Eigenvalues of Non-backtracking Walks in a Cycle with Random Loops

Ana Pop *

August 21, 2007

Abstract

In this paper we take a very special model of a random non-regular graph and study its non-backtracking spectrum. We study graphs consisting of a cycle with some random loops added; the graphs are not regular and their non-backtracking spectrum does not seem to be confined to some one-dimensional set in the complex plane. The non-backtracking spectrum is required in some applications, and has no straightforward connection to the usual adjacency matrix spectrum for general graphs, unlike the situation for regular graphs. Experimentally, the random graphs' spectrum appears similar in shape to its deterministic counterpart, but differs because the eigenvalues are visibly clustered, especially with a mysterious gap around $Re(\lambda) = 1$.

*Department of Computer Science, Princeton University, Princeton, NJ 08540 apop@princeton.edu

1 Introduction

Consider a graph, $G = (V, E)$. A graph can be traversed from a node to any other node through a walk of varied lengths. A non-backtracking walk in a graph G is one in which no step is the reverse step along the same edge as the previous step. Let A be G's $n \times n$ adjacency matrix. Let $B = B(G)$ be the $2n \times 2n$ matrix indexed on the directed edges of G (each edge of G gives rise to two directed edges) such that there is a 1 or a 0 in $B_{e,e'}$ according to whether or not e, e' forms a non-backtracking walk of length 2. For finite graphs, the eigenvalues of B are essentially the solutions to $\det(\mathrm{I}\lambda^2 - \mathrm{A}\lambda + (\mathrm{Q})) = 0$ [KS00], where D is the diagonal $n \times n$ matrix of vertex degrees and $Q = D - I$.

Non-backtracking walks arise in numerous situations. For example, the trace method of Broder and Shamir for estimating the second eigenvalue of random regular graphs [BS87, Fri91, Fri] applies to non-backtracking closed walks. In the regular case, the non-backtracking spectrum is easy to determine, but for non-regular graphs the spectrum is difficult to formulate. Thus we seek to understand the non-backtracking matrix B better. Number theorists are also interested in non-backtracking walks because of their relations to Zeta functions [ST96, ST00, ST]. Also, using non-backtracking walks is natural in certain applications such as congestion in computer networks. In data transmissions, error correcting codes called low density parity check (LDPC) codes exist to determine whether information was sent correctly. LDPC codes rely on solutions to a system of linear equations with unknown variables. The system can be iteratively solved by replacing variables one at a time into the equations. Since loops are undesirable on variables, non-backtracking walks should be used [HLW, HG].

In this paper, we will analyze the eigenvalues of the non-bactracking matrix for two families of graphs. Both are formed by adding loops to the cycle on n vertices. The first family is formed by adding the loops at evenly-spaced intervals, the second by adding the loops at randomly-spaced intervals. In this paper, we will use the term *evenly spaced k-graph* to refer to graphs that have loops every k vertices and the term *evenly spaced k-graph plot* to refer to the eigenvalue plot of the non-backtracking edge matrix derived from the evenly spaced k-graph. Similar terms will be used for the randomly spaced case. Special cases for these terms are when there are loops at no vertices and loops at all vertices, which will be expressed with $k = 0$ and $k = n$, respectively.

2 Tests Performed

Eigenvalues of the non-backtracking matrix B were plotted in Matlab in the complex plane and the resulting clusters were examined. The results are presented in this section and are discussed in the next section.

Evenly spaced k-graphs plots are shown in Figures 1 to 3. We show the position of the eigenvalues of the non-backtracking matrix in the complex plane for $k = 2, 4, 20$.

In addition to computing the eigenvalues from the non-backtracking matrix, a polynomial formula for the eigenvalues can also be derived using Fourier analysis. The case for an evenly spaced 2-graph is outlined as follows.

