# LINEAR $N$-GRAPHS

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Abstract. We call a simple graph $G$ a *linear $N$-graph* if its ordinary (vertex) chromatic number equals the linear chromatic number of its neighborhood complex $\mathcal{N}(G)$. We prove that the linearity is preserved under taking joins and multiplying vertices, and give a complete characterization of linear $N$-trees.

## 1. Preliminaries

An *(abstract) simplicial complex* $\Delta$ on a finite set $X$ is a family of subsets of $X$ satisfying the following properties.

(i) $\{x\} \in \Delta$ for all $x \in X$.
(ii) If $F \in \Delta$ and $H \subset F$, then $H \in \Delta$.

The elements of $\Delta$ are called *faces*, the *dimension* of a face $F$ is $\dim(F) = |F| - 1$, and the *dimension* of $\Delta$ is defined to be $\dim(\Delta) = \max\{\dim(F) \colon F \in \Delta\}$. The 0 and 1-dimensional faces of $\Delta$ are called *vertices* and *edges* while maximal faces are called *facets*. We denote by $\mathcal{F}_\Delta$ and $\mathcal{F}_\Delta(x)$ the set of facets of $\Delta$ and those containing a given vertex $x \in X$ respectively.

We next introduce some basics of graph theory, and we refer readers to [3] for more details.

By a simple graph $G$, we will mean an undirected graph without loops or multiple edges. If $G$ is a graph, $V(G)$ and $E(G)$ (or simply $V$ and $E$) denote its vertex and edge sets. An edge between $u$ and $v$ is denoted by $e = uv$. If $U \subset V$, the graph induced on $U$ is written $G_U$, and $G - U$ denotes the graph induced by $V - U$. In particular, we abbreviate $G - \{v\}$ to $G - v$. We denote by $|V|$ and $|E|$ the order and size of $G$, while $d(v)$ denotes the degree of a given vertex $v \in V$. A subset $U \subseteq V$ is called an *independent set* if no two vertices in $U$ forms an edge in $G$. We recall that a graph $G$ is said to be a *tree* if it is connected and contains no cycle.

There are various ways to associate a simplicial complex to a given simple graph $G = (V, E)$; however we consider here only one of them, known as the neighborhood complex of $G$. The set $\mathcal{N}(u) = \{v \in V \colon uv \in E\}$ is called the (open) *neighborhood* of $u$ in $G$, and the *neighborhood complex* $\mathcal{N}(G)$ of $G = (V, E)$ is defined to be the simplicial complex on $V$ such that a subset $F \subseteq V$ is a face of $\mathcal{N}(G)$ if and only if

$$\bigcap_{u \in F} \mathcal{N}(u) \neq \emptyset.$$

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We illustrate in Figure 1 a simple graph and its neighborhood complex.

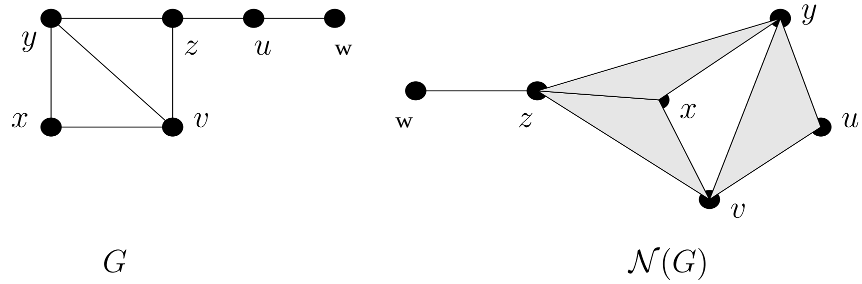


FIGURE 1. A simple graph and its neighborhood complex.

