

Review of “The Circuit Partition Polynomial with Applications and Relation to the Tutte and Interlace Polynomials” by Andrea Austin

This is an interesting paper that highlights relationships among various polynomials related to graphs. It should be published, subject to the following corrections, and the reviewer commends the author on an ambitious project.

Corrections:

- In general, insert line spaces after definitions.
- Page 1, paragraph 4, line 3. Change “calculate this for strings in general” to “enumerate Eulerian circuits”
- Page 1, paragraph 4, line 6. Change “apply the CPP” to “calculate the CPP”
- Page 3. Remove definition 3 and the paragraph following. The definition is a little vague and isn’t used in the rest of the paper.
- Page 3, Figure 3. The figure is actually a multigraph, unless your definition of graph allows for multiple edges between pairs of vertices. Either change definition 1 to include multiple edges, or define multigraphs.
- Page 3, Figure 4. Same comment as Figure 3.
- Page 4, Figure 5. Same comment as Figure 3.
- Page 4, 1st paragraph after Figure 5. Change “to the research” to “to research” and “represented an oriented” to “represented by an oriented”
- Page 4, 1st paragraph after Definition 7. There are actually five triples in the sequence (bba is left out). However in Figure 6, bba is there but aba is not. So Figure 6 needs to be fixed.
- Page 5, line 2. change “construction” to “constructing”
- Page 5, line 11. Change “graphs represent” to “graph represents”
- Page 5, line 12. Insert a comma after “reconstructions”
- Page 5, paragraph 3, line 2. Change “We determine” to “We would like to determine”
- Page 5, paragraph 3, line 4. Change “apply the CPP” to “calculate the CPP”
- Page 5, paragraph 3, last sentence. This sentence is a little awkward. Change to something like “Also, as we will see later, computing the CPP allows us to compute other polynomials, which gives us more information about de Bruijn graphs.”
- Page 5, Definition 8. “Planar graph” is used here without it being previously defined. Because the paper assumes no prior knowledge of graph theory, a sentence about planar graphs would be appropriate. Also, it’s not clear why medial graphs are being introduced here, so mention that you will use these in subsequent sections in relation to the Martin polynomial. Also,  $G_m$  is actually a multigraph, so see the comment regarding Figure 3 above.

- Page 6, Definition 9. Technically, a bridge doesn't connect two components of a graph since the bridge and the two components are one component before you remove the bridge. Also, removing a bridge results in the graph having more components than before, but the graph may already have been disconnected. So I'd change the definition to something like "A *bridge* in a graph  $G$  is an edge whose removal results in a graph with more components than  $G$ ."
- Page 6, Definition 11. A cut vertex disconnects the component from which it is removed, but the graph itself may already be disconnected, so change the definition to reflect this.
- Page 6, Figure 8. The vertex labeled as a cut vertex isn't a cut vertex. Removing it results in a graph with one vertex, which is, of course, connected. You need at least three vertices to have a cut vertex.
- Page 6, Definition 12. Don't capitalize "Graph State." Also, you haven't defined "valent"
- Page 6. Since you don't use Proposition 1 or Proposition 2 until later, move these right before Theorem 1 (and rename them Proposition 2 and Proposition 3, since they'll now follow your Proposition 3, which you'll rename Proposition 1). You can say something like "First we'll need the following two facts about the Martin polynomial," insert the two propositions, and then go into Theorem 1.
- Page 6, Proposition 1. Start by saying "Let  $m(G; x+1)$  represent the Martin polynomial of the graph  $G$ ." Also, insert either a space or a dot between the  $x$  and  $m(G; x+1)$  so that the  $x$  and  $m$  don't run together.
- Page 7, last line before Figure 10. Change "Eulerian component" to "Eulerian state"
- Page 8, second paragraph. You have defined the CPP, but I'm not sure what you mean by "demonstrated its results." It would help the reader here if you go back to the big picture and reiterate that the CPP counts the number of Eulerian circuits, and more importantly, say why (it may not be clear since so far you've talked about Eulerian graph states, but haven't explicitly related this to Eulerian circuits). Also, if each graph state represents an Eulerian circuit, then the total number of Eulerian circuits is  $2^n$  where  $n$  is the number of vertices (two choices for the split at each vertex). Since you were interested in counting the number of Eulerian circuits, you should mention this, and then mention that the CPP further refines this by grouping the circuits in terms of components. A sentence or two about why this grouping is useful would be appropriate (or at least say that the reason you group things the way you do in the CPP is to allow you to use it to compute the Martin polynomial). Also mention that substituting 1 into the CPP gives the total number of Eulerian circuits.
- Page 8, section IV, first paragraph. Change "when compared to..." to "is equal to the Tutte polynomial of the graph  $G$ ."
- Page 9, Figure 13. Same comment as Figure 3 (multigraph)