We first convert the adjacency matrix to the non-backtracking matrix by forming

$$\begin{pmatrix} 0 & I \\ -Q & A \end{pmatrix}$$

So we know that

$$(I\lambda^2 - A\lambda + Q)f = 0 \tag{1}$$

We assume that $f : \{0, 1, ..., n-1\} \to \mathbb{C}$ satisfies $f(i+2) = \zeta f(i)$ with $\zeta^{n/2} = 1$ (we write $f(n)$, $f(n+1)$ for $f(0)$, $f(1)$); these assumptions are justified by Fourier analysis. Figure 4 shows the generalization of the graph that was used to determine the equation.

Then equation (1) is equivalent to the two equations for $f_1 = f(1)$ and $f_2 = f(2)$

$$\lambda^2 f_1 - \lambda(\zeta^{-1} f_2 + 2f_1 + f_2) + 3f_1 = 0 \text{ and } \lambda^2 f_2 - \lambda(\zeta f_1 + f_1) + f_2 = 0$$

Using these equations, the final result is the polynomial in λ of degree 4 that, when plotted, yields the same result as in Figure 1.

$$\lambda^4 - 2\lambda^3 + \lambda^2(2 - \zeta^{-1} - \zeta) - 2\lambda + 3 = 0$$

Similar algebraic equations can be derived for an evenly spaced k-graph for any positive integer k.

Several graphs were studied with the following total number of random loops: $n/2, n/4, n/20$, as well as loops at every prime number. Figures 5 to 7 show randomly spaced k-graph plots depicting the position of the eigenvalues of the non-backtracking matrix in the complex plane.

