof can be applied to all functions that are finite sums of sines and cosines, except those that are of the form $a \cos x + b \sin x$, or, equivalently, $A \sin(x + B)$. The next section shows that even infinite sums do not work.

# 4 Main Result

**Theorem 1** *Let $\{f_1(x), f_2(x), ..., f_n(x)\}$ be an orthogonal dilation basis of the set of square-integrable periodic functions on $[0, 2\pi]$ with integral 0. Then $n = 2$; moreover, $f_1(x) = a_1 \sin x + b_1 \cos x$ and $f_2(x) = c(a_1 \cos x - b_1 \sin x)$ for some constants $a_1$, $b_1$, and $c$.*

**Proof** Since we are assuming that $\{f_i(jx)\}_{i,j}$ is a basis, there is an expansion

$$\sin(x) = \sum_{i=1}^{n} \sum_{j=1}^{\infty} a_{ij} f_i(jx).$$

The coefficients $a_{ij}$ are given by

$$a_{ij} = \frac{\langle \sin x,\ f_i(jx) \rangle}{\langle f_i(jx),\ f_i(jx) \rangle} = \frac{\int_0^{2\pi} \sin x\, f_i(jx) dx}{\int_0^{2\pi} f_i^2(jx) dx}, \quad (6)$$

since an element in a Hilbert space with an orthogonal basis is equal to the sum of its projections on the basis vectors.

Expand $f_i(x)$ in a Fourier series:

$$f_i(x) = \sum_{k=1}^{\infty} A_{ik} \cos kx + B_{ik} \sin kx.$$

Let us look at the integral in the numerator of Equation (6). When $j > 1$,

$$\int_0^{2\pi} \sin x\, f_i(jx) dx = \int_0^{2\pi} \sin x \left( \sum_{k=1}^{\infty} A_{ik} \cos jkx + B_{ik} \sin jkx \right) dx \quad (7)$$

$$= \sum_{k=1}^{\infty} \int_0^{2\pi} \sin x \left( A_{ik} \cos jkx + B_{ik} \sin jkx \right) dx = \sum_{k=1}^{\infty} 0 = 0, \quad (8)$$

since $\sin x$ is orthogonal to $\sin jx$ and $\cos jx$ for $j \geq 2$. It follows that $a_{ij} = 0$ when $j \geq 2$. Therefore,

$$\sin x = \sum_{i=1}^{n} d_i f_i(x) \tag{9}$$

for some constants $d_i$.

Similarly,

$$\cos x = \sum_{i=1}^{n} c_i f_i(x) \tag{10}$$

for some constants $c_i$.

Now, as $\sin x$ and $\cos x$ do form an orthogonal dilation basis, and because we have already shown that there is no constant term in the Fourier series, we can say that

$$f_1(x) = \sum_{j=1}^{\infty} a_j \cos jx + b_j \sin jx. \tag{11}$$

Then

$$a_j = \frac{1}{\pi} \int_0^{2\pi} f_1(x) \cos jx \, dx \tag{12}$$

$$= \frac{1}{\pi} \int_0^{2\pi} f_1(x)(\sum_{i=1}^{n} c_i f_i(jx)) \tag{13}$$

$$= \frac{1}{\pi} \sum_{i=1}^{n} c_i \int_0^{2\pi} f_1(x) f_i(jx) dx.$$

If $j > 1$, then

$$a_j = \sum_{i=1}^{n} 0 = 0. \tag{14}$$

A similar argument can be made for $b_j$ - that is, $b_j = 0$ for all $j > 1$. Therefore,

$$f_1(x) = a_1 \cos x + b_1 \sin x = A \sin(x + B) \tag{15}$$

for some $a_1$, $b_1$, $A$, and $B$.

A similar argument can be made for each of the $f_i(x)$, so each is a linear combination of $\sin x$ and $\cos x$.

However, that means that each of these functions is a linear combination of $\sin x$ and $\cos x$. Since this is a two-dimensional space, there can be only two functions, $f_1$ and $f_2$. The fact that $f_1$ are $f_2$ are orthogonal easily implies the theorem. This completes the proof.

Putting the constant function back in, as described at the end of section 2, we obtain the following.

**Corollary 1** *Let $f_1(x), f_2(x), ..., f_n(x)$ be an orthogonal dilation basis of the set of square-integrable periodic functions on $[0, 2\pi]$. Then $n = 3$, and the functions are given, up to order, by $f_1(x) = c_1$, $f_2(x) = a_1 \sin x + b_1 \cos x$, and $f_2(x) = c_2(a_1 \cos x - b_1 \sin x)$ for some constants $c_1$, $a_1$, $b_1$, and $c_2$.*

## References


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