## 2. LINEAR COLORINGS OF SIMPLICIAL COMPLEXES AND LINEAR $N$-GRAPHS

The notion of linear colorings of simplicial complexes was introduced by Civan and Yalçın [1] as a combinatorial tool to study the topology of simplicial complexes. Since it is possible to associate simplicial complexes to simple graphs, the method of linear coloring seems to be useful for understanding the structures of graphs as well. We here explore this connection in terms of the neighborhood complexes of graphs. We refer readers to [1] for more details.

**Definition 2.1.** Let $\Delta$ be a simplicial complex with vertex set $X$. A mapping $\alpha\colon X \to [k]$ is called a *linear coloring* of $\Delta$ if for every $i \in [k]$, the set $\mathcal{F}_i = \{\mathcal{F}(v) : \alpha(v) = i\}$ is linearly ordered by inclusion, where $[k] = \{1, \ldots, k\}$. In this case, the map $\alpha$ is said to be a $k$-linear coloring of $\Delta$. The least integer $k$ for which $\Delta$ admits a $k$-linear coloring is called the *linear chromatic number* of $\Delta$ and denoted by $\mathrm{lchr}(\Delta)$.

An alternative way to decide whether a given coloring of a simplicial complex $\Delta$ is linear can be stated as follows. A coloring $\alpha\colon X \to [k]$ is a linear coloring of $\Delta$ if and only if whenever $\alpha(x) = \alpha(y)$ for some $x, y \in X$, then one of the inclusions $\mathcal{F}_\Delta(x) \subseteq \mathcal{F}_\Delta(y)$ or $\mathcal{F}_\Delta(y) \subseteq \mathcal{F}_\Delta(x)$ holds ([1], Prop. 2.9). In Figure 2, we present a linear coloring of a simplicial complex $\Delta$ and a coloring of it which is not linear. We note that the coloring given in Figure 2 (b) is not linear, since there is no inclusion relation between the facet sets of vertices colored by 3.

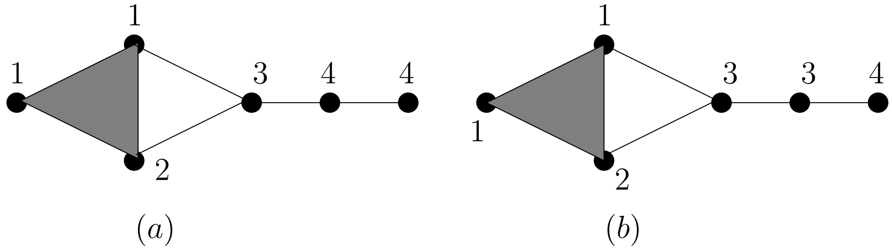


FIGURE 2. A linear coloring $(a)$ of a simplicial complex $\Delta$ and a non-linear coloring $(b)$.

Let $G = (V, E)$ be a simple graph. We recall that a (vertex) coloring of $G$ is a surjective mapping $\nu\colon V \to [n]$ such that $\nu(x) \neq \nu(y)$ whenever $(x, y) \in E$, and the least integer $n$ for which $G$ admits a (vertex) coloring is called the *vertex chromatic number* and denoted by $\chi(G)$.

Civan and Yalçın [1] prove that any linear coloring of the neighborhood complex of a given graph $G$ is in fact a vertex coloring of $G$.

**Proposition 2.2.** ([1], Prop. 8.1) *Let $G = (V, E)$ be a simple graph and let $\mathcal{N}(G)$ denote its neighborhood complex. If $\kappa\colon V \to [k]$ is a $k$-linear coloring of $\mathcal{N}(G)$, then $\kappa$ is a coloring of the underlying graph $G$.*

It follows that for any graph $G$, we have $\mathrm{lchr}(\mathcal{N}(G)) \geq \chi(G)$ ([1], Prop. 8.2). Furthermore, it can be seen that not every vertex coloring of a graph $G$ turns out to be a linear coloring of $\mathcal{N}(G)$. However, Civan and Yalçın [1] provide a characterization when a vertex coloring of a simple graph is linear whose converse does not hold in general.