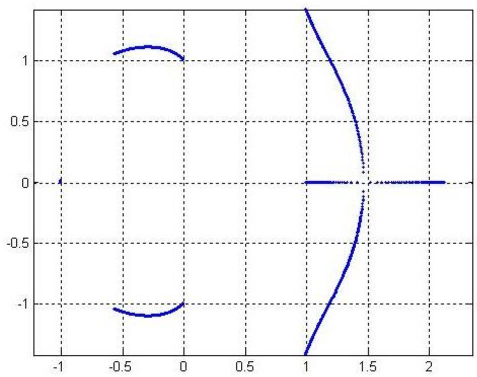

Figure 1: $n = 1000$, loops at every 2nd vertex

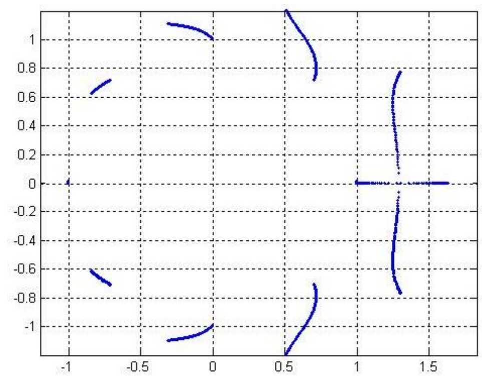

Figure 2: $n = 1000$, loops at every 4th vertex

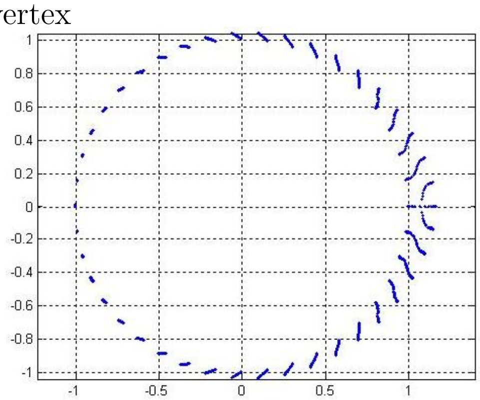

Figure 3: $n = 1000$, loops at every 20th vertex

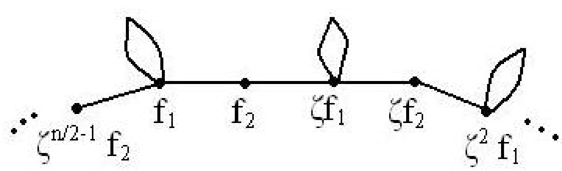

Figure 4: Generalization of graph with loops at every 2nd vertex

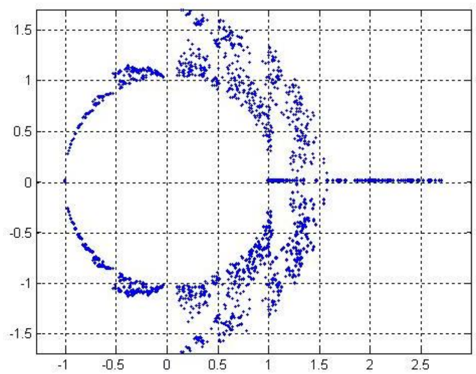

Figure 5: $n = 1000$, $n/2$ loops positioned randomly

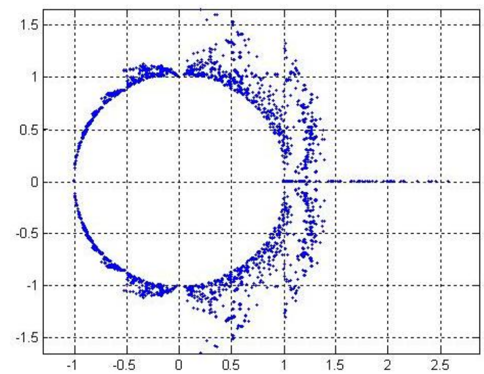

Figure 6: $n = 1000$, $n/4$ loops positioned randomly

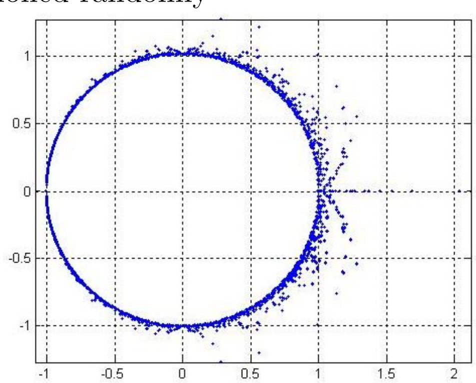

Figure 7: $n = 1000$, $n/20$ loops positioned randomly

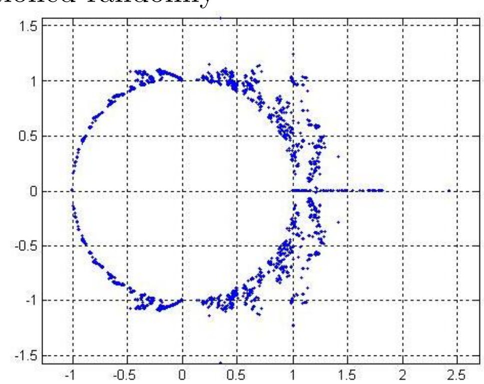

Figure 8: $n = 1000$, loops at every prime number

3 Analysis of Test Results

The figures presented in the previous section show that for a fixed k, the evenly spaced k-graph spectrum is confined to a one-dimensional set in the complex plane while the randomly spaced k-graph plots do not seem to be confined as such.

Referring to Figures 5 to 7, the non-backtracking spectra of graphs with randomly spaced loops yield eigenvalue clusters in the shape of the evenly spaced k-graph plot. They also approach the unit circle as the total number of loops decreases. In Figure 8 we plot the non-backtracking graph with loops at every prime number (in hopes that this will mimic randomness). Our analysis of Figures 1 to 7 leads us to a conjecture and a theorem.

Conjecture 3.1. *Let k be fixed and let Y_n be the non-backtracking spectrum of the n-cycle plus n/k evenly spaced loops graph in the complex plane with multiplicities. Similarly, let Z_n be the non-backtracking spectrum of the n-cycle plus n/k randomly spaced loops graph. As measures on the complex plane (meaning the measure of region, R, is the number of points in R divided by the total number of points), we expect Y_n to have a weak limit L, and Z_n to weakly converge in probability to L.*

In Conjecture 3.1, by convergence weakly in probability we mean that for any R and any $\epsilon > 0$, $\mathrm{Prob}(|\mathrm{Z_n(R)} - \mathrm{L(R)}| \geq \epsilon) \leq \epsilon$.

The following theorem was developed joint with Shlomo Hoory.