- Page 9, definition 14. You need some initial conditions. I'm assuming that  $T(G; x, y) = x$  if  $G$  has a single edge (not a loop) and  $T(G; x, y) = y$  if  $G$  has a single loop. You should say this. Also, do you mean  $T(G; x, y) = xT(G - e; x, y)$  if  $e$  is a bridge? And similarly if  $e$  is a loop? ( $T(G; x, y) = xT(G; x, y)$  doesn't make sense).
- Page 9, proposition 3. As a matter of style, keep  $G$  and  $H$  in the same order: " $T(G * H) = T(G)T(H)$  if  $G * H$  is the disjoint union or one-point join of  $G$  and  $H$ ." Also, the next sentence should be changed slightly, since a one-point join is not a cut vertex, it is the resulting graph obtained by joining the two original graphs at a single vertex (you know what you mean, you just need to fix the language).
- Page 9, last paragraph of the proof of Proposition 3. Change "performing" to "calculating" (two occurrences). You don't "perform" a polynomial...
- Page 10, 4th line above Figure 16. You say that you're using Definition 14 to calculate the Martin polynomial, but Definition 14 defines the Tutte polynomial, so it would be helpful to explain explicitly how you get  $m((\vec{G})_m; x) = x$ .
- Page 11, first line. Change "Tutte and Martin" to "Tutte and Martin polynomials" and explain how you get them.
- Page 11, case 2. You refer to corollary 1, but this doesn't appear until later in the paper. Also, is there an  $x$  missing in Proposition 2? You seem to use

$$m(G; x) = x \cdot m(H_1; x)m(H_2; x)$$

but Proposition 2 states

$$m(G; x) = m(H_1; x)m(H_2; x)$$

- Page 12, second to last paragraph of Case 3. "Based on our definition, we know that  $m(H_2) = 1$ ? Why? And which definition are you referring to?"
- Page 12, last line. Remove the colon.
- Page 12, first line after Figure 21. Change "apply the Tutte polynomial" to "construct the Tutte polynomial"
- Page 12, after Figure 22. The word "so" shouldn't be italicized.
- Page 14, first two paragraphs. Delete the first sentence of each paragraph and combine the remaining two sentences into one paragraph.
- Page 14, Section V, paragraph 1. You haven't actually defined a " $k$ -component circuit partition." How do these relate to Eulerian graph states?
- Page 14, Section V, last sentence of second paragraph. Delete "we must know"
- Page 14, Definition 15, second line. Change " $v$  and  $w$ " to " $v$  or  $w$ "

- Page 14, Figure 23. The first two pictures don't make sense to me. The three blue vertices are all connected in both figures. Are the dotted lines to mean that there should be two edges between these vertices, and you're swapping one edge for another? If so, why two edges between each pair of vertices? This would make sense for a digraph, but you're defining it for a general graph. So either make the edges directed (and change the definition to include digraphs), or perhaps change the figure so that two of the three edges are present in the first picture, and the third edge is present in the second picture.
- Page 14, last line. Change "cursively" to "recursively"
- Page 15, second paragraph. Change the sentence "Now that we have noted..." to "To better understand the interlace polynomial, it is beneficial to see its relation to the CPP."
- Page 15, Proposition 4. You refer to a 4-regular Eulerian digraph. Previously you've referred to these as 2-in 2-out digraphs. You should use the same term throughout for consistency. Also, and more importantly, you haven't defined a circle graph or chord diagram determined by a circuit  $C$ .
- Page 16. Explain how you are constructing the chord diagram and circle graph.
- Page 16, first sentence after Figure 28. Change "when we apply the interlace polynomial to  $H$ " to "when we construct the interlace polynomial of  $H$ " and then say how you are actually getting  $x^2 + x$ .