**Proposition 2.3.** ([1], Prop. 8.4) *A coloring $\nu\colon V \to [k]$ of $G = (V, E)$ is a $k$-linear coloring of $\mathcal{N}(G)$ if either $\mathcal{N}(v) \subseteq \mathcal{N}(u)$ or $\mathcal{N}(u) \subseteq \mathcal{N}(v)$ holds for every $x, y \in V$ with $\nu(x) = \nu(y)$.*

The central objects of our study are those simple graphs whose chromatic numbers equal to the linear chromatic numbers of their neighborhood complexes.

**Definition 2.4.** A simple graph $G$ is called a *linear $N$-graph* if $\chi(G) = \mathrm{lchr}(\mathcal{N}(G))$.

**Example 2.5.** The complete graphs and complete $r$-partite graphs are examples of linear $N$-graphs. Examining the graphs depicted in Figure 3, $G_1$ is a linear $N$-graph with $\chi(G_1) = \mathrm{lchr}(\mathcal{N}(G_1)) = 2$, while $G_2$ is not a linear $N$-graph, since $\chi(G_2) = 3$ and $\mathrm{lchr}(G_2) = 4$.

We next show that the linearity is preserved under some operations on graphs. We first recall that for any two disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, their join $G_1 \vee G_2$ is obtained from the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ by joining all the vertices of $G_1$ to all the vertices of $G_2$.

**Proposition 2.6.** *If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are linear $N$-graphs, so is their join $G_1 \vee G_2$.*

*Proof.* We let $V_1 = \{v_1, \ldots, v_n\}$ and $V_2 = \{u_1, \ldots, u_m\}$. Since $G_1$ and $G_2$ are both linear $N$-graphs, we have the equalities $\chi(G_1) = \mathrm{lchr}(\mathcal{N}(G_1)) = k_1$ and $\chi(G_2) = \mathrm{lchr}(\mathcal{N}(G_2)) = k_2$. On the other hand, it is well known that $\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2)$; hence, we need only to verify that $\mathrm{lchr}(\mathcal{N}(G_1 \vee G_2)) = k_1 + k_2$. Since the linear chromatic number is an upper bound for the vertex chromatic number, it is enough to show that $k_1 + k_2$ colors are sufficient to color $\mathcal{N}(G_1 \vee G_2)$ linearly. We note that any facet of $\mathcal{N}(G_1 \vee G_2)$ is of the form $F \cup V_i$ for some $F \in \mathcal{F}_{\mathcal{N}(G_j)}$, where $1 \leq i, j \leq 2$ and $i \neq j$. Therefore, any two vertices of $\mathcal{N}(G_i)$ having the same color can be linearly colored by the same color in $\mathcal{N}(G_1 \vee G_2)$, which proves that $\mathrm{lchr}(\mathcal{N}(G_1 \vee G_2)) \leq k_1 + k_2$. $\square$