Theorem 3.2. *The non-backtracking eigenvalues of a cycle plus some loops are always to the right of $\lambda = -1$.*

Proof. We replace A, Q in equation (1) by real numbers a, b as follows. Let Q be the non-negative diagonal matrix and $A = C + Q$ where C is the adjacency matrix of a cycle. The smallest possible eigenvalue of a cycle is -2, so $\frac{\langle Cf,f\rangle}{\langle f,f\rangle} \geq -2$. Then,

$$a = \frac{\langle Af, f\rangle}{\langle f, f\rangle} \geq -2$$

$$b = \frac{\langle Qf, f\rangle}{\langle f, f\rangle} \geq 0$$

And we can write

$$\lambda^2 - a\lambda + b = 0$$

Since $a \geq -2$ and $b \geq 0$ then for sure $|a| \leq 1$ so $|a| \leq b+1$. By Kotani and Sunada [KS00] (our a is their λ and our b is their μ), if the eigenvalues are real,

$$\lambda \geq \frac{a - \sqrt{a^2 - 4(|a|-1)}}{2} \geq \frac{a - ||a|-2|}{2} \geq -1 \text{ if } a \in [-2, \infty)$$

If the eigenvalues are imaginary,

$$Re(\lambda) = \frac{a}{2} \geq -1 \qquad \diamondsuit$$

If we compare the evenly spaced $2, 4, 20$-graph plots with randomly spaced $2, 4, 20$-graph plots, we notice some interesting characteristics.

We observe that the evenly spaced k-graph plot points are organized in a T-shape on the right side and a number of wings, which can also be seen to a certain degree in its corresponding non-deterministic plot with the same number of total loops. This behaviour is noted on the $n/2$, $n/4$, and $n/20$ plots. Further, we see that the gaps evident in the randomly spaced k-graph plot mimic the evenly spaced k-graph plot, almost as if to complete the plot.

We expect the plot in the complex plane to have the point density, at a certain location, being a random variable with some "small" variation. For any n, k, let $Z_{n,k}$ be a random variable of the non-backtracking spectrum distribution of the n-cycle plus n/k randomly spaced loops graph. For any closed region R, $Z_{n,k}(R) =$ (number of points in R) / (number of total points) and takes a value between 0 and 1.

Question. *We want to know what is* $\int (Z_{n,k}(R) - \overline{Z_{n,k}(R)})^2 dP$ *where* P *is the region of probability. That is, what is the variance* $Var(Z_{n,k}(R))$*?*

If R is a closed region and we expect that $Z_{n,k}$ weakly converges in probability to L as postulated in Conjecture 3.1 then we expect a 50% variance above and below L. Determining this variance would give us a better idea of the clusters formed in the randomly spaced k-graph plots in relation to the evenly spaced k-graph plots.

References

[BS87] A. Z. Broder and E. Shamir. On the second eigenvalue of random regular graphs (preliminary version). In *FOCS*, pages 286294. IEEE, 1987.

[Fri] J. Friedman. A proof of Alons second eigenvalue conjecture. Memoirs of the A.M.S., to appear.

[Fri91] J. Friedman. On the second eigenvalue and random walks n random d-regular graphs. *Combinatorica*, 11(4):331362, 1991.

[HG] S. Hoory and J. Goldberger. A continuous version of regular ldpc codes. submitted to IEEE Transactions on Information Theory.

[HLW] S. Hoory, N. Linial, and A. Wigderson. Expander graphs and their applications. Bulletin of the AMS, 43(4):439-561, October 2006.

[KS00] M. Kotani and T. Sunada. Zeta functions of finite graphs. *J. Math. Sci. Univ. Tokyo*, 7(1):725, 2000.

[ST] H. Stark and A. Terras. Zeta functions of finite graphs and coverings, part iii. http://math.ucsd.edu/ aterras/BrauerSiegel.pdf.

[ST96] H. Stark and A. Terras. Zeta functions of finite graphs and coverings. *Advances in Math.*, 121:124165, 1996.

[ST00] H. Stark and A. Terras. Zeta functions of finite graphs and coverings, part ii. *Advances in Math.*, 154:132195, 2000.