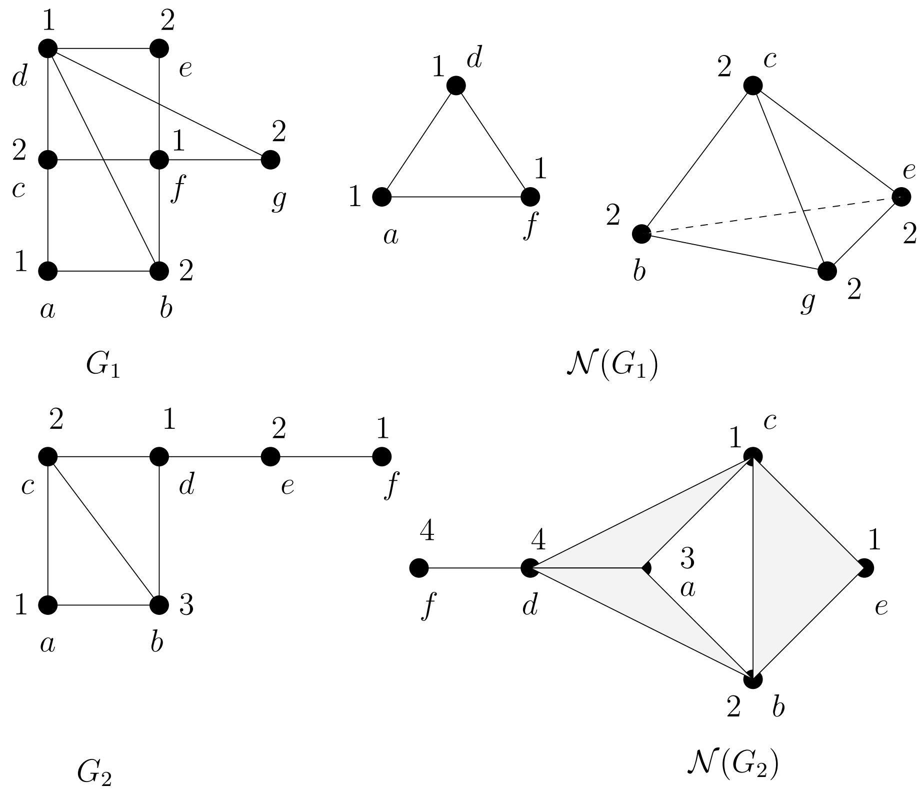


FIGURE 3. A linear $N$-graph $G_1$ and a non-linear $N$-graph $G_2$.

Let $G = (V, E)$ be a simple graph with $|V| = n$, and let $\mathbb{N}$ denote the set of positive integers. For a given vector $\mathfrak{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$, we define $G \circ \mathfrak{m}$ to be the graph constructed by substituting for each $v_i \in V$ an independent set of $m_i$ vertices $v_i^1, v_i^2, \ldots, v_i^{m_i}$ and joining $v_i^s$ with $v_j^t$ if and only if $v_i$ and $v_j$ are adjacent in $G$. It is more customary to allow the vectors $\mathfrak{m}$ to have zero coordinates to construct induced subgraphs of a given graph $G$ by multiplication of vertices from $G$. However, we here only consider the restricted case ([2]).

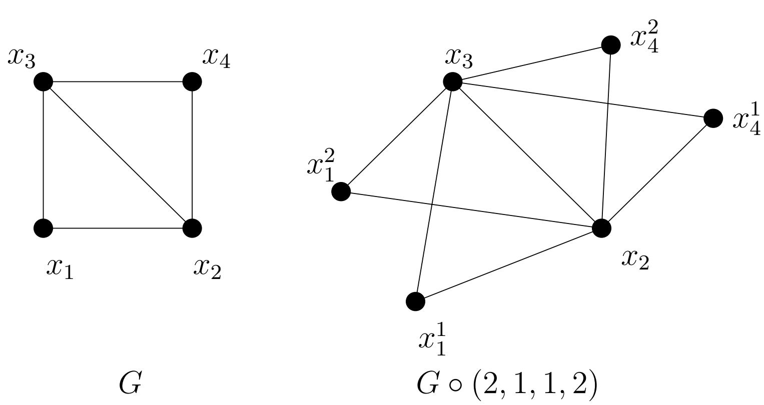


FIGURE 4. A multiplication of vertices of a graph.

**Proposition 2.7.** *If $G=(V,E)$ is a linear $N$-graph, so is $G\circ\mathfrak{m}$ for all $\mathfrak{m}\in\mathbb{N}^n$.*

*Proof.* We note that $\chi(G)=\chi(G\circ\mathfrak{m})$ for any $\mathfrak{m}\in\mathbb{N}^n$, where $|G|=n$. Therefore, it is enough to show that if $\chi(G)=k$, then $k$ colors are sufficient to color $\mathcal{N}(G\circ\mathfrak{m})$ linearly. Let $\kappa$ be a $k$-linear coloring of $\mathcal{N}(G)$ and let $\mathfrak{m}=(m_1,\dots,m_n)\in\mathbb{N}^n$ be given. We write $V=\{x_1,\dots,x_n\}$. Now, for any $i,j$ with $1\leq i,j\leq m_t$, we have $\mathcal{N}(x_i^t)=\mathcal{N}(x_j^t)$ in $G\circ\mathfrak{m}$. Thus, the mapping $\overline{\kappa}\colon V(G\circ\mathfrak{m})\to[k]$ defined by $\overline{\kappa}(x_i^t):=\kappa(x_i)$ is linear. This completes the proof. □

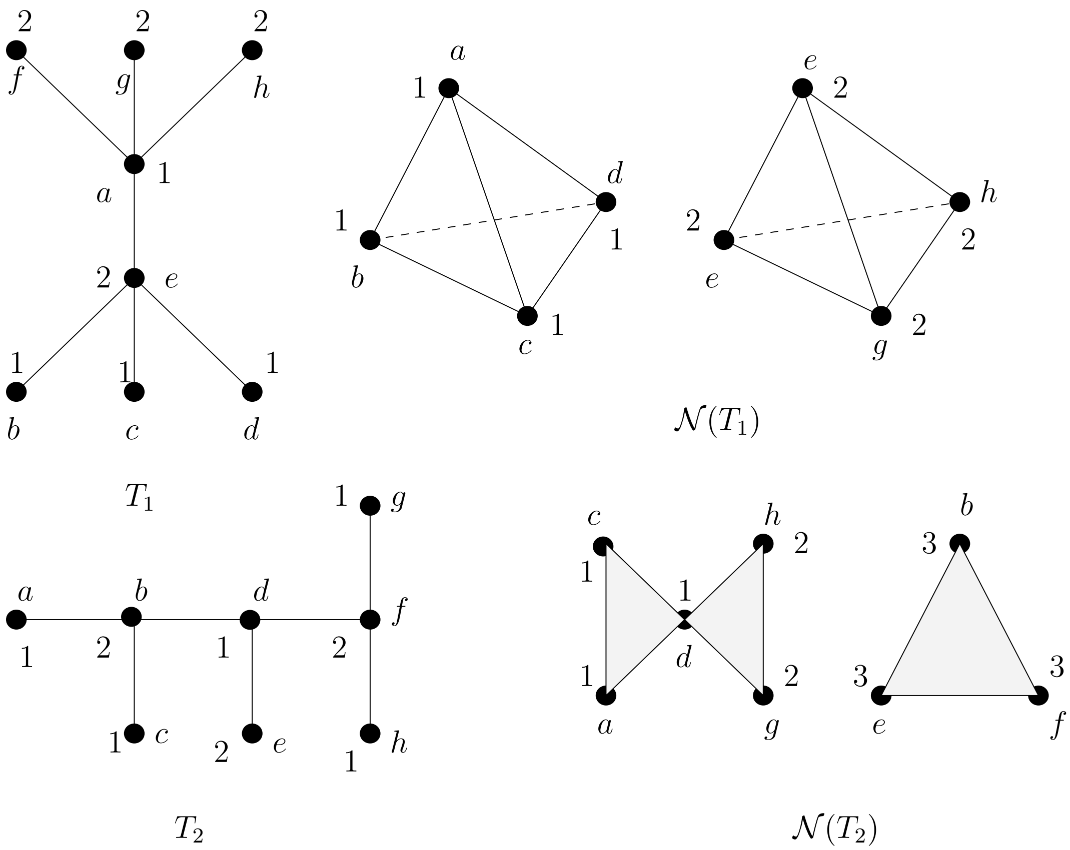


FIGURE 5. A linear $N$-tree $T_1$ and a non-linear $N$-tree $T_2$.

**Theorem 2.8.** *A tree $T$ is a linear $N$-graph if and only if $|\{v\in V(T)\colon d(v)>1\}|\leq 2$.*

*Proof.* Since there is nothing to prove if $|E|=0$ or 1, we suppose that $|E|>1$. Assume first that $T=(V,E)$ is a linear $N$-graph so that $\chi(T)=\mathrm{lchr}(\mathcal{N}(T))=2$, and let $V=V_1\cup V_2$ be the bipartition of vertices of $T$ with respect to a 2-linear coloring $\kappa\colon V\to[2]$, where we write $V_1=\{v_1^1,\dots,v_k^1\}$ and $V_2=\{v_1^2,\dots,v_l^2\}$ such that $\kappa(v_j^i)=i$ for any $i=1,2$ and $j\in[k]$ or $j\in[l]$.

By the linearity of $\kappa$, we may assume without loss of generality that the inclusions

$$\mathcal{F}_{\mathcal{N}(T)}(v_1^1)\subseteq\mathcal{F}_{\mathcal{N}(T)}(v_2^1)\subseteq\ldots\subseteq\mathcal{F}_{\mathcal{N}(T)}(v_k^1),$$
$$\mathcal{F}_{\mathcal{N}(T)}(v_1^2)\subseteq\mathcal{F}_{\mathcal{N}(T)}(v_2^2)\subseteq\ldots\subseteq\mathcal{F}_{\mathcal{N}(T)}(v_l^2)$$

hold. It follows that if $F_1$ and $F_2$ are facets of $\mathcal{N}(T)$ containing $v_1^1$ and $v_1^2$ respectively, then we must have $V_1 \subseteq F_1$ and $V_2 \subseteq F_2$ by the above inclusions. We note that $F_1 \cap F_2 = \emptyset$ for any such two facets, since $\kappa$ is also a vertex coloring of $T$. We further claim that any of these facets is the neighborhood of only one vertex of $T$. Otherwise, assume that $F_1 = \mathcal{N}(v_r^2)$ and $F_1 = \mathcal{N}(v_s^2)$ for some $r, s \in [l]$ with $r \neq s$. Then, for example, the set $\{v_1^1, v_r^2, v_k^1, v_s^2, v_1^1\}$ forms a cycle in $T$ which is impossible. Therefore, the sets $F_1 = V_1$ and $F_2 = V_2$ are the only facets of $\mathcal{N}(T)$, i.e., $\mathcal{F}(\mathcal{N}(T)) = \{V_1, V_2\}$ which forces $|\{v \in V(T) \colon d(v) > 1\}| \leq 2$.

For the converse, suppose that $|\{v \in V(T) \colon d(v) > 1\}| \leq 2$. We examine each possibility separately. If $|\{v \in V(T) \colon d(v) > 1\}| = 0$, then we must have either $|E| = 0$ or 1 so that the equality $\chi(T) = \mathrm{lchr}(\mathcal{N}(T))$ is clear. Let $|\{v \in V(T) \colon d(v) > 1\}| = 1$, and let $v \in V$ be the vertex with $d(v) > 1$. It follows that $\mathcal{N}(v) = V \backslash \{v\}$, since $T$ is connected. Thus, $\mathcal{F}(\mathcal{N}(T)) = \{\{v\}, V \backslash \{v\}\}$ from which we deduce that $\chi(T) = \mathrm{lchr}(\mathcal{N}(T)) = 2$. Finally, let $|\{v \in V(T) \colon d(v) > 1\}| = 2$, and let $u, v \in V$ be the vertices satisfying $d(u) > 1$ and $d(v) > 1$. We note in this case that $uv \in E$ and $\mathcal{F}(\mathcal{N}(T)) = \{\mathcal{N}(u), \mathcal{N}(v)\}$; hence, $\chi(T) = \mathrm{lchr}(\mathcal{N}(T)) = 2$. □

**Example 2.9.** In Figure 5, we present linear and non-linear $N$-trees with their neighborhood complexes.

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## References

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